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Simulation of ruin probabilities

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In this paper we describe the simulation of ruin probabilities using a new simulation technique based on a martingale transformation.

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1. Introduction

To study the evaluation of an insurance company's portfolio, we can, besides making analytical calculations, perform simulation of the risk process. These simulation models offer a powerful tool for analyzing actuarial problems. For instance, instead of calculating the upper (or lower) bounds on the probability that an insurance company is ruined before (or after) some time t, one can approximate the ruin probabilities using simulation techniques. This last method is growing in importance because of the advancement of computer science which allows for new techniques to be used.

Let $\{N_t\colon t \in \mathbb{R}_+\}$ be the random process that counts the claims of an insurance portfolio and let $\{X_n\colon n \in \mathbb{N}\}$ be a sequence of positive and i.i.d. random variables representing the sizes of the successive claims. We suppose that the claim number process $\{N_t\colon t \in \mathbb{R}_+\}$ is a homogeneous Poisson process with the risk parameter λ (>0). The moments of occurrence of the successive claims are represented by $T_n (n \in \mathbb{N}_0)$ with $T_0 \equiv 0$, and the interoccurrence times of the claims are represented by $U_n (n \in \mathbb{N})$, i.e.

$$U_n = T_n - T_{n-1}. \tag{1.1}$$

The interoccurrence times $\{U_n\colon n \in \mathbb{N}\}$ form a sequence of i.i.d. random variables, which are exponentially distributed with parameter λ. The surplus of the insurance company at some time t is now given by

$$Z_t^{(k)} = k + pt - S_t, \quad t \in \mathbb{R}_+, \tag{1.2}$$

where k is the initial reserve and p the constant premium density, i.e.

$$p = (1+\theta)\lambda \mathrm{E}[X_1],$$

where θ represents the safety loading. Furthermore, $\{S_t\colon t \in \mathbb{R}_+\}$ is the risk process with

$$S_t = \sum_{n=1}^{N_t} X_n, \quad t \in \mathbb{R}_+. \tag{1.3}$$

All these random variables are defined on some probability space $(\Omega, \mathscr{A}, P)$.

Simple methods exist to simulate the risk process. See, for instance, Knuth (1973) and Morgan (1984). The ruin time is defined as usual:

$$\begin{aligned} R_k &= \inf\{t \geq 0\colon Z_t^{(k)} < 0\}, \\ &= \infty \quad \text{if } Z_t^{(k)} \geq 0 \quad \text{for every } t \geq 0. \end{aligned} \tag{1.4}$$

Suppose we simulated the risk process n times and that ruin occurred r times. Then we get the following approximation:

$$P(R_k > t) \approx 1 - r/n. \tag{1.5}$$

It is clear that when the number of simulations n increases the result will be more accurate. A disadvantage of this method is the fact that for some values of the parameters the number r will be very small. If we then run the simulation program a second time, a small difference in r can cause a great difference in the approximation of the ruin probability. Furthermore, it is also possible that the claim sizes are difficult to simulate. Therefore we will try, in those cases, to use a martingale equivalent probability distribution Q instead of P to simulate the risk process. For instance, a probability distribution Q such that the number of ruins that occur in the time interval $[0, t]$ and under this probability distribution is greater than the number of ruins that occur in the same time interval and under the original probability distribution P. A transformation on the obtained number of ruins r under Q will then provide us with an accurate approximation of the ruin probability.

This procedure is described in Section 2. In Section 3 we will consider some examples and illustrate these numerically.

2. The simulation procedure

In the first part of this section we consider the definitions and properties we need to construct our simulation procedure.

Definition 2.1. Take $T \in \mathbb{R}_+$. Then

$$\mathscr{H}_t = \sigma\{S_u: 0 \le u \le t\} \tag{2.1}$$

and

$$\mathscr{H}_\infty = \sigma\{S_t: t \in \mathbb{R}_+\}. \tag{2.2}$$

Definition 2.2. Let P and Q be two probability distributions on $(\Omega, \mathscr{H}_\infty)$. If for each $0 \le t < \infty$, P and Q have the same null sets in $\mathscr{H}_t$, i.e.

$$\{N \in \mathscr{H}_t: P(N) = 0\} = \{N \in \mathscr{H}_t: Q(N) = 0\},$$

then we say that P and Q are *progressively equivalent*.

In the following $\mathrm{E}_Q[.]$ denotes the expectation under Q and $\mathrm{E}_P[.]$ denotes the expectation under P. Let β be a Borel-measurable mapping from $\mathbb{R}_+$ into $\mathbb{R}$ such that

$$\mathrm{E}_P[\mathrm{e}^{\beta(X_n)}] < \infty. \tag{2.3}$$

We put

$$S_t^\beta = \sum_{n=1}^{N_t} \beta(X_n), \quad t \in \mathbb{R}_+. \tag{2.4}$$

Let us consider the following random process on $(\Omega, \mathscr{H}_\infty, P)$:

$$M_t^\beta = \exp\{S_t^\beta\} / \mathrm{E}_P\left[\exp\{S_t^\beta\}\right]. \tag{2.5}$$

Proposition 2.1. *The random process $\{M_t^\beta: t \in \mathbb{R}_+\}$ is a strictly positive martingale on $(\Omega, \mathscr{H}_\infty, P)$ w.r.t. $\{\mathscr{H}_t: t \in \mathbb{R}_+\}$. Moreover,*

$$M_t^\beta = \exp\{S_t^\beta - \lambda t \mathrm{E}_P[\mathrm{e}^{\beta(X_1)} - 1]\}. \tag{2.6}$$

Proof. The random process $\{S_t^\beta: t \in \mathbb{R}_+\}$ has independent increments since it is a compound Poisson process. Furthermore, it is well known that a real random process with independent increments $\{Y_t: t \in \mathbb{R}_+\}$ generates the following two types of martingales:

(1) if each Y_t is integrable, then

$$\{Y_t - \mathrm{E}[Y_t]: t \in \mathbb{R}_+\}$$

is a martingale

(2) if r is a real number such that $0 < \mathrm{E}[\exp\{rY_t\}] < \infty$ for some t belonging to $\mathbb{R}_+$, then

$$\left\{\frac{\exp\{rY_t\}}{\mathrm{E}[\exp\{rY_t\}]}: t \in \mathbb{R}_+\right\}$$

is a martingale.

This proofs the first part of the proposition.

Expression (2.6) is an immediate consequence of Proposition 2.2 in Boogaert and Haezendonck (1989).

Proposition 2.2. *If P and Q are two progressively equivalent probability distributions on $(\Omega, \mathscr{H}_\infty)$, then the corresponding probability distributions Q_{X_1} and P_{X_1} of the random variable X_1 are equivalent.*

Proof. See Delbaen and Haezendonck (1989).

Proposition 2.3. *Let β be a Borel-measurable mapping from $\mathbb{R}_+$ into $\mathbb{R}$ which fulfils condition (2.3), then there exists a unique probability distribution Q on $(\Omega, \mathscr{H}_\infty)$ determined by*

$$Q(A) = \int_A M_t^\beta \,\mathrm{d}P \tag{2.7}$$

for all $0 \le s \le t$ and for all A belonging to $\mathscr{H}_s$.

This probability distribution satisfies the following properties:

(1) Q and P are progressively equivalent;
(2) $\{S_t: t \in \mathbb{R}_+\}$ is still a compound Poisson process on $(\Omega, \mathscr{H}_\infty, Q)$;
(3) $\lambda' = \mathrm{E}_Q[N_1] = \lambda \mathrm{E}_P[\exp\{\beta(X_1)\}]$;
(4) for all A belonging to $\mathscr{R}_+$,

$$Q_{X_1}(A) = \frac{1}{\mathrm{E}_P[\exp\{\beta(X_1)\}]} \times \int_A \exp\{\beta(x)\} \,\mathrm{d}P_{X_1}(x); \tag{2.8}$$

(5) the random process $\{1/M_t^\beta: t \in \mathbb{R}_+\}$ is a martingale on $(\Omega, \mathscr{H}_\infty, Q)$.

Proof. See Delbaen and Haezendonck (1989)

Now, if we can find a Borel-measurable mapping β from $\mathbb{R}_+$ into $\mathbb{R}$ with

$$\mathrm{E}_P[\exp\{\beta(X_1)\}] < \infty$$

and such that

$$\mathrm{E}_Q[N_1]\mathrm{E}_Q[X_1] = \lambda \mathrm{E}_P[X_1 \exp\{\beta(X_1)\}] \tag{2.9}$$

is large, then, during the simulation, the number of ruins under the probability distribution Q will be greater than the number of ruins under the probability distribution P. Using the foregoing proposition it is also possible to transform the probability distribution of the claim size in a more manageable form.

The probability of ruin before (or after) some time t can now be approximated as follows. Consider on $(\Omega, \mathscr{H}_\infty, Q)$ a sequence of i.i.d random variables $\{Y_n: n \in \mathbb{N}\}$ such that Y_n has the same probability distribution as $(1/M^\beta_{R_k})1_{(R_k \le t)}$. This means that

$$Q_{Y_n} = Q_{(1/M^\beta_{R_k})1_{(R_k \le t)}}. \tag{2.10}$$

The law of large numbers implies that

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i = \mathrm{E}_Q[Y_1]. \tag{2.11}$$

Therefore we successively find

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} Y_i &= \mathrm{E}_Q\left[\frac{1}{M^\beta_{R_k}} 1_{(R_k \le t)}\right] \\ &= \int_{(R_k \le t)} \frac{1}{M^\beta_{R_k}} \mathrm{d}Q \\ &= \int_{(R_k \le t)} \frac{1}{M^\beta_t} \mathrm{d}Q \\ &\quad \text{since } \left\{\frac{1}{M^\beta_t} : t \in \mathbb{R}_+\right\} \\ &\quad \text{is } a\ Q\text{-martingale} \\ &= \int_{(R_k \le t)} \frac{M^\beta_t}{M^\beta_t} \mathrm{d}P \quad \text{from (2.8)} \\ &= P(R_k \le t). \end{aligned} \tag{2.12}$$

Hence,

$$P(R_k > t) \approx 1 - \frac{\sum_{i=1}^{n} E_i}{n}, \tag{2.13}$$

where n is the number of simulations and where

$$\begin{aligned} E_i &= 1/M^\beta_{R_k} && \text{if } R_k \le t, \\ &= 0 && \text{if } R_k > t, \end{aligned}$$

or, equivalently,

$$\begin{aligned} \mathrm{E}_i &= \exp\left\{-S^\beta_{R_k} + \lambda R_k \mathrm{E}_P[\mathrm{e}^{\beta(X_1)} - 1]\right\} && \text{if } R_k \le t, \\ &= 0 && \text{if } R_k > t. \end{aligned}$$

3. Examples

In this section we will illustrate the preceding simulation procedure by some examples.

Example 3.1. We put

$$\bar{f} = \frac{x+1}{x+2} \mathrm{e}^{-x} 1_{[0,\infty]}(x). \tag{3.1}$$

Then we find that

$$\int_0^\infty \bar{f}(x)\, \mathrm{d}x = 1 - \mathrm{e}^2 \int_2^\infty \frac{\mathrm{e}^{-x}}{x} \mathrm{d}x = C.$$

Thus $C = 1 - \mathrm{e}^2 \mathrm{E}_i(2) \approx 1 - \mathrm{e}^2 * 0.04890$ (see for instance Schaum's Mathematical Handbook, p. 251).

We now suppose that the claim size X_1 has a density function given by

$$f_{X_1}(x) = (1/C)\bar{f}(x). \tag{3.2}$$

Then we find that

$$\mathrm{E}_P[X_1] = 2(1-C)/C. \tag{3.3}$$

Consider the following Borel-measurable mapping β:

$$\beta(x) = \ln((x+2)/(x+1)) + x(1-\epsilon), \tag{3.4}$$

where ϵ is strictly positive. Then

$$\mathrm{E}_P[\exp\{\beta(X_1)\}] = 1/C\epsilon \tag{3.5}$$

and therefore finite. This implies that Proposition 2.3. holds and thus

$$\mathrm{E}_Q[N_1] = \lambda' = \lambda/C\epsilon \tag{3.6}$$

and

$$\begin{aligned} Q_{X_1}(A) &= C\epsilon \int_A \frac{x+2}{x+1} \mathrm{e}^{x(1-\epsilon)}\, \mathrm{d}P_{X_1}(x) \\ &= \epsilon \int_A \mathrm{e}^{-\epsilon x}\, \mathrm{d}x. \end{aligned} \tag{3.7}$$

For the simulation of the interoccurrence times and the claim sizes we use a classical procedure. But now the claim size X_1 is exponentially distributed with parameter ϵ and the claim number process $\{N_t: t \in \mathbb{R}_+\}$ is a Poisson process with risk parameter $\lambda/C\epsilon$. Finally we get

$$P(R_k > t) = 1 - \frac{\sum_{i=1}^{n} E_i}{n},$$

Table 1

ϵ	$\theta = 0.05$	$\theta = 0.10$
$t = 1, k = 0$		
1	0.526	0.527
0.9	0.524	0.528
0.8	0.523	0.527
$t = 5, k = 0$		
1	0.255	0.287
0.9	0.259	0.284
0.8	0.254	0.278
$t = 5, k = 1$		
1	0.430	0.461
0.9	0.445	0.451
0.8	0.432	0.469
$t = 5, k = 5$		
1	0.870	0.872
0.9	0.865	0.875
0.8	0.867	0.874

where

$$E_i = \exp\left\{-S_{R_k}^{\beta} + \lambda R_k(1/C\epsilon - 1)\right\} \quad \text{if } R_k \le t,$$
$$= 0 \quad \text{if } R_k > t.$$

Numerical results can be found in Table 1.

Example 3.2. We now suppose that the claim size X_1 is Pareto distributed with parameters $\alpha(>0)$ and $\gamma(>0)$, i.e.

$$f_{X_1}(x) = (\alpha\gamma^{\alpha}/x^{\alpha+1})1_{[\gamma,\infty]}(x). \tag{3.8}$$

In Morgan (1984) we find a method to simulate such random variable. We will now show that the simulation of such risk process becomes easier using the procedure of Section 2. Put

$$\beta(x) = (\alpha+1)\ln(x) - \epsilon(x-\gamma), \tag{3.9}$$

where ϵ is a strictly positive constant. Then

$$\mathrm{E}_P[\mathrm{e}^{\beta(X_1)}] = \alpha\gamma^{\alpha}/\epsilon \tag{3.10}$$

and therefore finite. Proposition 2.3 then implies

$$\mathrm{E}_Q[N_1] = \lambda' = \lambda(\alpha\gamma^{\alpha}/\epsilon) \tag{3.11}$$

and

$$Q_{X_1}(A) = \epsilon\int_A \mathrm{e}^{-\epsilon x}\,\mathrm{d}x. \tag{3.12}$$

This means that the claim size is exponentially distributed with parameter ϵ under the probability

Table 2

ϵ	$\theta = 0.05$	$\theta = 0.10$
$t = 1, k = 0$		
1	0.473	0.482
0.9	0.473	0.481
0.8	0.467	0.477
$t = 5, k = 0$		
1	0.277	0.297
0.9	0.271	0.290
0.8	0.268	0.288
$t = 25, k = 0$		
1	0.953	0.954
0.9	0.942	0.943
0.8	0.953	0.954

distribution Q. We get

$$P(R_k > t) \approx 1 - \frac{\sum_{i=1}^{n} E_i}{n},$$

where

$$E_i = \exp\left\{-S_{R_k}^{\beta} + \lambda R_k(\alpha\gamma^{\alpha}/\epsilon - 1)\right\} \quad \text{if } R_k \le t,$$
$$= 0 \quad \text{if } R_k > t.$$

Numerical results for $\alpha = 1$, $\gamma = 2$ can be found in Table 2.

Example 3.3. We now suppose that the claim size X_1 is gamma distributed with parameters $a(>0)$ and $b(>0)$, i.e.

$$f_{X_1}(x) = (b^a/\Gamma(a))x^{a-1}\,\mathrm{e}^{-bx}1_{[0,\infty]}(x). \tag{3.13}$$

We will now show how simulation of such a risk process can be done using the procedure of Section 2. Put

$$\beta(x) = (1-a)\ln(x) + bx - \epsilon x, \tag{3.14}$$

where ϵ is a strictly positive constant. Then

$$\mathrm{E}_P[\mathrm{e}^{\beta(X_1)}] = b^a/\Gamma(a)\epsilon \tag{3.15}$$

and therefore finite. Proposition 2.3 then implies

$$\mathrm{E}_Q[N_1] = \lambda' = \lambda b^a/\Gamma(a)\epsilon \tag{3.16}$$

and

$$Q_{X_1}(A) = \epsilon\int_A \mathrm{e}^{-\epsilon x}\,\mathrm{d}x. \tag{3.17}$$

This means that the claim size is exponentially distributed with parameter ϵ under the probability

Table 3

ϵ	$\theta = 0.05$	$\theta = 0.10$
$t = 1, k = 0$		
1	0.527	0.535
0.9	0.541	0.547
0.8	0.530	0.535
$t = 5, k = 0$		
1	0.275	0.290
0.9	0.273	0.287
0.8	0.271	0.290
$t = 25, k = 0$		
1	0.134	0.155
0.9	0.134	0.155
0.8	0.131	0.154
$t = 1, k = 10$		
1	1.000	1.000
0.9	0.999	0.999
0.8	0.999	0.999

distribution Q. We get

$$P(R_k > t) \approx 1 - \frac{\sum_{i=1}^{n} E_i}{n},$$

where

$$E_i = \exp\left\{ -S_{R_k}^{\beta} + \lambda R_k \left(b^a / \Gamma(a) \epsilon - 1 \right) \right\} \quad \text{if } R_k \le t,$$
$$= 0 \quad \text{if } R_k > t.$$

If we take $a = 1$ (3.18) reduces to

$$f_{X_1}(x) = b\ \mathrm{e}^{-bx} 1_{[0,\infty]}(x).$$

So the above method can be used to improve simulation results for the exponential claim distribution. Numerical results for $b = 1$ can be found in Table 3.

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