The Power-Series Algorithm for a Wide Class of Markov Processes

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Abstract: The Power-Series Algorithm has been used to calculate the steady-state distribution of various queueing models with a multi-dimensional birth-and-death structure. In this paper, the method is generalized to a much wider class of Markov processes, including for example very general networks of queues and all kinds of non-queueing models. Also, the theoretical justification of the method is improved by deriving sufficient conditions for the steady-state probabilities and moments to be analytic. To do this, a lemma is derived that ensures ergodicity of a Markov process with generator Λ if the set of balance equations $\pi\Lambda = o$ has a solution π that satisfies $\sum_i \pi_i = 1$ and $\sum_i |\pi_i \Lambda_{ii}| < \infty$ but that need not be non-negative.
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1 Introduction

The Power-Series Algorithm (PSA) is a method to calculate the steady-state distribution of a multi-dimensional Markov process. If the process is ergodic, this steady-state distribution is determined by a set of balance and normalization equations. But since the number of equations is equal to the size of the state space these equations can be difficult to solve. The PSA aims to be an efficient way to solve them.

The basic idea is like a homotopy: the original Markov process is transformed with a parameter γ in such a way that for $\gamma = 1$ the transformed Markov process is the original Markov process and for $\gamma = 0$ the transformed Markov process is easy to analyze while the information from the problem at $\gamma = 0$ can be used to solve the problem at $\gamma = 1$. If an appropriate transformation is used, the steady-state probabilities of the transformed process are analytic functions of γ at $\gamma = 0$ and the coefficients of the power-series expansions can be calculated recursively. The

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steady-state distribution of the original process is then found by evaluating these power series at $\gamma = 1$. This way, the original difficult set of equations is replaced by a larger number of easily solvable equations. The basic idea of the PSA stems from Keane (see [8]). It has been applied to queueing models with queues in parallel [8, 2], the shortest-queue model [4] and various polling models [3, 6]. An overview can be found in [5]. For all these models the transformation parameter γ can be interpreted as the load of the system.

Koole [11] suggests a type of transformations that can handle Markov processes with a single recurrent class, provided that the steady-state probabilities of the transformed process are analytic in the transformation parameter. In the present paper, a specific transformation is proposed that not only transforms the transition rates but also the transition structure. Sufficient conditions on the Markov process are derived for this transformation to be appropriate and the steady-state probabilities to be analytic. The transformation is a generalization of the transformation used in the papers mentioned in the previous paragraph, so the results in this paper also apply to those models. With this generalized transformation the method is applicable to a much wider class of Markov processes, including for example queueing networks with very general arrival, service and routing processes [10]. Unfortunately, the transformation parameter γ no longer has a clear interpretation for all processes. As a consequence of this, only the value $\gamma = 1$ is of interest and no longer the whole range $\gamma \in [0, 1]$. In queueing models, this could be overcome by using more than one transformation parameter. For example, in networks of queues a parameter ρ could be used to transform the arrival process and a parameter σ for the routing process. However, this clear categorization of transitions is not always possible and using several parameters leads to power-series expansions in more than one variable which gives rise to more numerical problems.

Besides extending the class of Markov processes that can be handled, also the theoretical justification of the algorithm is improved by showing that, under certain conditions, the steady-state probabilities as functions of γ are indeed analytic in γ in a neighbourhood of $\gamma = 0$. This is the basic assumption of the method, but could so far only be proved for some specific models [8, 9]. The proof is comparable to the proof that was given for the $BMAP/PH/1$ queue [9] but now in a much more general setting.

The structure of the paper is as follows. In section 2 the notation for the considered Markov processes is introduced. In section 3 the algorithm to calculate the steady-state distribution and moments is described, assuming analyticity. Section 4 states sufficient conditions for the steady-state distribution and moments to be analytic for small values of the transformation parameter. Finally, in section 5 some conclusions are drawn.

2 The Markov process

Let $\{(\mathcal{N}_t, \mathcal{I}_t); t \geq 0\}$ be a continuous-time Markov process on state space $\Omega = \mathbb{N}^S \times \{1, \ldots, I\}$. In a queueing context, S could be the number of queues, $\mathcal{N}_t$ the queue-length process and $\mathcal{I}_t$

a supplementary variable to model for example non-exponentiality of the arrival and service process. Throughout this paper queueing terminology will be used, although the considered Markov processes need not be queueing processes.

The process has the following transition rates:

$$(n,i) \to (n+b,j) \quad \text{with rate} \quad \alpha_{bj}(n,i), \quad \text{for } b \in \mathbb{Z}^S,$$

for all (n,i), $(n+b,j) \in \Omega$. It is assumed that the Markov process is irreducible, non-instantaneous and ergodic. Therefore, the steady-state probabilities

$$p(n,i) = \Pr\{\ (\mathcal{N},\mathcal{I}) = (n,i)\ \} = \lim_{t\to\infty} \Pr\{\ (\mathcal{N}_t,\mathcal{I}_t) = (n,i)\ \}$$

exist for all $(n,i) \in \Omega$. They are uniquely determined by the balance and normalization equations

$$\begin{aligned} p(n)\bar{A}(n) &= \sum_{b\in\mathbb{Z}^S} p(n-b)A_b(n-b), \quad \text{for } n \in \mathbb{N}^S, \\ \sum_{n\in\mathbb{N}^S} p(n)e &= 1, \end{aligned}$$

where

$$\begin{aligned} p(n) &= \{\ p(n,i)\ \}_{1\le i\le I}\,, \\ A_b(n) &= \{\ \alpha_{bj}(n,i)\ \}_{1\le i,j\le I}\,, \\ \bar{A}(n) &= \{\ \bar{\alpha}(n,i)\ 1(i=j)\ \}_{1\le i,j\le I}\,, \\ \bar{\alpha}(n,i) &= \sum_{b\in\mathbb{Z}^S}\sum_{1\le j\le I} \alpha_{bj}(n,i) < \infty, \end{aligned}$$

for $n \in \mathbb{N}^S$ and $b \in \mathbb{Z}^S$. The vector e is a column vector of ones with appropriate size. All other vectors are row vectors. The vector o is a row vector of zeros with appropriate size.

The set of balance equations can be very large or infinite and hence difficult to solve. The PSA transforms the Markov process with a parameter γ in such a way that for $\gamma = 1$ the transformed process is equal to the original process. The steady-state distribution of the transformed process can be regarded as a function of γ. If the transformation is chosen in an appropriate way these functions will be analytic functions of γ at $\gamma = 0$ and the coefficients of the power-series expansions can be calculated recursively. The steady-state distribution of the original process is then found by evaluating these power series at $\gamma = 1$.

In the rest of this paper, the transformed Markov process for arbitrary values of $\gamma \in [0,1]$ will be called the γ-process. To specify the γ-process, define the following subsets of $\mathbb{Z}^S$:

$$Z_< = \left\{\ b \in \mathbb{Z}^S\ \middle|\ be < 0\ \right\},$$

$$Z_\ge = \left\{\ b \in \mathbb{Z}^S\ \middle|\ be \ge 0\ \right\}.$$

The transitions in $Z_<$ decrease the total queue length and will be called the downward transitions; those in $Z_\ge$ will be called upward transitions. In the γ-process, the downward transitions will be the same as in the original Markov process; the upward transitions will be transformed. For each upward transition b, define a scalar r_b and a transition-vector d_b as

$$
\begin{aligned}
r_b &= \begin{cases} be, & \text{if } be > 0, \\ 1, & \text{if } be = 0, \end{cases} \\
d_b &= [\, b \,]_- .
\end{aligned}
$$

The number r_b is equal to the total increase of the queue lengths caused by transition b if this increase is positive and equal to 1 if the total queue length remains constant. The operator $[\, x \,]_-$ denotes the element-wise minimum of the zero vector and x. Hence, $d_b \in -\mathbb{N}^S$ and $d_b = o$ if and only if $b \in \mathbb{N}^S$. For γ in [0,1], the γ-process is the process on Ω with the following transitions:

$$
\begin{array}{llcll}
(n,i) \to (n+b,j) & \text{with rate} & & \alpha_{bj}(n,i), & \text{for } b \in Z_<, \\
(n,i) \to (n+b,j) & \text{with rate} & \gamma^{r_b} & \alpha_{bj}(n,i), & \text{for } b \in Z_\geq, \\
(n,i) \to (n+d_b,j) & \text{with rate} & (\, 1-\gamma^{r_b} \,) & \alpha_{bj}(n,i), & \text{for } b \in Z_\geq,
\end{array}
$$

for all (n,i), $(n+b,j)$, $(n+d_b,j) \in \Omega$. In this γ-process, the transitions in $Z_<$ have the same rate as in the original Markov process. The transitions in $Z_\geq$ are still possible but the original rate is multiplied by γ^{r_b} and the corresponding transition d_b is made with the original rate multiplied by $(1-\gamma^{r_b})$. So from each state, the total transition rate is the same as in the original Markov process, but each upward transition is replaced by a downward transition or a selfloop with probability $(1-\gamma^{r_b})$. The 1-process (that is the γ-process with $\gamma = 1$) is the same as the original Markov process; the 0-process (the γ-process with $\gamma = 0$) is a Markov process with only downward transitions and selfloops. If the γ-process is ergodic, the steady-state distribution is uniquely determined by the balance and normalization equations:

$$
\begin{aligned}
p(\gamma,n)\bar{A}(n) &= \sum_{b \in Z_<} p(\gamma,n-b) \quad A_b(n-b) \\
&+ \sum_{b \in Z_\geq} \gamma^{r_b} \quad p(\gamma,n-b) \quad A_b(n-b) \\
&+ \sum_{b \in Z_\geq} (\, 1-\gamma^{r_b} \,) \; p(\gamma,n-d_b) \quad A_b(n-d_b), \\
\sum_{n \in \mathbb{N}^S} p(\gamma,n)e &= 1,
\end{aligned}
\tag{1}
$$

for $n \in \mathbb{N}^S$ and $\gamma \in [0,1]$.

The most important part of the transformation is that the rates of transitions that increase the total number of customers by r are multiplied by γ^r ($r = be > 0$). As a consequence of this, the steady-state probabilities satisfy

$$
p(\gamma,n) \in \mathcal{O}(\gamma^{ne}), \quad \text{for } \gamma \downarrow 0,\; n \in \mathbb{N}^S, \tag{2}
$$

which will be proved in section 4. It will also be shown that the $p(\gamma,n)$ are analytic in γ, so that they can be represented by their power-series expansions (3). Property (2) then implies that the coefficients corresponding to γ^r are zero for all states with more than r customers in the system, so for each fixed r there are only finitely many non-zero coefficients.

The second part of the transformation is that transitions that keep the total number in the system constant ($be = 0$) are multiplied by γ. Without this, in the r-th step of the algorithm one large set of equations would have to be solved with, in general, size $I \times \binom{r+S-1}{S-1}$, while now $\binom{r+S-1}{S-1}$ sets of equations with size I need to be solved, which is usually much easier.

Finally, the extra transitions d_b are added, for $b \in Z_{\geq}$, unlike in all previous algorithms where only the non-zero transition rates were transformed. These transitions are added to extend the class of Markov processes that can be handled. In section 3, it is shown that the algorithm is well defined if the 0-process has a single recurrent class consisting of only empty states. Since the extra transitions are non-positive, these extra transitions extend the class of models for which this assumption is satisfied. For example, without these extra transitions only feed-forward networks could be analyzed, while now networks with general Markovian routing can be studied [10].

Example. Consider a network with 2 queues. Customers arrive simultaneously at both queues according to a Poisson process with rate λ. At queue 1 the service rate is μ_1 and after service all customers go to queue 2. At queue 2 the service rate is μ_2 and after service all customers leave the network. For this model the transition diagram is given in figure 1.

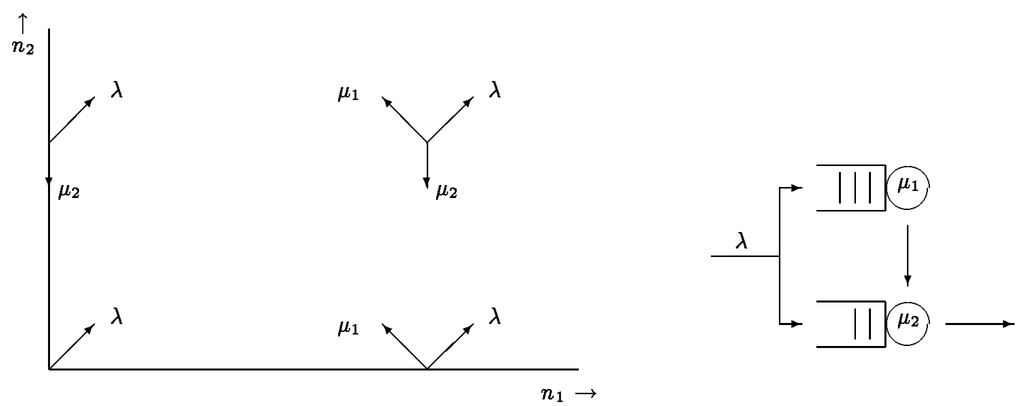

Figure 1: The untransformed queueing network

The parameters are:

b	$\alpha_b(n)$	r_b	d_b
$\binom{1}{1}$	λ	2	$\binom{0}{0}$
$\binom{-1}{1}$	$\mu_1\ 1(n_1 \geq 1)$	1	$\binom{-1}{0}$
$\binom{0}{-1}$	$\mu_2\ 1(n_2 \geq 1)$	–	–

In the transformed model, the arrival rate is multiplied by γ^2. The associated transition d_b is a selfloop with rate $(1-\gamma^2)\lambda$, which doesn't influence the steady-state behaviour. After service at queue 1, customers now go to queue two with probability γ and leave the network with probability $1-\gamma$.

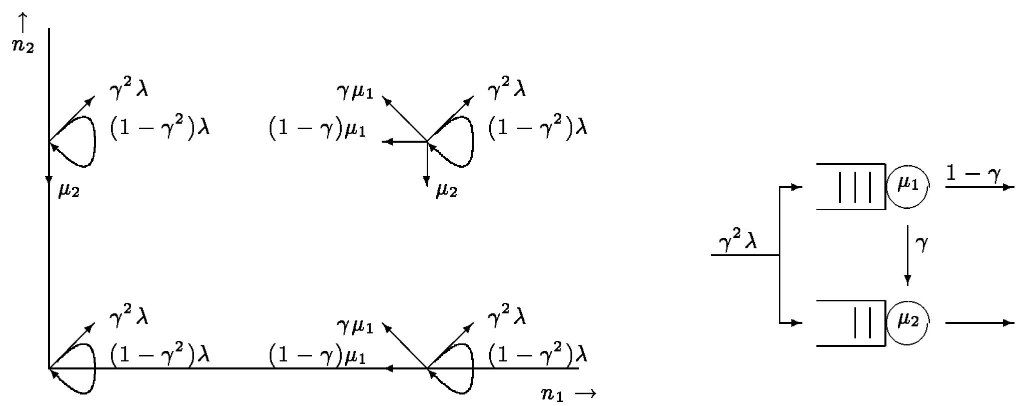

Figure 2: The transformed queueing network

3 The Power-Series Algorithm

In this section an algorithm will be proposed to calculate the expansions of the steady-state probabilities of the γ-process. For now, assume that these probabilities are analytic functions of γ at $\gamma = 0$ satisfying the order property (2), so that their power-series expansions exist and converge:

$$p(\gamma, n) = \sum_{r \geq ne} \gamma^r u(r, n), \quad \text{for } n \in \mathbb{N}^S. \tag{3}$$

In section 4 it will be shown that this assumption is justified under mild conditions on the transition rates. From the expansions of the steady-state probabilities, the expansions of moments can be obtained:

$$\mathrm{E}_\gamma\{\, f(\mathcal{N}, \mathcal{I})\,\} = \sum_{(n,i)\in\Omega} p_i(\gamma, n)\, f(n,i) = \sum_{r\geq 0} \gamma^r \sum_{(n,i)\in\Omega:\; ne\leq r} u_i(r,n)\, f(n,i), \tag{4}$$

for functions $f : \Omega \to \mathbb{R}$. Examples are:

$$\begin{aligned}
\Pr_\gamma\{\, \mathcal{N} = n \,\} &= \mathrm{E}_\gamma\{\, 1(\mathcal{N} = n)\,\} = \sum_{r\geq ne} \gamma^r u(r,n)e, && \\
\mathrm{E}_\gamma\{\, \mathcal{N}_s^t \,\} &= \sum_{r\geq 0} \gamma^r \sum_{ne\leq r} n_s^t\ u(r,n)e, && \text{for } 1 \leq s \leq S,\ t \geq 0, \\
\mathrm{E}_\gamma\{\, \mathcal{N}_s \mathcal{N}_t \,\} &= \sum_{r\geq 0} \gamma^r \sum_{ne\leq r} n_s n_t\ u(r,n)e, && \text{for } 1 \leq s,t \leq S.
\end{aligned} \tag{5}$$

In this paper, the calculation of the coefficients of (3) and (4) will be considered and the question whether these power series converge. An essential part of an efficient algorithm is formed by procedures to improve the convergence of these power series, like conformal mappings and the epsilon algorithm. For a discussion of these techniques, see [3].

The expansions (3) can be inserted into the balance and normalization equations (1). This renders equalities between functions of γ. Two analytic functions are only identical if all coefficients of their power-series expansions are identical. After some rearrangements this leads to the following equalities

$$\begin{aligned} u(r,n)B(n) &= \sum_{b\in Z_<} u(r,n-b) \quad A_b(n-b) \\ &+ \sum_{b\in Z_1} u(r-r_b,n-b) \quad A_b(n-b) \\ &+ \sum_{b\in Z_2} u(r,n-d_b) \quad A_b(n-d_b) \\ &- \sum_{b\in Z_1} u(r-r_b,n-d_b) \quad A_b(n-d_b), \\ u(r,o)e &= -\sum_{0<ne\le r} u(r,n)e \quad +1(r=0) \end{aligned} \tag{6}$$

for $n \in \mathbb{N}^S$ and $r \geq ne$, with

$$\begin{aligned} B(n) &= \bar{A}(n) - \sum_{b\in\mathbb{N}^S} A_b(n), && \text{for } n \in \mathbb{N}^S, \\ u(r,n) &= o, && \text{for } n \notin \mathbb{N}^S \text{ or } r < ne, \\ Z_1 &= Z_\geq \setminus \{o\}, \\ Z_2 &= Z_\geq \setminus \mathbb{N}^S. \end{aligned}$$

The coefficients $u(r,n)$ in the third summation in the right-hand side (RHS) have been brought to the left (those with $b \in \mathbb{N}^S$, so $d_b = o$). The coefficients $u(r-r_b,n)$ that cancel out in the second and fourth summation have been removed (those with $b = d_b = o$). For the I coefficients of the empty states, there are $I+1$ equations. One of them can be ignored, which comes down to ignoring one of the balance equations. For any matrix A, let A^* denote the matrix that is equal to A but with the first column removed. Then, ignoring the balance equation of state $(o,1)$, equation (6) for $n = o$ can be reduced to

$$\begin{aligned} u(r,o)B^*(o) &= \sum_{b\in Z_<} u(r,-b) \quad A_b^*(-b) \\ &+ \sum_{b\in Z_2} u(r,-d_b) \quad A_b^*(-d_b) \\ &- \sum_{b\in Z_1} u(r-r_b,-d_b) \quad A_b^*(-d_b), \\ u(r,o)e &= -\sum_{0<ne\le r} u(r,n)e, \end{aligned} \tag{7}$$

for $r > 0$ and

$$\begin{aligned} u(0,o)B^*(o) &= o, \\ u(0,o)e &= 1. \end{aligned} \tag{8}$$

The second summation in the RHS of (6) can be omitted in (7) because $-b \notin \mathbb{N}^S$ for all $b \in Z_1$. For $r = 0$ all summations in the RHS can be omitted because of the order property (2).

The coefficients $u(\tilde{r}, \tilde{n})$ in the RHS of both (6) and (7) satisfy either $\tilde{r} < r$ or they satisfy $\tilde{r} = r$ and $\tilde{n}e > ne$. Because $u(r, n) = o$ if $ne > r$, this implies that all coefficients can be calculated for increasing values of r and, for each fixed r, for decreasing values of ne starting with $ne = r$. So if the expansions of the steady-state probabilities (3) are to be calculated up to the coefficients of the R-th power of γ, the following algorithm can be used:

```
Power-Series Algorithm
calculate u(0, o) from (8),
for r := 1 to R do
        for N := r down to 1 do
                for all n ∈ IN^S with ne = N do
                        calculate u(r, n) from (6),
        calculate u(r, o) from (7).
```

This algorithm is well-defined if all sets of equations have a unique solution. Necessary and sufficient for this is the following assumption:

Assumption 0. *The matrices $B(n)$, for $n \neq o$, and $(B^*(o), e)$ are invertible.*

Suppose the size of the supplementary space I is equal to 1. Assumption 0 then reduces to the assumption that the scalar $B(n)$ is non-zero for all $n \neq o$. This is the same as assuming that, in the 0-process, each non-empty state has a positive transition rate. Since the 0-process has only downward transitions and selfloops, this means that the empty state is the only absorbing state and will eventually be reached. For $I \geq 1$, assumption 0 is equivalent to the following assumption:

Assumption 0'. *The 0-process has a single recurrent class consisting of only empty states, which will eventually be reached from any state in Ω.*

This can be shown by studying the 0-process in more detail. Besides the γ-processes on Ω, Markov processes will be considered on the finite sets $\Omega_n = \{n\} \times \{1, \ldots, I\}$ and $\bar{\Omega}_n = \{n\} \times \{1, \ldots, I, \Delta\}$, where Δ is an absorbing state.

First, consider the non-empty states. The diagonal elements of $B(n)$ are equal to the rates in the 0-process out of the states in Ω_n and the non-diagonal elements are equal to minus the rates in the 0-process from states in Ω_n to other states in Ω_n. Therefore, the elements of the vector $B(n)e$ are equal to the total rate in the 0-process of transitions from states in Ω_n to states not in Ω_n. Since the 0-process has no upward transitions, these transitions can only be downward and once the 0-process has left Ω_n it will never return. Aggregate all states not in Ω_n into a single state Δ. Starting from a state in Ω_n, the 0-process then reduces to a process on the finite state space $\bar{\Omega}_n$. Entering Δ corresponds to a downward transition from Ω_n in the 0-process. The process on $\bar{\Omega}_n$ has the following balance and normalization equations:

$$(\pi, \pi_\Delta) \begin{pmatrix} B(n) & -B(n)e \\ o & 0 \end{pmatrix} = (o, 0), \quad (\pi, \pi_\Delta) \begin{pmatrix} e \\ 1 \end{pmatrix} = 1.$$

If and only if $B(n)$ is invertible, it has the unique steady-state distribution (π, π_Δ) equal to

$$(\pi, \pi_\Delta) = (o, 1) \begin{pmatrix} B(n) & e \\ o & 1 \end{pmatrix}^{-1} = (o, 1) \begin{pmatrix} B^{-1}(n) & -B^{-1}(n)e \\ o & 1 \end{pmatrix} = (o, 1).$$

The assumption that $B(n)$ is invertible for $n \neq o$ is therefore equivalent to the assumption that, in each Markov process on $\bar{\Omega}_n$, the state Δ is the only absorbing class. This in turn is equivalent to the assumption that, in the 0-process, all non-empty states are transient and the empty states will eventually be reached.

Next, consider the 0-process after it has reached an empty state. Once the 0-process is in the set Ω_o, it will never leave Ω_o. On Ω_o, the steady-state distribution is determined by the balance and normalization equations

$$\pi B(o) = o, \quad \pi e = 1. \tag{9}$$

This set of equations uniquely determines the steady-state distribution if and only if the Markov process on the finite state space Ω_o has only one recurrent class, but also if and only if the matrix $(B^*(o), e)$ is invertible. Therefore, assumption 0 and assumption 0' are indeed equivalent.

There are several ways to overcome the difficulties that arise when assumption 0 is not satisfied. In [6], the problem was solved by changing the order of calculation of the coefficients. In [9, 10] it was possible to solve the difficulties by obtaining additional equations. These additional equations can, for example, come from independency or symmetry properties of the model. These approaches can not be applied in general.

Assumption 0' suggests yet another approach. According to this assumption, problems arise when the 0-process has several recurrent classes. So the solution to the problem could be to add more transitions such that the 0-process has only one recurrent class consisting of all empty states. For instance, this can be done by adding the following transitions to the γ-process:

$$\begin{array}{lll} (n,i) \to (\, [\, n - e^T \,]_+, i \,) & \text{with rate} \quad \delta(1-\gamma), & \text{for } 1 \leq i \leq I,\ n \neq o, \\ (o,i) \to (\, o, i \bmod I + 1 \,) & \text{with rate} \quad \delta(1-\gamma), & \text{for } 1 \leq i \leq I, \end{array}$$

for some fixed $\delta > 0$. The operator $[\, x \,]_+$ denotes the element-wise maximum of the zero-vector and x. This way, all non-empty states have a transition to a state with less customers, so for $\gamma = 0$ the non-empty states can not be recurrent. For the empty states, a set of cyclic transitions is added so that all the empty states form one recurrent class. The γ-process with $\gamma = 1$ is still equal to the original process. Setting up the balance equations and the recurrence relations for the coefficients of the power-series expansions shows that indeed for this extended transformation all coefficients can be calculated recursively with an algorithm similar to the algorithm described before. Because of the extra transitions, this new transformed process differs more from the original process than the γ-process described in section 2. For some simple models (satisfying assumption 0') this led to less efficient algorithms. This loss of efficiency could probably be reduced if the extra transitions for the non-empty states are only added to states (n, i) for which $B(n)$ is not invertible, but no detailed numerical experiments were made.

Adding the transitions above could also be done for another reason. It was assumed that $\Omega = \mathbb{N}^S \times \{1, \ldots, I\}$. Suppose that Ω is a proper subset of $\mathbb{N}^S \times \{1, \ldots, I\}$. Because of the extra transitions d_b, for $b \in Z_{\geq}$, it could happen that in the γ-process there is a transition to a state not in Ω. From this state no transition rate has been defined. Adding the above transitions solves this problem.

4 Analyticity of the steady-state probabilities and moments

The basic assumption of the algorithm is that the steady-state probabilities of the γ-process are analytic functions of γ at $\gamma = 0$, satisfying the order property (2). It will be shown in two steps that this assumption is justified. First, in theorem 1, it is proved that the power series produced by the algorithm converge for small values of γ. Therefore, the functions corresponding to these power series are well-defined analytic functions at $\gamma = 0$, and they satisfying the order property. Then, in theorem 2, it is proved that these functions are the steady-state probabilities of the γ-process. In theorem 3, it is shown that the expectation of $f(\mathcal{N}, \mathcal{I})$ is analytic at $\gamma = 0$ if the function $f(n, i)$ grows at most exponentially in n.

The theorems only make statements for γ in a neighbourhood of $\gamma = 0$, while the value $\gamma = 1$ is the value for which the γ-process is equal to the original Markov process. Therefore, the importance of the theorems is more theoretical than practical. In practice, techniques like conformal mappings and the epsilon algorithm [3] need to be applied to extend the convergence region of the expansions. A necessary condition for these techniques to be helpful is that the steady-state probabilities or expectations are analytic in a neighbourhood of $\gamma = 0$, which is what the theorems state.

In the proofs below, the following equalities are repeatedly used:

$$\# \left\{ n \in \mathbb{N}^S \;\middle|\; ne = r \right\} = \binom{r+S-1}{S-1},$$

$$\sum_{n \in \mathbb{N}^S} p^{ne} = (1-p)^{-S} = \sum_{r \geq 0} p^r \binom{r+S-1}{S-1},$$

for $S \geq 1$ and $|p| < 1$. The last equality follows from the fact that the probabilities of a negative binomial distribution sum up to 1.

Let $\mathcal{E}$ be the set of all functions $f : \Omega \rightarrow \mathbb{R}$ for which positive constants a and b exist such that

$$|f(n,i)| \leq ab^{ne} \quad \text{for all } (n,i) \in \Omega. \tag{10}$$

The set $\mathcal{E}$ contains all functions that grow at most exponentially in n. Since the growth rate b is allowed to exceed 1, this set of functions is very large.

For an I by J matrix A, define the norm

$$\|A\| = \max_{1 \leq i \leq I} \sum_{1 \leq j \leq J} |A_{ij}|.$$

For arbitrary matrices A and B, with well-defined sum or product, the following inequalities hold:

$$\|A + B\| \leq \|A\| + \|B\|, \quad \|AB\| \leq \|A\|\|B\|,$$

known as the triangle inequality and consistency. Notice that $\|x\| = xe = 1$ if x is a stochastic row vector, that $\|e\| = 1$ and that $\|A^*\| \leq \|A\|$.

Define, for all $n \in \mathbb{N}^S$, the scalars

$$\begin{aligned} a(n) &= \sum_{b \in Z_<} \|A_b(n-b)\| + \sum_{b \in Z_1} \|A_b(n-b)\| \\ &+ \sum_{b \in Z_2} \|A_b(n-d_b)\| + \sum_{b \in Z_1} \|A_b(n-d_b)\|, \end{aligned}$$

$$b(n) = \begin{cases} \left\|B^{-1}(n)\right\|, & \text{for } n \in \mathbb{N}^S \backslash \{o\}, \\ \left\|(B^*(o), e)^{-1}\right\|, & \text{for } n = o. \end{cases}$$

The scalar $a(n)$ is the sum of the norms of the matrices in the RHS of (6) and (7) and $b(n)$ is the norm of the inverse of the matrices in the left-hand side (LHS). Assumption 1 is sufficient to ensure convergence of the power series produced by the algorithm for γ small enough:

Assumption 1. $\sup_{n \in \mathbb{N}^S} a(n)b(n) < \infty$.

The scalar $a(n)$ is increasing in the rates into states $(n, .)$ and $b(n)$ increasing in the inverse of the rates out of states $(n, .)$. The assumption requires that the rate-in is not allowed to be too large compared to the rate-out, so it is related to a stability condition. A sufficient condition for the supremum to be finite is that there are only finitely many different values of $a(n)$ and $b(n)$. For example, consider a network of queues where the arrival process does not depend on the queue lengths and the service process only depends on whether queues are empty or not. Then there can be only 2^S different values of $a(n)b(n)$, so assumption 1 is satisfied. For the same reason, similar models with a finite number of servers at each queue are included. But, for example, also the $M/M/\infty$ queue satisfies assumption 1. Near the end of this section, an example is given that does not satisfy assumption 1.

Theorem 1. *Under assumptions 0 and 1 and in a neighbourhood of $\gamma = 0$, all power series produced by the algorithm converge.*

Proof. A sequence $\bar{u}(r,n)$ will be obtained such that

$$\|u(r,n)\| \leq \bar{u}(r,n), \quad \text{for } n \in \mathbb{N}^S,\ r \geq ne, \tag{11}$$

and such that the series

$$\sum_{r \geq ne} \gamma^r \bar{u}(r,n) \tag{12}$$

converge in a neighbourhood of $\gamma = 0$, for all $n \in \mathbb{N}^S$. If such a convergent majorant exists, the power series produced by the algorithm are absolutely convergent in the convergence region of the majorant.

Taking norms in (6) and (7) renders

$$\begin{aligned}
\|u(r,n)\| &\leq a(n)b(n)\max\Big\{ \sup_{b\in Z_<} \|u(r,n-b)\|, \quad \sup_{b\in Z_1} \|u(r-r_b,n-b)\|, \\
&\qquad \sup_{b\in Z_2} \|u(r,n-d_b)\|, \quad \sup_{b\in Z_1} \|u(r-r_b,n-d_b)\| \Big\}, \\
\|u(r,o)\| &\leq a(o)b(o)\max\Big\{ \sup_{b\in Z_<} \|u(r,-b)\|, \sup_{b\in Z_2} \|u(r,-d_b)\|, \sup_{b\in Z_1} \|u(r-r_b,-d_b)\| \Big\} \\
&+ b(o) \sum_{0<ne\leq r} \|u(r,n)\|,
\end{aligned}$$

for $r \geq ne \geq 1$. Define

$$c_0 = \sup_{n\in\mathbb{N}^S} a(n)b(n)$$

and let the numbers $\bar{u}(r,n)$ be such that

$$\begin{aligned}
\bar{u}(r,n) &\geq c_0\max\Big\{ \sup_{b\in Z_<} \bar{u}(r,n-b), \quad \sup_{b\in Z_1} \bar{u}(r-r_b,n-b), \\
&\qquad \sup_{b\in Z_2} \bar{u}(r,n-d_b), \quad \sup_{b\in Z_1} \bar{u}(r-r_b,n-d_b) \Big\}, \\
\bar{u}(r,o) &\geq c_0\max\Big\{ \sup_{b\in Z_<} \bar{u}(r,-b), \sup_{b\in Z_2} \bar{u}(r,-d_b), \sup_{b\in Z_1} \bar{u}(r-r_b,-d_b) \Big\} \\
&+ b(o) \sum_{0<ne\leq r} \bar{u}(r,n), \\
\bar{u}(0,o) &\geq \|u(0,o)\|,
\end{aligned} \tag{13}$$

for $r \geq ne \geq 1$. Then it follows from assumption 1 that $\|u(r,n)\| \leq \bar{u}(r,n)$, for all $n \in \mathbb{N}^S$, $r \geq ne$. These three inequalities (13) are indeed satisfied by the sequence

$$\bar{u}(r,n) = c_1^{-ne} c_2^{1(n=o)} c_3^r \; 1(r \geq ne,\ n \in \mathbb{N}^S),$$

with

$$\begin{aligned} c_1 &> \max\{1, c_0\}, \\ c_2 &= 1 + b(o)\left(\tfrac{c_1}{c_1-1}\right)^S, \\ c_3 &= c_1^2 c_2. \end{aligned}$$

The first inequality in (13) holds because of the following four inequalities:

$$\begin{aligned} &c_0 \sup_{b\in Z_<} c_1^{-ne+be} c_3^r = \bar{u}(r,n) \sup_{b\in Z_<} c_0 c_1^{be} \le \bar{u}(r,n), \\ &c_0 \sup_{b\in Z_1} c_1^{-ne+be} c_2^{1(n=b)} c_3^{r-r_b} = \bar{u}(r,n) \sup_{b\in Z_1} c_0 c_1^{be} c_2^{1(n=b)} c_3^{-r_b} \\ &\qquad\qquad \le \bar{u}(r,n) \sup_{b\in Z_1} (c_1^2 c_2)^{r_b} c_3^{-r_b} = \bar{u}(r,n), \\ &c_0 \sup_{b\in Z_2} c_1^{-ne+d_b e} c_3^r = \bar{u}(r,n) \sup_{b\in Z_2} c_0 c_1^{d_b e} \le \bar{u}(r,n), \\ &c_0 \sup_{b\in Z_1} c_1^{-ne+d_b e} c_3^{r-r_b} = \bar{u}(r,n) \sup_{b\in Z_1} c_0 c_1^{d_b e} c_3^{-r_b} \le \bar{u}(r,n), \end{aligned}$$

for all $r \ge ne \ge 1$. The second inequality in (13) is shown in two steps. In a similar way as for the first inequality, it can be shown that

$$c_0 \max\left\{ \sup_{b\in Z_<} \bar{u}(r,-b), \sup_{b\in Z_2} \bar{u}(r,-d_b), \sup_{b\in Z_1} \bar{u}(r-r_b,-d_b) \right\} \le c_3^r. \tag{14}$$

Also, it is true that

$$b(o) \sum_{0<ne\le r} \bar{u}(r,n) \le b(o) c_3^r \sum_{n\in \mathbb{N}^S} c_1^{-ne} = b(o) c_3^r \left(\frac{c_1}{c_1-1}\right)^S. \tag{15}$$

The sum of (14) and (15) is equal to $\bar{u}(r,o)$:

$$c_3^r + b(o) c_3^r \left(\frac{c_1}{c_1-1}\right)^S = c_2 c_3^r = \bar{u}(r,o),$$

for all $r \ge 1$, so also the second inequality in (13) holds. Finally, that the third inequality of (13) holds can be seen from the discussion following assumption 0'. Comparing (8) and (9) shows that $u(0,o)$ is a stochastic vector, so

$$||u(0,o)|| = 1 < c_2 = \bar{u}(0,o).$$

Therefore, it has been checked that all three inequalities (13), and thus (11), are satisfied.

What remains to be proved is that the series (12) converge in a neighbourhood of $\gamma = 0$. This is indeed true:

$$\sum_{r\ge ne} \gamma^r \bar{u}(r,n) = c_1^{-ne} c_2^{1(n=o)} \sum_{r\ge ne} \gamma^r c_3^r = \frac{c_2^{1(n=o)} \left(\frac{\gamma c_3}{c_1}\right)^{ne}}{1-\gamma c_3},$$

for all $n \in \mathbb{N}^S$ and $|\gamma| < c_3^{-1}$. Therefore, $\bar{u}(r,n)$ is a convergent majorant of the sequences produced by the algorithm, which proves the theorem. □

Define the vector-functions $q(\gamma, n)$ as the functions determined by the power series produced by the algorithm:

$$q(\gamma, n) = \sum_{r \geq ne} \gamma^r u(r, n), \quad \text{for } n \in \mathbb{N}^S.$$

By theorem 1, these $q(\gamma, n)$ are well-defined analytic functions of γ, for γ small enough, and the lower bound on the radius of convergence is uniform for all $n \in \mathbb{N}^S$. They are the desired steady-state probabilities if they satisfy the balance-equations and the γ-process is ergodic, which will be proved in theorem 2. The usual ergodicity theorems can not be used here, because it is not known in advance that the functions $q(\gamma, n)$ are non-negative. The following ergodicity lemma does not make this non-negativity assumption:

Lemma. *Let the Markov process $\{\mathcal{X}_t;\ t \geq 0\}$ on state space C be irreducible with generator $\{\Lambda_{ij};\ i, j \in C\}$, and $\lambda_i = -\Lambda_{ii} = \sum_{j \in C \setminus \{i\}} \Lambda_{ij} < \infty$, for $i \in C$. If a solution $\{\pi_i;\ i \in C\}$ exists such that for some $i_0 \in C$:*

$$\pi_i \lambda_i = \sum_{j \in C \setminus \{i\}} \pi_j \Lambda_{ji}, \quad \textit{for } i \in C \setminus \{i_0\}, \tag{16}$$

$$\sum_{i \in C} \pi_i = 1, \tag{17}$$

$$\sum_{i \in C} |\pi_i| \lambda_i < \infty, \tag{18}$$

then the Markov process is ergodic with steady-state distribution $\{\pi_i;\ i \in C\}$.

Proof. Summing the balance equations (16) over all $i \in C \setminus \{i_0\}$, reversing the order of summation and subtracting $\sum_{i \in C} \pi_i \lambda_i$, which is both justified because of (18), renders (16) for $i = i_0$. Therefore, the balance equations are satisfied for all $i \in C$.

The jump chain is the embedded discrete time Markov chain with transition probabilities $r_{ij} = \frac{\Lambda_{ij}}{\lambda_i} 1(i \neq j)$ for $i, j \in C$. Since the Markov process is irreducible, this jump chain is also irreducible. With $\phi_i = \pi_i \lambda_i$, for $i \in C$, the balance equations (16) and equation (18) can be rewritten as

$$\phi_i = \sum_{j \in C} \phi_j r_{ji}, \quad \text{for } i \in C,$$

$$\sum_{i \in C} |\phi_i| < \infty.$$

By theorem I.7.1 in Chung [7] this implies that $\phi_i > 0$ for $i \in C$ and that the jump chain is ergodic. But then also $\pi_i > 0$ for $i \in C$ and the Markov process is non-explosive by proposition II.2.4 in Asmussen[1]. From the normalization (17) it follows that $\{\pi_i;\ i \in C\}$ is a probability distribution. By theorem II.4.3 in Asmussen [1] this implies that the Markov process is ergodic with steady-state distribution $\{\pi_i;\ i \in C\}$. □

This lemma is a generalization of theorem II.4.4 in Asmussen [1]. In that theorem, it is assumed beforehand that the solution $\{\pi_i;\ i \in C\}$ is a probability distribution and therefore non-negative.

In the proof of theorem 2, it will be shown that the functions $q(\gamma, n)$ satisfy the conditions of the lemma if assumption 2 is satisfied:

Assumption 2. $\bar{\alpha} \in \mathcal{E}$.

This assumption requires that in the 1-process, and therefore in each γ-process, the total transition rate from state (n, i) grows at most exponentially in n (see definition (10)). Since the growth rate is allowed to be arbitrarily large, this assumption is very weak.

Theorem 2. *Under assumptions 0 to 2 and in a neighbourhood of $\gamma = 0$, the γ-process is ergodic with steady-state distribution $\{q_i(\gamma, n);\ (n, i) \in \Omega\}$.*

Proof. For $\gamma = 0$, it will be clear from assumption 0' that the process will always end up in the single finite recurrent class of empty states, so it is ergodic. Comparing (8) and (9) shows that the steady-state probabilities are equal to $q_i(0, n) = u_i(0, o)1(n = o)$, for $(n, i) \in \Omega$. Therefore, the theorem holds for $\gamma = 0$. For $\gamma > 0$ but sufficiently small, it will be checked in the rest of the proof whether the γ-process and the functions $q_i(\gamma, n)$ satisfy the conditions of the lemma.

The 1-process was assumed to be irreducible and non-instantaneous. For $\gamma > 0$, each transition in the 1-process is also possible in the γ-process, so the γ-process is also irreducible. The total transition rate in the γ-process from each state is constant in γ (or non-decreasing in γ if selfloops are ignored) and therefore finite for all $\gamma \in (0, 1]$.

By assumption 2, positive constants a_1 and b_1 exist such that

$$\bar{\alpha}(n, i) \leq a_1 b_1^{ne} \quad \text{for all } (n, i) \in \Omega.$$

As a consequence of this, the following inequality holds:

$$\sum_{b \in Z} \|A_b(n)\| \leq \sum_{1 \leq i \leq I} \sum_{b \in \mathbb{Z}^S} \sum_{1 \leq j \leq I} \alpha_{bj}(n, i) \leq I\ a_1\ b_1^{ne}, \tag{19}$$

for all $Z \subseteq \mathbb{Z}^S$ and $n \in \mathbb{N}^S$. This will be used in the rest of the proof.

That the balance equations (16) are satisfied can be shown as follows. Rearranging equations (6) for $n \neq o$, renders

$$\begin{aligned} u(r, n)\bar{A}(n) \;=\; & \sum_{b \in Z_<} u(r, n - b) \, A_b(n - b) \\ + & \sum_{b \in Z_\geq} u(r - r_b, n - b) \, A_b(n - b) \\ + & \sum_{b \in Z_\geq} u(r, n - d_b) \, A_b(n - d_b) \\ - & \sum_{b \in Z_\geq} u(r - r_b, n - d_b) \, A_b(n - d_b), \end{aligned}$$

for $r \geq ne$. Multiplying both sides by γ^r, summing over $r \geq ne$ and changing the order of summations renders

$$\begin{aligned}
\sum_{r\geq ne} \gamma^r u(r,n)\bar{A}(n) &= \sum_{b\in Z_<} \sum_{r\geq ne} \gamma^r \; u(r,n-b) \; A_b(n-b) \\
&+ \sum_{b\in Z_\geq} \gamma^{r_b} \sum_{r\geq ne-r_b} \gamma^r \; u(r,n-b) \; A_b(n-b) \\
&+ \sum_{b\in Z_\geq} \sum_{r\geq ne} \gamma^r \; u(r,n-d_b) \; A_b(n-d_b) \\
&- \sum_{b\in Z_\geq} \gamma^{r_b} \sum_{r\geq ne-r_b} \gamma^r \; u(r,n-d_b) \; A_b(n-d_b).
\end{aligned} \tag{20}$$

Since $u(r,n) = 0$ for all $r < ne$, this is equivalent to

$$\begin{aligned}
q(\gamma,n)\bar{A}(n) &= \sum_{b\in Z_<} q(\gamma,n-b) \; A_b(n-b) \\
&+ \sum_{b\in Z_\geq} \gamma^{r_b} \; q(\gamma,n-b) \; A_b(n-b) \\
&+ \sum_{b\in Z_\geq} (1-\gamma^{r_b}) \; q(\gamma,n-d_b) \; A_b(n-d_b),
\end{aligned}$$

which coincides with the balance equations of the γ-process (1) for the non-empty states. A similar approach applied to equations (7) and (8) leads to the balance equations of all empty states, except state $(o,1)$. So (16) is satisfied, with $i_0 = (o,1)$, if changing the order of summations was justified. This is indeed the case when assumption 2 is satisfied, because then the four terms in the RHS of (20) are all absolutely convergent for γ small enough. In the first term the order of the summations over r and b can be reversed because:

$$\begin{aligned}
&\sum_{b\in Z_<} \sum_{r\geq ne} \gamma^r \|u(r,n-b)A_b(n-b)\| \\
&\quad\leq \sum_{b\in \mathbb{Z}^S} \sum_{r\geq ne-be} \gamma^r \; c_1^{-ne+be} c_2 c_3^r \; I a_1 b_1^{ne-be} \; 1(n-b\in \mathbb{N}^S) \\
&\quad= \tfrac{c_2 I a_1}{1-\gamma c_3} \left(1-\tfrac{\gamma c_3 b_1}{c_1}\right)^{-S},
\end{aligned}$$

for $|\gamma| < c_3^{-1}$ and $|\gamma| < c_1(c_3 b_1)^{-1}$. Inequality (19) was used with $Z = \{b\}$. In the second term of (20) the order of summations can be reversed because the summation over b is finite:

$$\#\left\{\, b\in Z_\geq \;\middle|\; n-b\in \mathbb{N}^S \,\right\} = \binom{ne+S}{S}.$$

In the third term of (20) the order of summations can be reversed because

$$\begin{aligned}
&\sum_{b\in Z_\geq} \sum_{r\geq ne} \gamma^r \|u(r,n-d_b)A_b(n-d_b)\| \\
&\quad\leq \sum_{d\in -\mathbb{N}^S} \left\{ \sum_{b\in Z_\geq : d_b = d} \|A_b(n-d)\| \right\} \sum_{r\geq ne-de} \gamma^r \|u(r,n-d)\| \\
&\quad\leq \sum_{d\in -\mathbb{N}^S} I a_1 b_1^{ne-de} \sum_{r\geq ne-de} \gamma^r c_1^{-ne+de} c_2 c_3^r \\
&\quad= \tfrac{I a_1 c_2}{1-\gamma c_3} \left(\tfrac{\gamma c_3 b_1}{c_1}\right)^{ne} \left(1-\tfrac{\gamma b_1 c_3}{c_1}\right)^{-S},
\end{aligned}$$

for $|\gamma| < c_3^{-1}$ and $|\gamma| < c_1(c_3b_1)^{-1}$. That in the fourth term of (20) the order of summations can be reversed can be shown in a similar way as for the third term. This completes the part of the proof that shows that condition (16) holds.

That the normalization equation (17) is satisfied can be shown as follows. Rearranging the normalization parts of equations (7) and (8) renders

$$\sum_{0\le ne\le r} u(r,n)e = 1(r=0),$$

for $r \ge 0$. Multiplying both sides by γ^r, summing over $r \ge 0$ and changing the order of the summations over r and n renders

$$\sum_{n\in\mathbb{N}^S}\sum_{r\ge ne}\gamma^r u(r,n)e = \sum_{(n,i)\in\Omega} q_i(\gamma,n) = 1.$$

Here, the order of summations can be reversed because

$$\begin{aligned}\sum_{n\in\mathbb{N}^S}\sum_{r\ge ne}\gamma^r|u(r,n)e| &\le \sum_{n\in\mathbb{N}^S}\sum_{r\ge ne}\gamma^r c_1^{-ne}c_2c_3^r \\ &= \tfrac{c_2}{1-\gamma c_3}\left(1-\tfrac{\gamma c_3}{c_1}\right)^{-S},\end{aligned} \tag{21}$$

for $|\gamma| < c_3^{-1} < c_1c_3^{-1}$.

Finally, condition (18) is satisfied because

$$\begin{aligned}\sum_{(n,i)\in\Omega}|q_i(\gamma,n)|\bar{\alpha}(n,i) &\le I\sum_{n\in\mathbb{N}^S}\sum_{r\ge ne}\gamma^r\ c_1^{-ne}c_2c_3^r\ a_1b_1^{ne} \\ &= \tfrac{Ic_2a_1}{1-\gamma c_3}\left(1-\tfrac{\gamma c_3b_1}{c_1}\right)^{-S} < \infty,\end{aligned}$$

for $|\gamma| < c_3^{-1}$ and $|\gamma| < c_1(c_3b_1)^{-1}$. It has been shown that the γ-process and the functions $q_i(\gamma,n)$ satisfy all conditions of the lemma, which finishes the proof of theorem 2. □

The γ-process is such that upward transitions in the 1-process are replaced by downward transitions or selfloops. From this, it will be obvious in many applications that if the 1-process is ergodic, then the γ-process is ergodic for all γ in [0,1]. However, this is not true in general as can be seen from the next example.

Example. Consider the Markov process on $\mathbb{N}^2$ illustrated in figure 3. When no rate is indicated, the rate equals 1. This process always ends in the cycle $(0,0)\to(3,1)\to(2,0)\to(1,1)\to(0,0)$, so it is ergodic. The transition-diagram of the corresponding γ-process (without the selfloops) is given in figure 4. For $\gamma = 0$, the process will always end up in the origin, so assumption 0 is satisfied. Assumption 1 is satisfied if $\sup_k \lambda_k < \infty$. Assumption 2 is satisfied if λ_k grows at most exponentially in k.

When the process is in a state $(2k,0)$, with $k \ge 1$, then in two steps the process will go either up to state $(2k+2,0)$ or down to state $(2k-2,0)$. The probability of going up is equal to

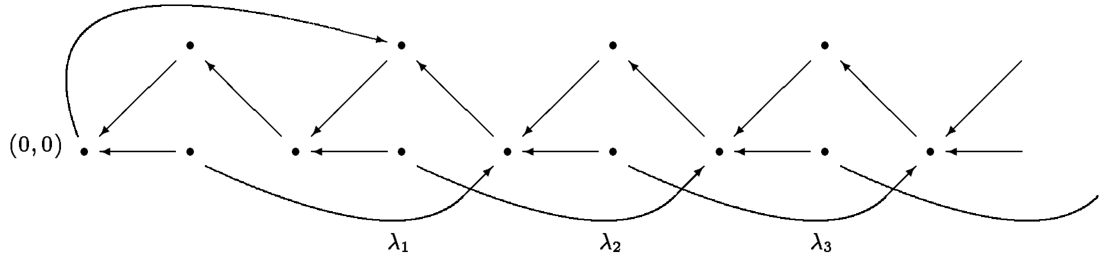

Figure 3: The untransformed process

$\pi_k(\gamma) = (1-\gamma)\ \lambda_k \gamma^3 (1+\lambda_k \gamma^3)^{-1}$, which is the probability of going first to state $(2k-1,0)$ and then to state $(2k+2,0)$.

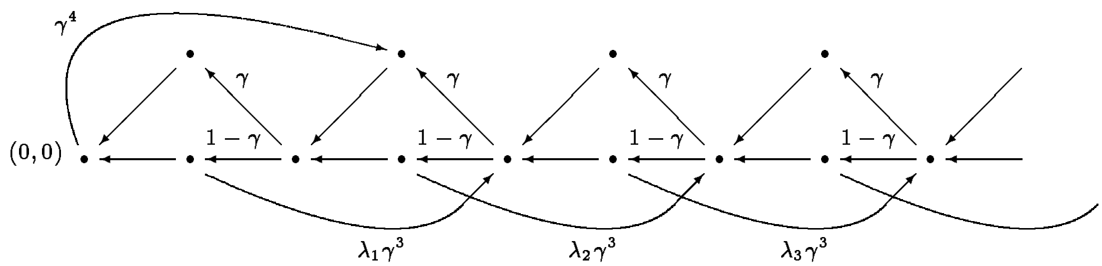

Figure 4: The transformed process

First, take $\lambda_k \equiv \lambda < 2048/27$. Then $\pi_k(\gamma) < \frac{1}{2}$, for all $k \geq 1$ and γ in [0,1] and the γ-process is ergodic for all γ in [0,1].

Next, take $\lambda_k \equiv \lambda \geq 2048/27$. Then the equation $\pi_k(\gamma) = \frac{1}{2}$ has solutions γ_1 and γ_2 with $0 < \gamma_1 \leq \frac{3}{8} \leq \gamma_2 < \frac{1}{2}$. The γ-process is ergodic for $\gamma \in [0,\gamma_1) \cup (\gamma_2, 1]$, null-recurrent for $\gamma \in \{\gamma_1, \gamma_2\}$ and transient for $\gamma \in (\gamma_1, \gamma_2)$. Therefore, the γ-process is ergodic at $\gamma = 1$ (as was assumed in section 2) and in a neighbourhood of $\gamma = 0$ (as was proved in this section), but it is not ergodic in between.

Finally, take $\lambda_k = k$, for $k \geq 1$. Then $\lim_{k\to\infty} \pi_k(\gamma) = 1-\gamma > \frac{1}{2}$, for all γ in $(0, \frac{1}{2})$. On the other hand, $\pi_k(\gamma) < 1-\gamma \leq \frac{1}{2}$, for all $k \geq 1$ and γ in $[\frac{1}{2}, 1]$. Therefore, the γ-process is ergodic for $\gamma \in \{0\} \cup [\frac{1}{2}, 1]$ and transient for $\gamma \in (0, \frac{1}{2})$. That the γ-process is not ergodic in a neighbourhood of $\gamma = 0$ is not in contradiction with theorem 2, because assumption 1 is not satisfied.

In this example, the transient behaviour of the γ-process can be avoided by not considering the original process as a process on $\mathbb{N}^2$ but as a process on $\mathbb{N} \times \{0,1\}$. On $\mathbb{N}^2$, the transition $(2k,0) \to (2k-1,1)$ is an upward transition and hence redirected to the transition $(2k,0) \to (2k-1,0)$. From the state $(2k-1,0)$, the process has a high probability to go up to state

$(2k+2,0)$. This way, replacing upward transitions by downward transitions results in more visits to states from which large upward transitions are likely. Considered as a process on $\mathbb{N} \times \{0,1\}$, the transition $(2k,0) \to (2k-1,1)$ is downward instead of upward, and is therefore not redirected.

The next theorem proves that the expectation $\mathrm{E}_\gamma\{\ f(\mathcal{N},\mathcal{I})\ \}$ is analytic at $\gamma = 0$ if the function $f(n,i)$ grows at most exponentially in n (see definition (10)). This implies that the power-series expansions can be calculated in the way suggested by formula (4). The examples (5) in section 3 are all polynomial in n, so they satisfy this assumption.

Assumption 3. $f \in \mathcal{E}$.

Theorem 3. *Under assumptions 0 to 3 and in a neighbourhood of $\gamma = 0$, the expectation $E_\gamma\{\ f(\mathcal{N},\mathcal{I})\ \}$ is analytic in γ.*

Proof. By assumption 3, positive constants a_2 and b_2 exist such that

$$|f(n,i)| \leq a_2 b_2^{ne} \quad \text{for all } (n,i) \in \Omega.$$

The expectation is analytic because its power-series expansion (4) is absolutely convergent:

$$\begin{aligned} \sum_{r\geq 0} \gamma^r \left| \sum_{(n,i)\in\Omega:\ ne\leq r} u_i(r,n)\ f(n,i) \right| &\leq \sum_{r\geq 0} \gamma^r \sum_{ne\leq r} I\ c_1^{-ne} c_2 c_3^r\ a_2 b_2^{ne} \\ &= \tfrac{c_2 a_2 I}{1-\gamma c_3} \left(1 - \tfrac{\gamma c_3 b_2}{c_1} \right)^{-S}, \end{aligned}$$

for $|\gamma| < c_3^{-1}$ and $|\gamma| < c_1(c_3 b_2)^{-1}$. □

5 Conclusions

The applicability of the Power-Series Algorithm has been extended to a wide class of Markov processes. A transformation has been proposed that can be used in many cases, and it was shown that if it is not applicable then usually a similar transformation can be found that does apply. Sufficient conditions were derived to ensure that the steady-state probabilities are analytic in the transformation parameter. These conditions are weak and satisfied for most queueing models. Numerical results can be found in [10] and the significance and flexibility of the method is illustrated by the many different models it has been applied to.

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