NORTH-HOLLAND

Regular Graphs With Four Eigenvalues

Edwin R. van Dam
Tilburg University
Department of Econometrics
P.O. Box 90153
5000 LE Tilburg
The Netherlands

Dedicated to J. J. Seidel

Submitted by Willem Haemers

ABSTRACT

We study the connected regular graphs with four distinct eigenvalues. Properties and feasibility conditions of the eigenvalues are found. Several examples, constructions and characterizations are given, as well as some uniqueness and nonexistence results.

1. INTRODUCTION

Connected regular graphs having at most three distinct eigenvalues are very well classified by means of combinational properties: they are the complete and the strongly regular graphs. Distance-regular graphs of diameter d (or more generally, d-class association schemes) are generalizations of complete ($d = 1$) and strongly regular ($d = 2$) graphs from a combinatorial point of view. The adjacency matrices of these graphs have $d + 1$ distinct eigenvalues, but for $d > 2$ the converse is not true: not every regular graph with $d + 1$ distinct eigenvalues is distance-regular (or comes from a d-class

email: Edwin.vanDam@kub.nl

LINEAR ALGEBRA AND ITS APPLICATIONS 226–228:139–162 (1995)

0024-3795/95/$9.50
SSDI 0024-3795(94)00346-F

association scheme).

In this paper, we shall take a closer look at the connected regular graphs with four distinct eigenvalues. Already for those graphs, many examples exist that are not distance-regular (or from 3-class association schemes). Still we can deduce some nice properties. An important observation is that these graphs are walk-regular, which implies rather strong conditions for the possible spectra. Furthermore, we shall give several constructions, some characterizations, and uniqueness and nonexistence results. Many of the constructions use strongly regular graphs. As general references for these graphs, we use the papers by Seidel [21] and Brouwer and van Lint [3]. As general reference for spectra of graphs, we use the book by Cvetković, Doob, and Sachs [6].

Throughout this paper, we shall denote by $\{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \dots, [\lambda_t]^{m_t}\}$ the spectrum of a matrix with t distinct eigenvalues λ_i with multiplicities m_i. If the matrix is the adjacency matrix of a connected k-regular graph, then λ_1 denotes k, and has multiplicity $m_1 = 1$.

2. PROPERTIES OF THE EIGENVALUES

In this section, we shall derive some properties of the eigenvalues of graphs with four distinct eigenvalues. To obtain these we shall use some elementary lemmas about polynomials with rational or integral coefficients (for example see [10]).

By $Z[x]$ and $Q[x]$ we denote the rings of polynomials over the integers and rationals, respectively.

LEMMA 2.1. *If a monic polynomial $p(x) \in Z[x]$ has a monic divisor $q(x) \in Q[x]$, then also $q(x) \in Z[x]$.*

LEMMA 2.2. *If $a \pm \sqrt{b}$, with $a, b \in Q$, is an irrational root of a polynomial $p(x) \in Q[x]$, then so is $a \mp \sqrt{b}$, with the same multiplicity.*

The characteristic polynomial $c(x)$ of the adjacency matrix of a graph is monic and has integral coefficients. Using Lemmas 2.1 and 2.2, we now obtain the following results.

COROLLARY 2.3. *Every rational eigenvalue of a graph is integral.*

COROLLARY 2.4. *If $\frac{1}{2}(a \pm \sqrt{b})$ is an irrational eigenvalue of a graph, for some $a, b \in Q$, then so is $\frac{1}{2}(a \mp \sqrt{b})$, with the same multiplicity, and $a, b \in Z$.*

The minimal polynomial of the adjacency matrix A of a graph is the unique monic polynomial $m(x) = x^t + m_{t-1}x^{t-1} + \cdots + m_0$ of minimal degree such that $m(A) = O$.

LEMMA 2.5. *The minimal polynomial m of a graph has integral coefficients.*

Proof. The following short argument was pointed out by P. Rowlinson [personal communication]. The equation $m(A) = O$ can be seen as a system of n^2 (if n is the size of A) linear equations in the unknowns m_i, with integral coefficients. Since the system has a unique solution, this solution must be rational. (The solution can be found by Gaussian Elimination, and during this algorithm all entries of the system remain rational.) So the minimal polynomial has rational coefficients, and since it divides the characteristic polynomial, we find $m(x) \in \mathbb{Z}[x]$. ■

In the following, G will be a connected k-regular graph on v vertices having spectrum $\{[k]^1, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}, [\lambda_4]^{m_4}\}$. Now Lemma 2.1 implies that the polynomials p and q, defined by

$$p(x) = (x - \lambda_2)(x - \lambda_3)(x - \lambda_4) = \frac{m(x)}{x - k},$$

$$q(x) = (x - \lambda_2)^{m_2 - 1}(x - \lambda_3)^{m_3 - 1}(x - \lambda_4)^{m_4 - 1} = \frac{c(x)}{m(x)},$$

have integral coefficients. We shall use these polynomials in the proof of the following theorem.

THEOREM 2.6. *Let G be a connected k-regular graph on v vertices with spectrum $\{[k]^1, [\lambda_2]^{m_2}, [\lambda_3]^{m_3}, [\lambda_4]^{m_4}\}$, and let $m = (v - 1)/3$. Then $m_2 = m_3 = m_4 = m$ and $k = m$ or $k = 2m$, or G has two or four integral eigenvalues. Moreover, if G has exactly two integral eigenvalues, then the other two have the same mulitplicities and are of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbb{Z}$.*

Proof. Without loss of generality we may assume $m_2 \leqslant m_3 \leqslant m_4$. If all three are equal, then they must be equal to m, and $k + m(\lambda_2 + \lambda_3 + \lambda_4) = \operatorname{trace}(A) = 0$, where A is the adjacency matrix of G. Since $p(x) \in \mathbb{Z}[x]$, we have that $\lambda_2 + \lambda_3 + \lambda_4 \in \mathbf{Z}$, so k is a multiple of m. Since $v = 3m + 1$, it follows that $k = m$ or $k = 2m$.

If $m_2 = m_3 < m_4$, then $(x - \lambda_4)^{m_4 - m_2} = q(x)/p(x)^{m_2 - 1} \in \mathbf{Z}[x]$, so $\lambda_4 \in \mathbf{Z}$. Now it follows that $(x - \lambda_2)(x - \lambda_3) \in \mathbf{Z}[x]$, so λ_2 and λ_3 are both integral or of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbf{Z}$.

If $m_2 < m_3$, then $(x - \lambda_3)^{m_3 - m_2}(x - \lambda_4)^{m_4 - m_2} = q(x)/p(x)^{m_2 - 1} \in \mathbf{Z}[x]$. Now it follows that λ_3 and λ_4 are both integral or of the form $\frac{1}{2}(a \pm \sqrt{b})$, with $a, b \in \mathbf{Z}$, and if λ_3 and λ_4 are irrational, then $m_3 = m_4$. In both cases, it follows that λ_2 is integral. ■

Each of the three cases of Theorem 2.6 can occur. Small examples are given by the 7-cycle C_7 with (approximated) spectrum

$$\left\{[2]^1, [1.247]^2, [-0.445]^2, [-1.802]^2\right\},$$

the 6-cycle C_6 with spectrum $\{[2]^1, [1]^2, [-1]^2, [-2]^1\}$, and the complement of the union of two 5-cycles $(2C_5)^c$ with spectrum

$$\left\{[7]^1, \left[\tfrac{1}{2}(-1 + \sqrt{5})\right]^4, \left[\tfrac{1}{2}(-1 - \sqrt{5})\right]^4, [-3]^1\right\}.$$

Another important property of connected regular graphs with four distinct eigenvalues, which we shall use in Section 4.6, is that the multiplicities of the eigenvalues follow from the eigenvalues and the number of vertices (cf. [6, p. 161]). This follows from the following three equations, which uniquely determine m_2, m_3, and m_4:

$$1 + m_2 + m_3 + m_4 = v,$$

$$k + m_2\lambda_2 + m_3\lambda_3 + m_4\lambda_4 = 0,$$

$$k^2 + m_2\lambda_2^2 + m_3\lambda_3^2 + m_4\lambda_4^2 = vk.$$

The second equation follows from the trace of A, and the third from the trace of A^2, where A is the adjacency matrix of the graph.

Note that the eigenvalues alone do not determine the multiplicities. For example, the complement of the Cube has spectrum $\{[4]^1, [2]^1, [0]^3, [-2]^3\}$, while the line graph of the Cube has spectrum $\{[4]^1, [2]^3, [0]^3, [-2]^5\}$. This example is the smallest of an infinite class given by Doob [7, 8].

3. WALK-REGULAR GRAPHS AND FEASIBILITY CONDITIONS

A walk-regular is a graph G for which the number of walks of length r from a given vertex x to itself (closed walks) is independent of the choice of x, for all r (cf. [11]). Since this number equals A^r_{xx}, it is the same as saying that A^r has constant diagonal for all r, if A is the adjacency matrix of G. Note that a walk-regular graph is always regular. If G has v vertices and is connected k-regular with four distinct eigenvalues k, λ_2, λ_3, and λ_4, then $(A - \lambda_2 I)(A - \lambda_3 I)(A - \lambda_4 I) = (1/v)(k - \lambda_2)(k - \lambda_3)k - \lambda_4)J$; i.e., $h(A) = J$, where h is the Hoffman polynomial and J is the all-one matrix (cf. [16]). Since A^2, A, I, and J all have constant diagonal, we see that A^r has constant diagonal for every r. So G is walk-regular.

3.1. *Feasibility conditions*

If G is walk-regular on v vertices with degree k and spectrum $\{[\lambda_1]^{m_1}, [\lambda_2]^{m_2}, \ldots, [\lambda_t]^{m_t}\}$, the number of triangles through a given vertex x is independent of x, and equals

$$\Delta = \tfrac{1}{2} A^3_{xx} = \frac{Tr(A^3)}{2v} = \frac{1}{2v} \sum_{i=1}^{t} m_i \lambda_i^3.$$

This expression gives a feasibility condition for the spectrum of G, since Δ should be a nonnegative integer. In general, it follows that

$$\theta_r = \frac{1}{v} \sum_{i=1}^{t} m_i \lambda_i^r.$$

is a nonnegative integer. Since the number of closed walks of odd length r is even, θ_r should be even if r is odd. For even r, we can also sharpen the condition, since then the number of nontrivial closed walks (that is, those containing a cycle) is even. For example, if $r = 4$, the number of trivial closed walks through a given vertex (i.e., passing only one or two other vertices) equals $2k^2 - k$, so

$$\Xi = \frac{\theta_4 - 2k^2 + k}{2}$$

is a nonnegative integer, and it equals the number of quadrangles through a vertex.

In case we have four distinct eigenvalues, the following lemma will also be useful.

LEMMA 3.1. *If G is a connected k-regular graph with four distinct eigenvalues, such that the number of triangles through an edge is constant, then the number of quadrangles through an edge is also constant.*

Proof. Since G is connected and regular with four distinct eigenvalues, its adjacency matrix A satisfies the equation $A^3 + p_2 A^2 + p_1 A + p_0 I = pJ$, for some p_2, p_1, p_0, and p. Now $A^3_{xy} + p_2 \lambda_{xy} + p_1 = p$, for any two adjacent x, y with λ_{xy} common neighbors. Since the number of triangles through an edge is constant, say λ, we have $\lambda_{xy} = \lambda$, and so the number of walks of length 3 from x to y is equal to $A^3_{xy} = p - p_1 - p_2 \lambda$. Since there are $2k - 1$ walks which are trivial, the number of quadrangles containing edge $\{x, y\}$ equals $p - p_1 - p_2 \lambda - 2k + 1$, which is independent of the given edge. ■

Note that if ξ is the (constant) number of quadrangles through an edge, and if Ξ is the number of quadrangles through a vertex, then $\xi = 2\Xi / k$.

3.2. Simple eigenvalues

If a walk-regular graph has a simple eigenvalue $\lambda \neq k$, then we can say more on the structure of the graph. We shall prove that the graph admits a regular partition into halves with degrees $(\frac{1}{2}(k + \lambda), \frac{1}{2}(k - \lambda))$, that is, we can partition the vertices into two parts of equal size such that every vertex has $\frac{1}{2}(k + \lambda)$ neighbors in its own part and $\frac{1}{2}(k - \lambda)$ neighbors in the other part. As a consequence we obtain that $k - \lambda$ is even, a condition which was proven by Godsil and McKay [11]. This condition eliminates, for example, the existence of a graph with spectrum $\{[14]^1, [2]^9, [-1]^{19}, [-13]^1\}$. We also find other divisibility conditions.

LEMMA 3.2. *Let B be a symmetric matrix of size v, having constant diagonal and constant row sums r, and spectrum $\{[r]^1, [s]^1, [0]^{v-2}\}$, with $s \neq 0$; then v is even and (possibly after permuting rows and columns) B can be written as*

$$B = \begin{pmatrix} \dfrac{r+s}{v} J_{\frac{1}{2}v} & \dfrac{r-s}{v} J_{\frac{1}{2}v} \\ \dfrac{r-s}{v} J_{\frac{1}{2}v} & \dfrac{r+s}{v} J_{\frac{1}{2}v} \end{pmatrix}.$$

Proof. Consider the matrix $M = B - (r/v)J$; then M is symmetric, has constant diagonal, say x, row sums zero, and spectrum $\{[s]^1, [0]^{v-1}\}$. So, M has rank 1. By noticing that the determinant of all principal submatrices of size 2 must be zero, and using that M is symmetric and has constant diagonal, it follows that M only has entries $\pm x$. Since M has row sums zero, it follows that v is even and that we can write M as

$$M = \begin{pmatrix} xJ_{\frac{1}{2}v} & -xJ_{\frac{1}{2}v} \\ -xJ_{\frac{1}{2}v} & xJ_{\frac{1}{2}v} \end{pmatrix}.$$

Now B has nontrivial eigenvalues r and vx, so $s = vx$, and the result follows. ∎

THEOREM 3.3. *let G be a connected walk-regular graph on v vertices and degree k, having distinct eigenvalues $k, \lambda_2, \lambda_3, \dots, \lambda_t$, of which an eigenvalue unequal to k, say λ_j, has multiplicity 1. Then v is even and G admits a regular partition into halves with degrees $(\frac{1}{2}(k+\lambda_j), \frac{1}{2}(k-\lambda_j))$. Moreover, v is a divisor of*

$$\prod_{i\neq j}(k-\lambda_i) + \prod_{i\neq j}(\lambda_j-\lambda_i) \quad \textit{and} \quad \prod_{i\neq j}(k-\lambda_i) - \prod_{i\neq j}(\lambda_j-\lambda_i).$$

Proof. Let $b(x) = \prod_{i\neq j}(x-\lambda_i)$, and let $B = b(A)$; then it follows from Lemma 3.2 (B has constant diagonal since G is walk-regular) that v is even and

$$B = \begin{pmatrix} \dfrac{r+s}{v}J_{\frac{1}{2}v} & \dfrac{r-s}{v}J_{\frac{1}{2}v} \\ \dfrac{r-s}{v}J_{\frac{1}{2}v} & \dfrac{r+s}{v}J_{\frac{1}{2}v} \end{pmatrix},$$

where $r = \prod_{i\neq j}(k-\lambda_i)$ and $s = \prod_{i\neq j}(\lambda_j-\lambda_i)$.

Now $(\underline{1}, -\underline{1})^T$ is an eigenvector of B with eigenvalue s, and since this eigenvalue is simple, and A and B commute, it follows that $(\underline{1}, -\underline{1})^T$ is also an eigenvector of A, and the corresponding eigenvalue must then be λ_j. This implies that if we partition A the same way as we partitioned B, with

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

then $A_{11}\underline{1} = A_{22}\underline{1} = \frac{1}{2}(k + \lambda_j)\underline{1}$ and $A_{12}\underline{1} = A_{12}^T\underline{1} = \frac{1}{2}(k - \lambda_j)\underline{1}$. Since λ_j must be an integer, and $b(x) = m(x)/(x - k)(x - \lambda_j)$, where $m(x)$ is the minimal polynomial of G, it follows from Lemma 2.1 that b has integral coefficients, and so B is an integral matrix. But then $v \mid r + s$ and $v \mid r - s$. ■

COROLLARY 3.4. *If G is a connected walk-regular graph with degree k, and λ is a simple eigenvalue, then $k - \lambda$ is even.*

As a consequence of the divisibility conditions in Theorem 3.3, we derive that there are no graphs with spectrum $\{[8]^1, [2]^7, [-2]^9, [-4]^1\}$ (on 18 vertices), or $\{[13]^1, [5]^1, [1]^{22}, [-5]^8\}$ (on 32 vertices). These spectra satisfy all previously mentioned conditions.

4. EXAMPLES, CONSTRUCTIONS AND CHARACTERIZATIONS

4.1. *Distance-regular graphs and association schemes*

Distance-regular graphs (see [1]) and, more generally, association schemes will give us several examples of graphs with four distinct eigenvalues. The graphs can be obtained by taking the union of some classes (or just one class) as adjacency relation. In general, graphs from d-class association schemes have $d + 1$ eigenvalues, but sometimes some eigenvalues coincide. So most examples come from 3-class association schemes (see [19]), such as the Johnson scheme $J(n, 3)$ and the Hamming scheme $H(3, q)$.

An example coming from a 5-class association scheme is obtained by taking distance 3 and 5 in the dodecahedron as adjacency relation. The resulting graph has spectrum $\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$.

In general, distance-regularity is not determined by the spectrum of the graph. Haemers [13] proved that it is, provided that some additional conditions are satisfied. Haemers and Spence [15] found (almost) all graphs with the spectrum of a distance-regular graph with at most 30 vertices. Most of these graphs have four distinct eigenvalues.

4.1.1. Pseudocyclic association schemes. A d-class association scheme is said to be pseudocyclic if there are d eigenvalues with the same multiplicity. If the number of vertices q is a prime power and $q \equiv 1 \pmod{d}$, then the cyclotomic scheme, which has the d-th power cyclotomic classes of $GF(q)$ as classes, is an example. For $d = 3$ (and $q > 4$), this graph has four distinct eigenvalues and is obtained by making two elements of $GF(q)$ adjacent if their difference is a cube. The smallest example is the 7-cycle C_7. If the number of vertices is not a prime power, then only three pseudocyclic

3-class association schemes are known. On 28 vertices, Mathon [19] found one, and Hollmann [18] proved that there are precisely two. Furthermore, Hollmann [17] found one on 496 points.

4.1.2. *Bipartite graphs.* Examples of bipartite graphs with four distinct eigenvalues are the incidence graphs of symmetric 2-(v, k, λ) designs. It is proven by Cvetković, Doob, and Sachs [6, p. 166] that these are the only examples, i.e., a connected bipartite regular graph with four distinct eigenvalues must be the incidence graph of a symmetric 2-(v, k, λ) design. Moreover, it is distance-regular and its spectrum is

$$\left\{[k]^1, [\sqrt{k-\lambda}]^{v-1}, [-\sqrt{k-\lambda}]^{v-1}, [-k]^1\right\}.$$

4.2. *The complement of the union of strongly regular graphs*

If G has tv vertices and spectrum $\{[k]^t, [r]^{tf}, [s]^{tg}\}$, and is the union of t strongly regular graphs (all with the same spectrum and hence the same parameters), then the complement of G is a connected regular graph with spectrum

$$\left\{[tv-k-1]^1, [-s-1]^{tg}, [-r-1]^{tf}, [-k-1]^{t-1}\right\},$$

so it has four distinct eigenvalues (if $t > 1$).

Note that if a connected regular graph has four distinct eigenvalues, then its complement is also connected and regular with four distinct eigenvalues, or it is disconnected, and then it is the union of strongly regular graphs, all having the same spectrum.

4.3. *Product constructions*

If G is a graph with adjacency matrix A, then we denote by $G \otimes J_n$ the graph with adjacency matrix $A \otimes J_n$, and by $G \circledast J_n$ we denote the graph with adjacency matrix $(A + I) \otimes J_n - I$. If G is connected and regular, then so are $G \otimes J_n$ and $G \circledast J_n$. Note that $(G \otimes J_n)^c = G^c \circledast J_n$, where G^c is the complement of G.

If G has v vertices and spectrum $\{[k]^1, [r]^f, [0]^m, [s]^g\}$, where m is possibly zero, then $G \otimes J_n$ has vn vertices and spectrum

$$\left\{[kn]^1, [rn]^f, [0]^{m+vn-v}, [sn]^g\right\}.$$

Similarly, if G has v vertices and spectrum $\{[k]^1, [r]^f, [-1]^m, [s]^g\}$, where m is possibly zero, then $G \circledast J_n$ has vn vertices and spectrum

$$\left\{[kn + n - 1]^1, [rn + n - 1]^f, [-1]^{m + vn - v}, [sn + n - 1]^g\right\}.$$

So, if we have a strongly regular graph or a connected regular graph with four distinct eigenvalues of which one is 0 or -1, then this construction produces a bigger graph with four distinct eigenvalues. The following theorem is a characterization of $C_5 \otimes J_n$, from which its uniqueness and the uniqueness of its complement $C_5 \circledast J_n$ follows.

THEOREM 4.1. *Let G be a connected regular graph with four distinct eigenvalues and adjacency matrix A. If* $\operatorname{rank}(A) \leqslant 5$ *and G has no triangles* ($\Delta = 0$), *then G is isomorphic to $C_5 \otimes J_n$ for some n.*

Proof. Let G have v vertices and degree k. First we shall prove that G has diameter 2. Suppose G has diameter 3 and take two vertices x, y at distance 3. Let A be partitioned according to $G(x) \cup \{y\}$ and the remaining vertices. Then

$$A = \begin{pmatrix} O_{k+1, k+1} & N \\ N^T & B \end{pmatrix}.$$

Since $\operatorname{rank}(A) \leqslant 5$, it follows that $\operatorname{rank}(N) \leqslant 2$. Now write

$$N = \begin{pmatrix} \underline{1}_k & N_1 \\ 0 & N_2 \end{pmatrix} \quad \text{and} \quad N' = \begin{pmatrix} \underline{0}_k & N_1 \\ 1 & N_2 \end{pmatrix}.$$

Since the all-one vector is in the column space of N (N has constant row sums k), $\operatorname{rank}(N') \leqslant \operatorname{rank}(N)$, so $\operatorname{rank}(N_1) \leqslant 1$. But then $N_1 = (J_{k, k-1} \; O)$, and we have a subgraph $K_{k,k}$, so it follows that G is disconnected, which is a contradiction. So G has diameter 2.

Next let A be partitioned according to $G(x)$ and the remaining vertices. Then

$$A = \begin{pmatrix} O_{k,k} & N \\ N^T & B \end{pmatrix},$$

with $\operatorname{rank}(N) \leqslant 2$. If $\operatorname{rank}(N) = 1$, then $N = J_{k,k}$, and so G is a bipartite complete graph $K_{k,k}$, but then G only has three distinct eigenvalues. So $\operatorname{rank}(N) = 2$. Now write

$$N = \begin{pmatrix} J_{n,3k-v} & J_{n,v-2k} & O_{n,v-2k} \\ J_{k-n,3k-v} & O_{k-n,v-2k} & J_{k-n,v-2k} \end{pmatrix},$$

for some n. Note that since $\operatorname{rank}(N) = 2$, we have that all parts in N are nonempty. Since G has no triangles, it follows from Lemma 3.1 that the number of quadrangles ξ through an edge is constant. If we count the number of quadrangles through x (which corresponds to one of the first $3k - v$ columns of N) and a vertex y which corresponds to one of the first n rows of N (x and y are adjacent), then we see that

$$\xi = (n-1)(k-1) + (k-n)(3k-v-1)$$

$$= (k-1)^2 + (k-n)(2k-v).$$

On the other hand, if we count the number of quadrangles through x and a vertex z which corresponds to one of the last $k - n$ rows of N, then we see that

$$\xi = (k-n-1)(k-1) + n(3k-v-1) = (k-1)^2 + n(2k-v).$$

So $n = \frac{1}{2}k$ and since A has rank at most 5 and zero diagonal, it follows that A is the adjacency matrix of $C_5 \otimes J_n$. ■

COROLLARY 4.2. *For any n, $C_5 \otimes J_n$ and $C_5 \circledast J_n$ are uniquely determined by their spectra.*

By $IG(l, l-1, l-2)$ we denote the incidence graph of the unique (trivial) 2-$(l, l-1, l-2)$ design. It can be obtained by removing a complete matching from the complete bipartite graph $K_{l,l}$, and is the complement of the $l \times 2$ grid.

THEOREM 4.3. *For each l and n, the graph $IG(l, l-1, l-2) \circledast J_n$ is uniquely determined by its spectrum.*

Proof. Note that for $l = 1$ or 2, the statement is trivial. So suppose $l > 2$. Let G be a graph with adjacency matrix A and spectrum

$$\left\{[nl-1]^1, [2n-1]^{l-1}, [-1]^{2nl-l-1}, [-n(l-2)-1]^1\right\}.$$

Now let $B = (A-(2n-1)I)(A+I)$; then we can partition A and B according to Theorem 3.3 such that

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} n(l-2)J_{nl} & O_{nl} \\ O_{nl} & n(l-2)J_{nl} \end{pmatrix},$$

where A_{11} and A_{22} have row sums $n-1$ and A_{12} has row sums $nl-n$. If two vertices x and y from the same part of the partition are adjacent, then it follows that $A_{xy}^2 = n(l-2) + 2n - 2 = k - 1$, so x and y have the same neighbors. So x has $n-1$ neighbors, which have the same neighbors as x, so $G = H \circledast J_n$, for some graph H. Since H must have the same spectrum as $IG(l, l-1, l-2)$, and this graph is uniquely determined by its spectrum, G is isomorphic to $IG(l, l-1, l-2) \circledast J_n$. ■

If A is the adjacency matrix of a conference graph G, that is, a strongly regular graph which has parameters $(v = 4\mu + 1, k = 2\mu, \mu - 1, \mu)$, and spectrum $\{[k]^1, [\frac{1}{2}(-1+\sqrt{v})]^k, [\frac{1}{2}(-1-\sqrt{v})]^k\}$, then the graph with adjacency matrix

$$\begin{pmatrix} A & I \\ I & J - I - A \end{pmatrix}$$

has spectrum

$$\left\{[k+1]^1, [k-1]^1, \left[\tfrac{1}{2}\left(-1+\sqrt{v+4}\right)\right]^{2k}, \left[\tfrac{1}{2}\left(-1-\sqrt{v+4}\right)\right]^{2k}\right\}.$$

We shall call this graph the twisted double of G. We shall prove that this is the only way to construct a graph with this spectrum.

THEOREM 4.4. *Let $v = 4\mu + 1$ and $k = 2\mu$. Then G is a graph with spectrum $\{[k+1]^1, [k-1]^1, [\frac{1}{2}(-1+\sqrt{v+4})]^{2k}, [\frac{1}{2}(-1-\sqrt{v+4})]^{2k}\}$ if and only if G is the twisted double of a conference graph on v vertices.*

Proof. Let A be the adjacency matrix of G and let B be as in the proof of Theorem 3.3, then we find that

$$B = A^2 + A - (\mu + 1)I = \begin{pmatrix} \mu J & J \\ J & \mu J \end{pmatrix},$$

and that we can write A (A_{12} has row and column sums 1) as

$$A = \begin{pmatrix} A_{11} & I \\ I & A_{22} \end{pmatrix}, \quad \text{and so} \quad B = \begin{pmatrix} A_{11}^2 + A_{11} - \mu I & A_{11} + A_{22} + I \\ A_{11} + A_{22} + I & A_{22}^2 + A_{22} - \mu I \end{pmatrix}.$$

This implies that $A_{11}^2 + A_{11} - \mu I = \mu J$ and $A_{11} + A_{22} + I = J$, so A_{11} is the adjacency matrix of a strongly regular graph with parameters ($v = 4\mu + 1, k = 2\mu, \mu - 1, \mu$), and A_{22} is the adjacency matrix of its complement. ∎

Since the conference graphs on 9, 13, and 17 vertices are unique, also their twisted doubles are uniquely determined by their spectra. Since there is no conference graph on 21 vertices, there is also no graph on 42 vertices with spectrum $\{[11]^1, [9]^1, [2]^{20}, [-3]^{20}\}$.

There are 15 conference graphs on 25 vertices, of which only one is isomorphic to its complement (cf. [20]). Since complementary graphs rise to the same twisted double, it follows that there are precisely 8 graphs on 50 vertices with spectrum $\{[13]^1, [11]^1, \{\frac{1}{2}(-1 + \sqrt{29})]^{24}, [\frac{1}{2}(-1 - \sqrt{29})]^{24}\}$.

Let G and G' be graphs with adjacency matrices A and A', and eigenvalues λ_i, $i = 1, 2, \ldots, v$, and λ'_i, $i = 1, 2, \ldots, v'$, respectively. Then the graph with adjacency matrix $A \otimes I_{v'} + I_v \otimes A'$ has eigenvalues $\lambda_i + \lambda'_j$, $i = 1, 2, \ldots, v$, $j = 1, 2, \ldots, v'$. We shall denote this graph, which is sometimes called the sum [6] or the Cartesian product of G and G', by $G \oplus G'$.

If G is a strongly regular graph with spectrum $\{[k]^1, [r]^f, [s]^g\}$, and G' is the complete graph on m vertices, then $G \oplus G'$ is a graph with spectrum

$$\{[k + m - 1]^1, [k - 1]^{m-1}, [r + m - 1]^f, [r - 1]^{f(m-1)}, \\ [s + m - 1]^g, [s - 1]^{g(m-1)}\}.$$

So we get a graph with four distinct eigenvalues if $m = k - r = r - s$. Examples are $G \oplus K_m$, where G is the complete bipartite graph $K_{m,m}$ or the lattice graph $\mathrm{OA}(m, 2)$ (see Section 4.5.3 for a definition), and $G \oplus K_4$, where G is the Clebsch or the Shrikhande graph.

Also $K_m \oplus K_n$ $(m > n \geqslant 2)$ is a graph with four distinct eigenvalues: it is the same graph as the line graph of the complete bipartite graph $K_{m,n}$ (see Section 4.4).

4.4. *Line graphs and other graphs with least eigenvalue* -2

If G is a strongly regular graph $(k \neq 2)$ or a bipartite regular graph with four distinct eigenvalues (the incidence graph of a symmetric 2-design, cf. Section 4.1.2), then its line graph $L(G)$ has four distinct eigenvalues. If G is strongly regular with v vertices and spectrum $\{[k]^1, [r]^f, [s]^g\}$, then it is well known that $L(G)$ has $\frac{1}{2}vk$ vertices and spectrum

$$\left\{[2k-2]^1, [r+k-2]^f, [s+k-2]^g, [-2]^{\frac{1}{2}vk-v}\right\}.$$

If G is the incidence graph of a symmetric 2-design, with v vertices and spectrum $\{[k]^1, [r]^f, [-r]^f, [-k]^1]$, then $L(G)$ has $\frac{1}{2}vk$ vertices and spectrum

$$\left\{[2k-2]^1, [r+k-2]^f, [-r+k-2]^f, [-2]^{\frac{1}{2}vk-v+1}\right\}.$$

Also the line graph of the complete bipartite graph $K_{m,n}$ has four distinct eigenvalues (if $m > n \geqslant 2$): its spectrum is

$$\left\{[m+n-2]^1, [m-2]^{n-1}, [n-2]^{m-1}, [-2]^{mn-m-n+1}\right\}.$$

Now these graphs provide almost all connected regular graphs with four distinct eigenvalues and least eigenvalue at least -2. It was proven by Doob and Cvetković [9] that a regular connected graph with least eigenvalue greater than -2 is K_n or C_{2n+1} for some $n \geqslant 1$. So the only one with four distinct eigenvalues is C_7. Bussemaker, Cvetković, and Seidel [4] found all connected regular graphs with least eigenvalue -2, which are neither line graphs, nor cocktail-party graphs. Among them are 12 graphs with four distinct eigenvalues:

BCS_9:	one graph on 12 vertices with spectrum	$\{[4]^1, [2]^3, [0]^3, [-2]^5\}$,
BCS_{70}:	one graph on 18 vertices with spectrum	$\{[7]^1, [4]^2, [1]^5, [-2]^{10}\}$,
BCS_{153}–BCS_{160}:	eight graphs on 24 vertices with spectrum	$\{[10]^1, [4]^4, [2]^3, [-2]^{16}\}$,
BCS_{179}:	one graph on 18 vertices with spectrum	$\{[10]^1, [4]^2, [1]^4, [-2]^{11}\}$,
BCS_{183}:	one graph on 24 vertices with spectrum	$\{[14]^1, [4]^4, [2]^2, [-2]^{17}\}$.

Cocktail-party graphs are strongly regular, so we are left with line graphs. Now Doob [8] showed that if G has four distinct eigenvalues, least eigenvalue -2, and is the line graph of, say, H, then H is a strongly regular graph, or the incidence graph of a symmetric 2-design, or a complete bipartite graph $K_{m,n}$, with $m > n \geqslant 2$.

Furthermore, it is known (cf. [6, p. 175]) that $L(K_{m,n})$ is not characterized by its spectrum if and only if $\{m, n\} = \{6, 3\}$ or

$$\{m, n\} = \{2t^2 + t, 2t^2 - t\}$$

and there exists a symmetric Hadamard matrix with constant diagonal of order $4t^2$. In the first case, there is one cospectral graph: BCS_{70}.

If G is the line graph of the incidence graph of a symmetric 2-(v, k, λ) design, then the only possible cospectral graph is the line graph of the incidence graph of other symmetric 2-(v, k, λ) designs, unless $(v, k, \lambda) = (4, 3, 2.)$ In that case, there is one exception: BCS_9.

Note that the complement of a connected regular with least eigenvalue -2 is a graph with second largest eigenvalue 1.

4.5. Other graphs from strongly regular graphs

In the previous sections, we already used strongly regular graphs to construct other graphs. In this section, we shall construct graphs from strongly regular graphs having certain properties, like having large cliques or cocliques, having a spread, or a regular partition into halves.

4.5.1. Hoffman cocliques and cliques. If G is a nonbipartite strongly regular graph on v vertices, with spectrum $\{[k]^1, [r]^f, [s]^g\}$, and C is a coclique of size c meeting the Delsarte (Hoffman) bound, i.e., $c = -vs/(k - s)$, then the induced subgraph $G \setminus C$ is a regular, connected graph with spectrum

$$\{[k + s]^1, [r]^{f-c+1}, [r + s]^{c-1}, [s]^{g-c}\},$$

so it has four distinct eigenvalues if $c < g$. This is an easy consequence of a theorem by Haemers and Higman [14] on strongly regular decompositions of strongly regular graphs. Note that by looking at the complement of the graph, a similar construction works for cliques instead of cocliques.

For example, by removing a 3-clique (a line) in the generalized quadrangle GQ(2, 2), we obtain a graph with spectrum $\{[5]^1, [1]^6, [-1]^2, [-3]^3\}$. If we remove a 6-coclique from a strongly regular graph with parameters (26, 10, 3, 4) (these exist), then we get a graph with spectrum $\{[7]^1, [2]^8, [-1]^5, [-3]^6\}$.

4.5.2. *Spreads.* If G admits a spread, that is, a partition of the vertices into cliques of size $1 - k/s$ (i.e., meeting the Hoffman bound), then by removing the spread, that is, the edges in these cliques, we obtain a graph with spectrum

$$\left\{\left[k + \frac{k}{s}\right]^{1}, [r + 1]^{(k/\mu)(-s-1)}, \left[r + \frac{k}{s}\right]^{f-(k/\mu)(-s-1)}, [s + 1]^{g}\right\}.$$

Note that these graphs come from 3-class association schemes. For example, if we remove a spread from the generalized quadrangle GQ(2, 4), we get a distance-regular graph with spectrum $\{[8]^1, [2]^{12}, [-1]^8, [-4]^6\}$.

4.5.3. *Seidel switching.* Let G be a strongly regular graph on v vertices admitting a regular partition into halves with degrees

$$\left(\tfrac{1}{2}(k + s), \tfrac{1}{2}(k - s)\right),$$

so its adjacency matrix A can be written as

$$A = \begin{pmatrix} A_{11} & A_{12} \\ A_{12}^T & A_{22} \end{pmatrix},$$

where all parts have equal size and A_{11}, A_{22} have row sums $\frac{1}{2}(k + s)$. Now the graph with adjacency matrix

$$\begin{pmatrix} A_{11} & J - A_{12} \\ J - A_{12}^T & A_{22} \end{pmatrix}$$

has spectrum

$$\left\{\left[s + \tfrac{1}{2}v\right]^{1}, [r]^{f}, [s]^{g-1}, \left[k - \tfrac{1}{2}v\right]^{1}\right\}.$$

This operation on G is called Seidel switching. Note that we can interchange the role of r and s. It follows from Theorem 3.3 that this is the only way to construct a graph with this spectrum.

THEOREM 4.5. *If G is an $(s + \frac{1}{2}v)$-regular graph with four distinct eigenvalues on v vertices and with spectrum*

$$\left\{\left[s + \tfrac{1}{2}v\right]^{1}, [r]^{f}, [s]^{g-1}, \left[k - \tfrac{1}{2}v\right]^{1}\right\},$$

then v is even and G can be obtained by Seidel switching in a strongly regular graph with spectrum $\{[k]^1, [r]^f, [s]^g\}$ admitting a regular partition into halves with degrees $(\frac{1}{2}(k+s), \frac{1}{2}(k-s))$.

This theorem may be useful in case we want to prove uniqueness or nonexistence of certain graphs, such as in some of the following examples, where we find some infinite families of graphs with four distinct eigenvalues. The first two families are obtained from the lattice graphs OA$(n, 2)$ for even n. The lattice graph is the graph on the n^2 ordered pairs (i, j), with $i, j = 1, 2, \ldots, n$, where two vertices are adjacent if they agree in one of the coordinates. Its spectrum is $\{[2n-2]^1, [n-2]^{2n-2}, [-2]^{(n-1)^2}\}$.

If we take for one part of the partition the set $\{(i,j) \mid i, j = 1, \ldots, \frac{1}{2}n\} \cup \{(i,j) \mid i, j = \frac{1}{2}n + 1, \ldots, n\}$ then we have a regular partition into halves with degrees $(n-2, n)$. Thus by Seidel switching, we obtain a graph with spectrum

$$\left\{\left[\tfrac{1}{2}n^2 - 2\right]^1, [n-2]^{2n-2}, [-2]^{(n-1)^2-1}, \left[2n - \tfrac{1}{2}n^2 - 2\right]^1\right\}.$$

Note that in this case (in general) there are different ways to obtain regular partitions into halves with these degrees, and so possibly different graphs with this spectrum.

If we take for one part of the partition the set $\{(i,j) \mid i = 1, \ldots, n, j = 1, \ldots, \frac{1}{2}n\}$, then we have a regular partition into halves with degrees $(\frac{1}{2}(3n-4), \frac{1}{2}n)$. Thus we obtain a graph with spectrum

$$\left\{\left[\tfrac{1}{2}n^2 + n - 2\right]^1, [n-2]^{2n-3}, [-2]^{(n-1)^2}, \left[2n - \tfrac{1}{2}n^2 - 2\right]^1\right\},$$

so for $n \geqslant 6$, it has four distinct eigenvalues. The following theorem proves that this graph is uniquely determined by its spectrum.

THEOREM 4.6. *For each even $n \geqslant 6$, there is exactly one graph on n^2 vertices with spectrum*

$$\left\{\left[\tfrac{1}{2}n^2 + n - 2\right]^1, [n-2]^{2n-3}, [-2]^{(n-1)^2}, \left[2n - \tfrac{1}{2}n^2 - 2\right]^1\right\}.$$

Proof. According to the previous theorem, a graph having the required spectrum must be obtained by Seidel switching in a strongly regular graph with spectrum $\{[2n-2]^1, [n-2]^{2n-2}, [-2]^{(n-1)^2}\}$. For $n \neq 4$, the only graph with this spectrum is the lattice graph OA$(n, 2)$. Furthermore, we must have a regular partition into halves with degrees $(\frac{1}{2}(3n-4), \frac{1}{2}n)$. Now there is (up to isomorphism) exactly one way to do this: take a spread and split it into two equal parts. ∎

This partition can also be used for the graphs OA(n, m) for "arbitrary" m. This graph is obtained from an orthogonal array, that is, an $m \times n^2$ matrix M such that for any two rows a, b, we have that $\{(M_{ai}, M_{bi}) \mid i = 1, \dots, n^2\} = \{(i,j) \mid i,j = 1, \dots, n\}$. The graph has vertices $1, 2, \dots, n^2$, and two vertices v, w are adjacent if $M_{iv} = M_{iw}$ for some i. This graph is strongly regular with spectrum $\{[mn-m]^1, [n-m]^{m(n-1)}, [-m]^{(n-1)(n-m+1)}\}$. If we now take for one part of the partition the set $\{i \mid M_{1i} = 1, \dots, \frac{1}{2}n\}$, then we have a regular partition into halves with degrees

$$\left(n-1+(m-1)\left(\tfrac{1}{2}n-1\right), \tfrac{1}{2}(m-1)n\right).$$

Thus we obtain a graph with spectrum

$$\left\{\left[\tfrac{1}{2}n^2+n-m\right]^1, \left[n-m\right]^{m(n-1)-1}, \left[-m\right]^{(n-1)(n-m+1)}, \left[mn-\tfrac{1}{2}n^2-m\right]^1\right\}.$$

Another family of graphs can be obtained from the triangular graphs $T(n)$, for $n \equiv 1 \pmod 4$. The triangular graph $T(n)$ is the graph on the $\frac{1}{2}n(n-1)$ unordered pairs taken from the n symbols $1, 2, \dots, n$, where two pairs are adjacent if they have a symbol in common. Its spectrum is $\{[2n-4]^1, [n-4]^{n-1}, [-2]^{n(n-3)/2}\}$. For each $n \equiv 1 \pmod 4$, we now get a regular partition into halves with degrees $(n-3, n-1)$ by taking for one part the pairs $\{i, j\}$, $i \neq j$, with

$$i = 1, \dots, \tfrac{1}{4}(n-1), \qquad j = 2, \dots, \tfrac{1}{2}(n-1)+1, \quad \text{or}$$

$$i = \tfrac{1}{4}(n-1)+1, \dots, \tfrac{1}{2}(n-1),$$

$$j = \tfrac{1}{2}(n-1)+2, \dots, 3(n-1)/4+1, \quad \text{or}$$

$$i = \tfrac{1}{2}(n-1)+1, \dots, n-1, \qquad j = 3(n-1)/4+2, \dots, n.$$

For $n \equiv 1 \pmod 4$, we thus obtain a graph with spectrum

$$\left\{\left[\tfrac{1}{4}n(n-1)-2\right]^1, \left[n-4\right]^{n-1}, \left[-2\right]^{\frac{1}{2}n(n-3)-1}, \left[2n-\tfrac{1}{4}n(n-1)-4\right]^1\right\}.$$

Note that (in general) there are more ways to obtain such partitions, and so possibly different graphs with this spectrum. The following lemma shows that we need the restriction $n \equiv 1 \pmod 4$, and gives a property of the partitions.

LEMMA 4.7. *If the triangular graph $T(n)$ admits a regular partition into halves V_1 and V_2, with degrees $(n-3, n-1)$, then $n \equiv 1 \pmod 4$ and for each $i = 1, \dots, n$ we have that $|\{j \neq i \mid \{i,j\} \in V_1\}| = \frac{1}{2}(n-1)$.*

Proof. First, note that the number of vertices $\frac{1}{2}n(n-1)$ should be even, so that $n \equiv 0$ or $1 \pmod 4$. Now fix i, and let $m = |\{j \neq i \mid \{i,j\} \in V_1\}|$. If $\{i,j\} \in V_1$, then we must have that

$$|\{h \neq i,j \mid \{h,j\} \in V_1\}| + |\{h \neq i,j \mid \{i,h\} \in V_1\}| = n-3,$$

so $|\{h \neq j \mid \{h,j\} \in V_1\}| = n-1-m$. If $\{i,j\} \in V_2$, then we must have that

$$|\{h \neq i,j \mid \{h,j\} \in V_1\}| + |\{h \neq i,j \mid \{i,h\} \in V_1\}| = n-1,$$

and then also $|\{h \neq j \mid \{h,j\} \in V_1\}| = n-1-m$. Now it follows that

$$\begin{aligned} m + (n-1)(n-1-m) &= \sum_{j=1}^{n} |\{h \neq j \mid \{h,j\} \in V_1\}| \\ &= 2|V_1| = \tfrac{1}{2}n(n-1), \end{aligned}$$

which implies that $m = \frac{1}{2}(n-1)$, and since this must be an integer, we must have $n \equiv 1 \pmod 4$. ■

Since the triangular graph $T(n)$ is uniquely determined by its spectrum unless $n = 8$, Theorem 4.5 and Lemma 4.7 imply the following result.

THEOREM 4.8. *For each $n \equiv 0 \pmod 4$, $n \neq 8$, there is no graph with spectrum*

$$\left\{\left[\tfrac{1}{4}n(n-1)-2\right]^1, [n-4]^{n-1}, [-2]^{\frac{1}{2}n(n-3)-1}, \left[2n-\tfrac{1}{4}n(n-1)-4\right]^1\right\}.$$

The next lemma shows that the "other" regular partition into halves is not possible, which together with Theorem 4.5 proves Theorem 4.10.

LEMMA 4.9. *For each $n \neq 4$, the triangular graph $T(n)$ does not admit a regular partition into halves with degrees $(3n/2 - 4, \frac{1}{2}n)$.*

Proof. Suppose we have such a partition with halves V_1 and V_2. Note that now both n and $\frac{1}{2}n(n-1)$ must be even, so $n \equiv 0 \pmod 4$. So we may suppose that $n \geqslant 8$. Now fix i and let $m = |\{j \neq i \mid \{i,j\} \in V_1\}|$. Without loss of generality, we may assume that $m > 0$. Then we find that if $\{i,j\} \in V_1$, then

$$\left|\{h \neq j \mid \{h,j\} \in V_1\}\right| = 3n/2 - 2 - m.$$

If $\{i,j\} \in V_2$, then we must have that

$$\left|\{h \neq j \mid \{h,j\} \in V_1\}\right| = \tfrac{1}{2}n - m.$$

This implies that $m \leqslant \frac{1}{2}n$ unless there is no j with $\{i,j\} \in V_2$. So $m \leqslant \frac{1}{2}n$ or $m = n - 1$. Now let j be such that $\{i,j\} \in V_1$, and

$$m' = \left|\{h \neq j \mid \{h,j\} \in V_1\}\right|;$$

then also $m' \leqslant \frac{1}{2}n$ or $m' = n - 1$. Without loss of generality, we may assume that $m \geqslant m'$, and since $m + m' = 3n/2 - 2$, we must have $m = n - 1$ and $m' = \frac{1}{2}n - 1$. Since $m' \geqslant 3$, there is an $h \neq i, j$ such that $\{i,j\} \in V_1$ and $\{j, h\} \in V_1$. Now let $m'' = |\{g \neq h \mid \{h, g\} \in V_1\}|$; then $m + m'' = 3n/2 - 2 = m' + m''$, so $m = m'$, which is a contradiction. ∎

THEOREM 4.10. *For each $n \neq 4$, there is no graph with spectrum $\{[\frac{1}{4}n(n-1) + n - 4]^1, [n-4]^{n-2}, [-2]^{\frac{1}{2}n(n-3)}, [2n - \frac{1}{4}n(n-1) - 4]^1\}$.*

For all parameter sets of strongly regular graphs on at most 63 vertices, except for $T(9)$ and $OA(6,2)$, we shall now give an example of how we can obtain a graph with four distinct eigenvalues, using Seidel switching. The only graphs we have to consider are the strongly regular graphs on 40 vertices with spectrum $\{[12]^1, [2]^{24}, [-4]^{15}\}$, the Hoffman-Singleton graph, which is the unique graph on 50 vertices with spectrum $\{[7]^1, [2]^{28}, [-3]^{21}\}$, and the Gewirtz graph, which is the unique graph on 56 vertices with spectrum $\{[10]^1, [2]^{35}, [-4]^{20}\}$.

Now there is one generalized quadrangle $GQ(3,3)$ (a strongly regular graph on 40 vertices) with a spread, and by splitting it into two equal parts, we have a regular partition into halves with degrees $(7,5)$. Thus we obtain a graph with spectrum $\{[22]^1, [2]^{23}, [-4]^{15}, [-8]^1\}$.

Haemers [12, ex. 6.2.2] constructed a strongly regular graph on 40 vertices admitting a regular partition into halves with degrees $(4, 8)$. This yields a graph with spectrum $\{[16]^1, [2]^{24}, [-4]^{14}, [-8]^1\}$.

Since it is possible to partition the vertices of the Hoffman-Singleton graph into two halves such that the induced subgraphs on each of the halves is the union of five pentagons (cf. [3]), we have a regular partition into two halves with degrees $(2, 5)$, and so we can construct a graph with spectrum $\{[22]^1, [2]^{28}, [-3]^{20}, [-18]^1\}$.

Since it is possible to split the Gewirtz graph into two Coxeter graphs (cf. [2]), we have a regular partition into two halves with degrees $(3, 7)$, and so we obtain with spectrum $\{[24]^1, [2]^{35}, [-4]^{19}, [-18]^1\}$.

The Gewirtz graph also contains a regular graph on 28 vertices of degree 6 (cf. [2]), and so we have a regular partition into two halves with degrees $(6, 4)$. Thus we obtain a graph with spectrum $\{[30]^1, [2]^{34}, [-4]^{20}, [-18]^1\}$.

4.5.4. Subconstituents. Let G be a strongly regular graph with parameters (v, k, λ, μ) and spectrum $\{[k]^1, [r]^f, [s]^g\}$. For any vertex x, we denote by $G(x)$ the induced subgraph on the set of neighbors of x. By $G_2(x)$ we denote the induced subgraph on the vertices which are not adjacent to x. These (regular) graphs are called subconstituents of G with respect to x.

Cameron, Goethals, and Seidel [5] proved that there is a one–one correspondence between the restricted eigenvalues $\notin \{r, s\}$ of the subconstituents of G, such that corresponding eigenvalues have the same restricted multiplicity, and add up to $r + s$. Here we call an eigenvalue restricted if it has an eigenvector orthogonal to the all-one vector. Its restricted multiplicity is the dimension of its eigenspace, which is orthogonal to the all-one vector.

This implies that if $\lambda = 0$, so $G(x)$ is a graph without edges, and hence has spectrum $\{[0]^k\}$; then $G_2(x)$ is a $(k - \mu)$-regular graph with restricted eigenvalues $r + s$, and possibly r and s, with multiplicities $k - 1$, and say m_r and m_s, respectively. Since $\mu = -(r + s)$, we find that $m_r = f + k$ and $m_s = g - k$, so $G_2(x)$ has spectrum $\{[k + r + s]^1, [r]^{f-k}, [r + s]^{k-1}, [s]^{g-k}\}$. For example, the Gewirtz graph is a strongly regular graph with $\lambda = 0$ and spectrum $\{[10]^1, [2]^{35}, [-4]^{20}\}$, so Gewirtz$_2(x)$ is a graph with spectrum $\{[8]^1, [2]^{25}, [-2]^9, [-4]^{10}\}$. Also the Hoffman-Singleton graph Ho-Si is a strongly regular graph with $\lambda = 0$, and its spectrum is $\{[7]^1, [2]^{28}, [-3]^{21}\}$, so Ho-Si$_2(x)$ is a graph with spectrum $\{[6]^1, [2]^{21}, [-1]^6, [-3]^{14}\}$.

If $\lambda = r$ and $G(x)$ is the union of $(r + 1)$-cliques, so it has spectrum $\{[r]^{k/(r+1)}, [-1]^{rk/(r+1)}\}$; then $G_2(x)$ is a $(k - \mu)$-regular graph with restricted eigenvalues $r + s + 1$, and possibly r and s, with multiplicities $rk/(r + 1)$, and say m_r and m_s, respectively. Since $\mu = -s$, we find that $m_r = f - k$ and $m_s = g - rk/(r + 1) - 1$, so $G_2(x)$ has spectrum $\{[k + s]^1, [r]^{f-k}, [r + s + 1]^{rk/(r+1)}, [s]^{g-rk/(r+1)-1}\}$. Examples of such graphs can be found when G is the graph of a generalized quadrangle.

4.6. Covers

In this section, we shall construct n-covers of $C_3 \otimes J_n$, $C_3 \circledast J_n = K_{3n}$, $C_5 \circledast J_n$, $C_6 \circledast J_n$, and Cube $\circledast J_n$, having four distinct eigenvalues.

Let C be the $n \times n$ circulant matrix defined by $C_{ij} = 1$ if $j = i + 1 \pmod n$, and $C_{ij} = 0$ otherwise. Then let A and B be the $n^2 \times n^2$ matrices defined by

$$A = \begin{pmatrix} I & I & \cdots & I \\ C & C & \cdots & C \\ \vdots & \vdots & & \vdots \\ C^{n-1} & C^{n-1} & \cdots & C^{n-1} \end{pmatrix} \quad \text{and}$$

$$B = \begin{pmatrix} I & C & \cdots & C^{n-1} \\ C^{n-1} & I & \ddots & \vdots \\ \vdots & \ddots & \ddots & C \\ C & \cdots & C^{n-1} & I \end{pmatrix}.$$

Furthermore, let $D = (J_n - I_n) \otimes I_n$. Then the graphs with adjacency matrices

$$A_3 = \begin{pmatrix} O & A & A^T \\ A^T & O & A \\ A & A^T & O \end{pmatrix}, \qquad B_3 = \begin{pmatrix} D & A & A^T \\ A^T & D & A \\ A & A^T & D \end{pmatrix},$$

$$B_5 = \begin{pmatrix} D & A & O & O & A^T \\ A^T & D & A & O & O \\ O & A^T & D & A & O \\ O & O & A^T & D & A \\ A & O & O & A^T & D \end{pmatrix}$$

are n-covers of $C_3 \otimes J_n$, $C_3 \circledast J_n$, and $C_5 \circledast J_n$, respectively.

The graphs with adjacency matrices

$$B_6 = \begin{pmatrix} D & O & O & O & A^T & A^T \\ O & D & O & A & O & D+I \\ O & O & D & A & D+I & O \\ O & A^T & A^T & D & O & O \\ A & O & D+I & O & D & O \\ A & D+I & O & O & O & D \end{pmatrix},$$

$$B_8 = \begin{pmatrix} D & O & O & O & O & D+I & B & B \\ O & D & O & O & D+I & O & B & B \\ O & O & D & O & B & B & O & D+I \\ O & O & O & D & B & B & D+I & O \\ O & D+I & B & B & D & O & O & O \\ D+I & O & B & B & O & D & O & O \\ B & B & O & D+I & O & O & D & O \\ B & B & D+I & O & O & O & O & D \end{pmatrix}$$

are n-covers of $C_6 \circledast J_n$ and Cube $\circledast J_n$, respectively.

A_3 has spectrum, $\{[2n]^1, [n]^{3n-3}, [0]^{3(n-1)^2}, [-n]^{3n-1}\}$. The crucial step to show this is that $A_3(A_3^2 - n^2 I) = 2nJ$ (the multiplicities follow from the eigenvalues). For $n = 2$, we get the line graph of the cube, and for $n = 3$, we get a graph, which is cospectral (but not isomorphic) with the cubic lattice graph $H(3,3)$.

B_3 has spectrum $(\{3n - 1]^1, [-1]^{3n^2-6n+5}, [-1 + \frac{1}{2}n(1 \pm \sqrt{5})]^{3n-3}\}$. The crucial step here is that $(B_3 + I)((B_3 + I)^2 - n(B_3 + I) - n^2 I) = 5nJ$. For $n = 2$, we get the icosahedron.

Similarly, we find that B_5 has spectrum

$$\left\{[3n - 1]^1, [-1]^{5n^2-10n+5}, \left[-1 + \tfrac{1}{2}n\left(1 \pm \sqrt{5}\right)\right]^{5n-3}\right\},$$

B_6 has spectrum $\{[3n - 1]^1, [2n - 1]^{4n-2}, [-1]^{6n^2-6n+2}, [-n - 1]^{2n-1}\}$, and B_8 has spectrum $\{[4n - 1]^1, [2n - 1]^{6n-3}, [-1]^{8n^2-8n+3}, [-2n - 1]^{2n-1}\}$.

I wish to thank W. H. Haemers for several suggestions and remarks on this paper.

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Received 6 June 1994; final manuscript accepted 28 November 1994