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# Characterizing distance-regularity of graphs by the spectrum

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**Abstract**

We characterize the distance-regular Ivanov–Ivanov–Faradjev graph from the spectrum, and construct cospectral graphs of the Johnson graphs, Doubled Odd graphs, Grassmann graphs, Doubled Grassmann graphs, antipodal covers of complete bipartite graphs, and many of the Taylor graphs. We survey the known results on cospectral graphs of the Hamming graphs, and of all distance-regular graphs on at most 70 vertices.


*Keywords:* Distance-regular graphs; Eigenvalues; Cospectral graphs

## 1. Introduction

In this paper we give new results on the question whether distance-regularity of graphs is a property that is determined by the spectrum of the adjacency matrix of the graph. The answer

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to this question is no in general; however for particular distance-regular graphs the answer is affirmative, such as for the diameter two case (strongly regular graphs), cf. [7]. Here we settle the problem for the Ivanov–Ivanov–Faradjev graph, the Johnson graphs, the Doubled Odd graphs, the Grassmann graphs, the Doubled Grassmann graphs, the antipodal covers of complete bipartite graphs, and many of the Taylor graphs (for definitions of these graphs we refer the reader to [2]). We also give an overview of all results on cospectral graphs of the Hamming graphs and of all distance-regular graphs on at most 70 vertices.

Consider an arbitrary graph $G$ and let $\{X_0, \dots, X_m\}$ be a partition of the vertex set $V$. We say that (the symbol) $x_{i,j}$ is *well-defined* if each vertex in $X_i$ is adjacent to a constant number $x_{i,j}$, say, of vertices in $X_j$. If $x_{i,j}$ is well-defined for all $i, j \in \{0, \dots, m\}$, then the vertex partition is called *regular* (or *equitable*) and the $(m+1) \times (m+1)$ matrix $(x_{i,j})$ is called the *quotient matrix*. Given a vertex $x \in V$ with local diameter $d$, let $X_i = G_i(x)$ be the set of vertices at distance $i$ from $x$. Then $\{G_0(x), \dots, G_d(x)\}$ is called the *distance partition* with respect to $x$. If $G$ is connected and the distance partition is regular, with respect to every vertex, each with the same quotient matrix, then $G$ is (called) *distance-regular*. The intersection parameters of $G$ are $a_i = x_{i,i}$, $b_i = x_{i,i+1}$, $c_i = x_{i,i-1}$, $k_i = |X_i|$ and $k = k_1$ ($i = 0, \dots, d$; take $b_d = c_0 = 0$). These parameters satisfy the following obvious conditions

$$a_i + b_i + c_i = k, \qquad k_{i-1}b_{i-1} = k_i c_i \quad (i = 1, \dots, d),$$
$$k_0 = c_1 = 1, \qquad b_0 = k_1 = k.$$

Thus all parameters of $G$ can be obtained from the intersection array $\{b_0, \dots, b_{d-1}; c_1, \dots, c_d\}$.

For an arbitrary graph we say that $a_i$ is *well-defined* if for all distance partitions the $x_{i,i}$'s are well-defined and the same. For $b_i$, $c_i$ and $k_i$, this is defined similarly.

## 2. The Ivanov–Ivanov–Faradjev graph

We will show that the distance-transitive Ivanov–Ivanov–Faradjev graph is determined by its spectrum. This graph, constructed by Ivanov, Ivanov, and Faradjev [11], is a 3-fold antipodal cover of the $M_{22}$ graph. Brouwer [1] proved that both the $M_{22}$ graph and the Ivanov–Ivanov–Faradjev graph are uniquely determined distance-regular graphs with respective intersection arrays $\{7, 6, 4, 4; 1, 1, 1, 6\}$ and $\{7, 6, 4, 4, 4, 1, 1, 1; 1, 1, 1, 2, 4, 4, 6, 7\}$. In [7] it was (among other things) proved that the $M_{22}$ graph is the unique graph with spectrum $\{[7]^1, [4]^{55}, [1]^{154}, [-3]^{99}, [-4]^{21}\}$ by showing that a graph with that spectrum must be distance-regular. Here we will do the same for the Ivanov–Ivanov–Faradjev graph: we will show that a graph with spectrum

$$\left\{[7]^1, [5]^{42}, [4]^{55}, \left[\frac{-1+\sqrt{33}}{2}\right]^{154}, [1]^{154}, [0]^{198}, [-3]^{99}, \left[\frac{-1-\sqrt{33}}{2}\right]^{154}, [-4]^{21}\right\}$$

must be distance-regular, and hence is the Ivanov–Ivanov–Faradjev graph. The proof of this result uses the following two lemmas. The first is an obvious generalization of a lemma from [7], the second is a result by Brouwer [1] on the occurrence of Petersen graphs.

**Lemma 1.** [7] *Let $G$ and $G'$ be two graphs with the same spectrum, with $d+1$ distinct eigenvalues, and let $t \leqslant d$ be a positive integer. Suppose that in $G$ the parameters $a_i$, $b_i$, and $c_{i+1}$, $i = 0, \dots, t-1$ are well-defined, and that $c_t = c_{t-1}$. If in $G'$ the parameters $a'_i$, $b'_i$, and $c'_{i+1}$, $i = 0, \dots, t-2$ are well-defined, and the same as the corresponding intersection parameters of $G$, then also $a'_{t-1}$, $b'_{t-1}$, and $c'_t$ are well-defined, and the same as in $G$.*

**Proof.** The proof is a small alteration of that of Lemma 1 in [7]: replace $c'_t(x, y) \geqslant 1$ by $c'_t(x, y) \geqslant c_{t-1} = c_t$. $\square$

**Lemma 2.** [1] *Let $G$ be a graph such that the intersection parameters $a_1 = 0$, $a_2 = 2$, $c_2 = c_3 = 1$ are well-defined. Then any two vertices at distance two determine a unique induced Petersen graph.*

The latter lemma can also be found in greater generality in [2, Proposition 4.3.11]. The induced Petersen graph determined by two vertices $x$ and $y$ at distance two, let us call it $P(x, y)$, is obtained by taking the component of $G_2(x)$ containing $y$, and the vertices between $x$ and this component. An important additional remark (by Brouwer [1]) is that in our case, when the valency $k = 7$, any two distinct intersecting induced Petersen graphs intersect in an edge. Every edge, on the other hand, is contained in 3 Petersen graphs.

**Theorem 1.** *The Ivanov–Ivanov–Faradjev graph is uniquely determined by its spectrum.*

**Proof.** Let $G$ be a graph with adjacency matrix $A$, which has the same spectrum as the Ivanov–Ivanov–Faradjev graph IIF. From Lemma 1 (where $G$ plays the role of $G'$, and IIF the role of $G$) it follows that in $G$ the intersection parameters $k = b_0 = 7$, $c_1 = 1$, $a_1 = 0$, $b_1 = 6$, $c_2 = 1$, $a_2 = 2$, $b_2 = 4$, $c_3 = 1$ and the degrees $k = k_1 = 7$, $k_2 = 42$, and $k_3 = 168$ are well-defined. Thus Lemma 2 applies, and any two vertices $x$ and $y$ at distance 2 determine a unique induced Petersen graph $P(x, y)$.

Since $a_2 = 2$ and $c_3 = 1$, it is easy to see that $a_3(x, y) \geqslant 2$ for all $x$ and $y$ at distance 3. Since $\mathrm{Tr}(A^7)$ is determined by the spectrum, and counts the total number of closed walks of length 7 starting at $x$, over all vertices $x$, it now follows that $\sum_x \sum_{y \in G_3(x)} a_3(x, y)$ is determined by the spectrum, i.e., it is the same as in the Ivanov–Ivanov–Faradjev graph (here we used the fact that all intersection parameters "up to" $c_3$ are well-defined; and the total number of closed walks of length 7 can be expressed in terms of these intersection parameters and the above double sum). Thus the average $a_3(x, y)$ equals 2 (since we know that $k_3$ is well-defined too). From the lower bound above it thus follows that the intersection parameter $a_3 = 2$ is well-defined, and hence so is $b_3 = 4$.

At this point we can deduce that $G$ is a so-called walk-regular graph, i.e., $A^i$ has constant diagonal for all $i$. Indeed, from the fact that all intersection parameters "up to" $a_3$ are well-defined it follows that $A^i$ has constant diagonal for all $i \leqslant 7$. Together with the fact that the Hoffman-polynomial $h$ of $G$ has degree 8, and satisfies $h(A) = \frac{h(k)}{v} J$ [15], it now follows that $G$ is walk-regular. Thus $A^i_{xx}$, the number of closed walks of length $i$ starting at $x$, follows from the spectrum.

Let $x$ and $y$ be two vertices at distance 4, and let $z$ be a vertex at distance 2 from both $x$ and $y$. The Petersen graphs $P(x, z)$ and $P(z, y)$ intersect, hence intersect in an edge, hence in a vertex $w \neq z$ in $G_2(x) \cap G_2(y)$. But this implies that there is another path of length 4 between $x$ and $y$, that is, $c_4(x, y) \geqslant 2$. From counting edges between $G_3(x)$ and $G_4(x)$, we see that

$$\sum_{y \in G_4(x)} c_4(x, y) = k_3 b_3 = 2 \cdot 336.$$

From counting closed walks of length 8 starting at $x$, we find that

$$\sum_{y \in G_4(x)} c_4(x, y)^2 = 4 \cdot 336,$$

that is, it is the same as in the Ivanov–Ivanov–Faradjev graph (since the number of closed walks of length 8 starting at $x$ follows from the spectrum, and can be expressed in terms of the well-defined intersection parameters "up to" $b_3$ and the above sum of squares). From $c_4(x, y) \geqslant 2$ we now obtain

$$4 \cdot 336 \geqslant \sum_{y \in G_4(x)} 2 \cdot c_4(x, y) = 4 \cdot 336,$$

hence we have equality, and the intersection parameter $c_4 = 2$ and also $k_4 = 336$, are well-defined. Moreover, since all three neighbors of $y$ in $P(z, y)$ are at distance at most two from $z$, but only $c_4 = 2$ can be at distance 3 from $x$, it now follows that $a_4(x, y) \geqslant 1$. Similar as before ($a_3$), we derive that $a_4 = 1$ and hence $b_4 = 4$, are well-defined (by counting closed walks of length 9 through $x$).

We now arrive at the most technical part of our proof. We start by showing that $a_5(x, y) \geqslant 2$ for vertices $x$ and $y$ at distance 5. Let $x \sim x_1 \sim x_2 \sim v_1 \sim z \sim y$ be a path of length 5, and let $v_2$ be another neighbor of $z$ at distance three from $x$. The first step consists of showing that $P(v_1, y) \neq P(v_2, y)$: since the $b_4 = 4$ neighbors of $z$ in $G_5(x)$ are at distance at least three from $x_2$, it follows that $v_2 \in P(x_2, z)$. Thus $P(x_2, z) = P(v_1, v_2)$. This implies that $P(v_i, y) \neq P(v_1, v_2)$ $(i = 1, 2)$, and then also $P(v_1, y) \neq P(v_2, y)$. The second step consists of showing that $P(v_i, y)$ contains a neighbor of $y$ in $G_5(x)$ $(i = 1, 2)$. By symmetry, it suffices to show this for $i = 1$. So let us assume that all three neighbors $z, z_1, z_2$ of $y$ in $P(v_1, y)$ are at distance 4 from $x$. Since $P(x_1, v_1)$ intersects $P(v_1, y)$, they both contain an edge $\{v_1, v_3\}$ in $G_3(x)$. Without loss of generality we may assume that $z_1$ is adjacent to $v_3$. Now $z_2$ cannot be adjacent to $v_1$ or $v_3$. Let $w_1$ and $w_2$ be the common neighbors of $z_2$ with $v_1$ and $v_3$, respectively. Since $a_4 = 1$, at least one of $w_1$ and $w_2$, say $w_1$ (without loss of generality), must be in $G_3(x)$ (that is, not in $G_4(x)$). But then $P(v_1, y) = P(w_1, v_3) = P(x_1, v_1)$, since the latter also must contain the 2 neighbors ($v_3$ and $w_1$) of $v_1$ in $G_3(x)$, and this is a contradiction (since $x_1$ and $y$ are at distance 4). It is clear that from these two steps we find that $a_5(x, y) \geqslant 2$, and hence $c_5(x, y) \leqslant 5$.

Next we will show that $c_5(x, y) \neq 5$. Suppose, on the contrary, that $c_5(x, y) = 5$. Let $x_1, x_2, v_1, v_2, z$ be as above. Let $z_1, z_2, z_3, z_4$ be the other 4 neighbors of $y$ in $G_4(x)$, and let $y_1, y_2$ be the two neighbors of $y$ in $G_5(x)$. We may assume that $z_i, y_i \in P(v_i, y)$ $(i = 1, 2)$. The third Petersen graph through $\{z, y\}$ must then be $P(z, z_3) = P(z_3, z_4)$. Since $P(z_1, z_2)$ intersects $P(v_i, y)$ in an edge already, it does not contain $y_1, y_2$, or $z$. So without loss of generality we may assume that $z_3 \in P(z_1, z_2)$. Then the three Petersen graphs through $\{y, z_3\}$ must be $P(z_1, z_2)$, $P(z_3, z_4) = P(z, z_3)$, and $P(y_1, y_2)$. But earlier we saw that a Petersen graph through an edge ($\{z, y\}$) between $G_4(x)$ and $G_5(x)$ (in particular $P(y_1, y_2)$) contains at most two vertices in $G_5(x)$, a contradiction. So $c_5(x, y) \leqslant 4$.

As before, it now follows that $c_5 = 4$ and then $k_5 = 336$. From Lemma 1 it now follows that $a_5 = 2$, $b_5 = 1$, $c_6 = 4$, and $k_6 = 84$. In this case ($c_5 = c_6$) also $a_6(x, y) \geqslant a_5 = 2$, from which we obtain $a_6 = 2$, and $b_6 = 1$.

We now have left 16 vertices at distance at least 7 from $x$. Suppose there are $n_i$ vertices among those that have $i$ neighbors in $G_6(x)$. Since a vertex at distance exactly 7 from $x$ has at least $c_6 = 4$ neighbors in $G_6(x)$, it follows that $n_1 = n_2 = n_3 = 0$. Moreover,

$$\sum_{i=0}^{7} n_i = 16,$$

and by counting edges between $G_6(x)$ and $G_7(x)$ we find that

$$\sum_{i=0}^{7} i \cdot n_i = 84.$$

By counting closed walks of length 14 through $x$ (as before) we find that

$$\sum_{i=0}^{7} i^2 \cdot n_i = \sum_{y \in G_7(x)} c_7(x, y)^2 = 6^2 \cdot 14 = 504.$$

By taking a linear combination of the above three equations (with coefficients 14, $-11/2$, $1/2$), and substituting $n_1 = n_2 = n_3 = 0$, we deduce that

$$14n_0 - n_5 - n_6 = 14.$$

This implies that $n_0 \geqslant 1$. If $n_0 = 1$, then $n_5 = n_6 = 0$, and then it easily follows that $n_4 = 7$ and $n_7 = 8$. In this case there is one vertex at distance 8 from $x$, and it must be adjacent to all $n_4 = 7$ vertices $y$ that have $c_7(x, y) = 4$. But then the induced graph on these 7 vertices has edges, which implies that the graph contains triangles, a contradiction. Hence $n_0 \geqslant 2$. From the first equation it then follows that $n_5 + n_6 \leqslant 14$; but from the fourth equation we obtain $n_5 + n_6 \geqslant 14$, hence $n_5 + n_6 = 14$, $n_0 = 2$, and then $n_4 = n_7 = 0$, and it follows that $n_5 = 0$ and $n_6 = 14$. This implies that $c_7 = 6$ and $k_7 = 14$. As before we then see that $a_7 = 0$ and $b_7 = 1$, and then $c_8 = 7$ and $k_8 = 2$. Thus $G$ is distance-regular with the same intersection array as the Ivanov–Ivanov–Faradjev graph. Since this distance-regular graph is determined by its intersection array [1], $G$ must be the Ivanov–Ivanov–Faradjev graph. □

## 3. Cospectral graphs

For several families of graphs we shall now construct cospectral graphs. Some of these constructions use the following useful switching tool of Godsil and McKay [12].

**Lemma 3.** *Let $G$ be a graph and let $\Pi = \{D, C_1, \ldots, C_m\}$ be a partition of the vertex set of $G$. Suppose that for every vertex $x \in D$ and every $i \in \{1, \ldots, m\}$, $x$ has either 0, $\frac{1}{2}|C_i|$ or $|C_i|$ neighbors in $C_i$. Moreover, suppose that $\{C_1, \ldots, C_m\}$ is a regular partition of $G \setminus D$. Make a new graph $G'$ as follows. For each $x \in D$ and $i \in \{1, \ldots, m\}$ such that $x$ has $\frac{1}{2}|C_i|$ neighbors in $C_i$ delete the corresponding $\frac{1}{2}|C_i|$ edges and join $x$ instead to the $\frac{1}{2}|C_i|$ other vertices in $C_i$. Then $G$ and $G'$ have the same spectrum.*

By computer we searched for switching partitions $\Pi$ with sets $C_i$ of sizes 4, 6, and 8, in some of the distance-regular graphs mentioned in Table 1. This was repeatedly done also in the cospectral graphs so obtained. For the Johnson graph $J(7,3)$ we thus found 100 cospectral graphs, while for $J(8,3)$ we found no fewer than 33,525. For the Taylor graphs over $P(13)$, $GQ(2,2)$, $T(6)$, $P(17)$, and the complement of the Schläfli graph, we obtained 1173, 104,799, 3, 74,112, and 174,608 cospectral graphs, respectively. For the Hadamard graph on 48 vertices we found 79,469 cospectral graphs. (All numbers mentioned include the original distance-regular graphs.) For the Gosset graph, the Mathon graphs, and the strongly regular graphs minus a spread (on 40 and 64 vertices) no (other) cospectral graphs were found.

Table 1
Graphs cospectral with distance-regular graphs, $v \leqslant 70$

| $v$ | Spectrum | Intersection array | Name DRG | $dr_\Sigma$ | $gr_\Sigma$ | Ref. |
|---|---|---|---|---|---|---|
| 12 | $\{[5]^1, [\sqrt{5}]^3, [-1]^5, [-\sqrt{5}]^3\}$ | $\{5, 2, 1; 1, 2, 5\}$ | Icosahedron | 1 | 1 | [14] |
| 15 | $\{[4]^1, [2]^5, [-1]^4, [-2]^5\}$ | $\{4, 2, 1; 1, 1, 4\}$ | $L$(Petersen) | 1 | 1 | [14] |
| 16 | $\{[4]^1, [2]^4, [0]^6, [-2]^4, [-4]^1\}$ | $\{4, 3, 2, 1; 1, 2, 3, 4\}$ | $H(4, 2)$ | 1 | 2 | [15] |
| 18 | $\{[3]^1, [\sqrt{3}]^6, [0]^4, [-\sqrt{3}]^6, [-3]^1\}$ | $\{3, 2, 2, 1; 1, 1, 2, 3\}$ | Pappus, 3-cover $K_{3,3}$ | 1 | 1 | [14] |
| 20 | $\{[9]^1, [3]^5, [-1]^9, [-3]^5\}$ | $\{9, 4, 1; 1, 4, 9\}$ | $J(6, 3)$ | 1 | 9 | [14] |
| 20 | $\{[3]^1, [2]^4, [1]^5, [-1]^5, [-2]^4, [-3]^1\}$ | $\{3, 2, 2, 1, 1; 1, 1, 2, 2, 3\}$ | Desargues | 1 | 2 | [14] |
| 20 | $\{[3]^1, [\sqrt{5}]^3, [1]^5, [0]^4, [-2]^4, [-\sqrt{5}]^3\}$ | $\{3, 2, 1, 1, 1; 1, 1, 1, 2, 3\}$ | Dodecahedron | 1 | 1 | [14] |
| 21 | $\{[4]^1, [1+\sqrt{2}]^6, [1-\sqrt{2}]^6, [-2]^8\}$ | $\{4, 2, 2; 1, 1, 2\}$ | $GH(2, 1)$ | 1 | 1 | [14] |
| 24 | $\{[7]^1, [\sqrt{7}]^8, [-1]^7, [-\sqrt{7}]^8\}$ | $\{7, 4, 1; 1, 2, 7\}$ | Klein | 1 | 10 | [14] |
| 27 | $\{[6]^1, [3]^6, [0]^{12}, [-3]^8\}$ | $\{6, 4, 2; 1, 2, 3\}$ | $H(3, 3)$ | 1 | 4 | [14] |
| 27 | $\{[8]^1, [2]^{12}, [-1]^8, [-4]^6\}$ | $\{8, 6, 1; 1, 3, 8\}$ | $GQ(2, 4) \setminus$ spread | 2 | 13 | [14] |
| 28 | $\{[3]^1, [2]^8, [-1+\sqrt{2}]^6, [-1]^7, [-1-\sqrt{2}]^6\}$ | $\{3, 2, 2, 1; 1, 1, 1, 2\}$ | Coxeter | 1 | 1 | [14] |
| 28 | $\{[13]^1, [\sqrt{13}]^7, [-1]^{13}, [-\sqrt{13}]^7\}$ | $\{13, 6, 1; 1, 6, 13\}$ | Taylor($P(13)$) | 1 | $\geqslant 1173$ | Section 3 |
| 30 | $\{[3]^1, [2]^9, [0]^{11}, [-2]^9, [-3]^1\}$ | $\{3, 2, 2, 2; 1, 1, 1, 3\}$ | Tutte's 8-cage | 1 | 1 | [14] |
| 32 | $\{[15]^1, [3]^{10}, [-1]^{15}, [-5]^6\}$ | $\{15, 8, 1; 1, 8, 15\}$ | Taylor($GQ(2, 2)$) | 1 | $\geqslant 104799$ | Section 3 |
| 32 | $\{[15]^1, [5]^6, [-1]^{15}, [-3]^{10}\}$ | $\{15, 6, 1; 1, 6, 15\}$ | Taylor($T(6)$) | 1 | $\geqslant 3$ | Section 3 |
| 32 | $\{[4]^1, [2]^{12}, [0]^6, [-2]^{12}, [-4]^1\}$ | $\{4, 3, 3, 1; 1, 1, 3, 4\}$ | $IG(AG(2, 4) \setminus \mathrm{pc})$ | 1 | 1 | Proposition 7, [7] |
| 32 | $\{[5]^1, [\sqrt{5}]^8, [1]^{10}, [-\sqrt{5}]^8, [-3]^5\}$ | $\{5, 4, 1, 1; 1, 1, 4, 5\}$ | Wells | 1 | 3 | [7] |
| 32 | $\{[8]^1, [\sqrt{8}]^8, [0]^{14}, [-\sqrt{8}]^8, [-8]^1\}$ | $\{8, 7, 4, 1; 1, 4, 7, 8\}$ | Hadamard graph | 1 | 327 | [9] |
| 32 | $\{[5]^1, [3]^5, [1]^{10}, [-1]^{10}, [-3]^5, [-5]^1\}$ | $\{5, 4, 3, 2, 1; 1, 2, 3, 4, 5\}$ | $H(5, 2)$ | 1 | $\geqslant 2$ | Section 4 |
| 35 | $\{[4]^1, [2]^{14}, [-1]^{14}, [-3]^6\}$ | $\{4, 3, 3; 1, 1, 2\}$ | Odd(4) | 1 | 1 | [3,16] |
| 35 | $\{[12]^1, [5]^6, [0]^{14}, [-3]^{14}\}$ | $\{12, 6, 2; 1, 4, 9\}$ | $J(7, 3)$ | 1 | $\geqslant 100$ | Section 3 |
| 36 | $\{[5]^1, [2]^{16}, [-1]^{10}, [-3]^9\}$ | $\{5, 4, 4; 1, 1, 4\}$ | Sylvester | 1 | 1 | [3] |
| 36 | $\{[17]^1, [\sqrt{17}]^9, [-1]^{17}, [-\sqrt{17}]^9\}$ | $\{17, 8, 1; 1, 8, 17\}$ | Taylor($P(17)$) | 1 | $\geqslant 74112$ | Section 3 |
| 36 | $\{[6]^1, [\sqrt{6}]^{12}, [0]^{10}, [-\sqrt{6}]^{12}, [-6]^1\}$ | $\{6, 5, 4, 1; 1, 2, 5, 6\}$ | 3-cover $K_{6,6}$ | 1 | 40 | Section 3.4 |

(*continued on next page*)

Table 1 (*continued*)

| $v$ | Spectrum | Intersection array | Name DRG | $dr_\Sigma$ | $gr_\Sigma$ | Ref. |
|---|---|---|---|---|---|---|
| 40 | $\{[9]^1, [3]^{15}, [-1]^9, [-3]^{15}\}$ | $\{9, 6, 1; 1, 2, 9\}$ | $SRG \setminus$ spread | 3 | $\geqslant 3$ | [10] |
| 42 | $\{[6]^1, [2]^{21}, [-1]^6, [-3]^{14}\}$ | $\{6, 5, 1; 1, 1, 6\}$ | Ho-Si$_2(x)$ | 1 | 1 | [3] |
| 42 | $\{[13]^1, [\sqrt{13}]^{14}, [-1]^{13}, [-\sqrt{13}]^{14}\}$ | $\{13, 8, 1; 1, 4, 13\}$ | Mathon(Cycl(13,3)) | $\geqslant 1$ | $\geqslant 1$ | [17] |
| 45 | $\{[4]^1, [3]^9, [1]^{10}, [-1]^9, [-2]^{16}\}$ | $\{4, 2, 2, 2; 1, 1, 1, 2\}$ | $GO(2, 1)$ | 1 | 1 | [7] |
| 45 | $\{[6]^1, [3]^{12}, [1]^9, [-2]^{18}, [-3]^5\}$ | $\{6, 4, 2, 1; 1, 1, 4, 6\}$ | 3-cover $GQ(2, 2)$ | 1 | $\geqslant 2$ | [7] |
| 48 | $\{[12]^1, [\sqrt{12}]^{12}, [0]^{22}, [-\sqrt{12}]^{12}, [-12]^1\}$ | $\{12, 11, 6, 1; 1, 6, 11, 12\}$ | Hadamard graph | 1 | $\geqslant 79469$ | Section 3 |
| 50 | $\{[5]^1, [\sqrt{5}]^{20}, [0]^8, [-\sqrt{5}]^{20}, [-5]^1\}$ | $\{5, 4, 4, 1; 1, 1, 4, 5\}$ | $IG(AG(2, 5) \setminus \text{pc})$ | 1 | 1 | Proposition 7, [7] |
| 51 | $\{[16]^1, [4]^{17}, [-1]^{16}, [-4]^{17}\}$ | $\{16, 10, 1; 1, 5, 16\}$ | Mathon(Cycl(16,3)) | $\geqslant 1$ | $\geqslant 1$ | [17] |
| 52 | $\{[6]^1, [2+\sqrt{3}]^{12}, [2-\sqrt{3}]^{12}, [-2]^{27}\}$ | $\{6, 3, 3; 1, 1, 2\}$ | $GH(3, 1)$ | 1 | 1 | [13] |
| 52 | $\{[25]^1, [5]^{13}, [-1]^{25}, [-5]^{13}\}$ | $\{25, 12, 1; 1, 12, 25\}$ | Taylor($SRG(25, 12)$) | 4 | $\geqslant 249106$ | Section 3.3 |
| 54 | $\{[9]^1, [3]^{18}, [0]^{16}, [-3]^{18}, [-9]^1\}$ | $\{9, 8, 6, 1; 1, 3, 8, 9\}$ | 3-cover $K_{9,9}$ | 4 | $\geqslant 5$ | Section 3.4 |
| 56 | $\{[15]^1, [7]^7, [1]^{20}, [-3]^{28}\}$ | $\{15, 8, 3; 1, 4, 9\}$ | $J(8, 3)$ | 1 | $\geqslant 33525$ | Section 3 |
| 56 | $\{[27]^1, [9]^7, [-1]^{27}, [-3]^{21}\}$ | $\{27, 10, 1; 1, 10, 27\}$ | Gosset,Tayl(Schläfli) | 1 | $\geqslant 1$ | Section 3.3 |
| 56 | $\{[27]^1, [3]^{21}, [-1]^{27}, [-9]^7\}$ | $\{27, 16, 1; 1, 16, 27\}$ | Taylor(Co-Schläfli) | 1 | $\geqslant 174608$ | Section 3 |
| 57 | $\{[6]^1, [\frac{3}{2}+\frac{1}{2}\sqrt{5}]^{18}, [\frac{3}{2}-\frac{1}{2}\sqrt{5}]^{18}, [-3]^{20}\}$ | $\{6, 5, 2; 1, 1, 3\}$ | Perkel | 1 | 1 | [3,5] |
| 60 | $\{[11]^1, [\sqrt{11}]^{24}, [-1]^{11}, [-\sqrt{11}]^{24}\}$ | $\{11, 8, 1; 1, 2, 11\}$ | Mathon(Cycl(11,5)) | $\geqslant 1$ | $\geqslant 1$ | [17] |
| 60 | $\{[19]^1, [\sqrt{19}]^{20}, [-1]^{19}, [-\sqrt{19}]^{20}\}$ | $\{19, 12, 1; 1, 6, 19\}$ | Mathon(Cycl(19,3)) | $\geqslant 1$ | $\geqslant 1$ | [17] |
| 60 | $\{[29]^1, [\sqrt{29}]^{15}, [-1]^{29}, [-\sqrt{29}]^{15}\}$ | $\{29, 14, 1; 1, 14, 29\}$ | Taylor($SRG(29, 14)$) | 6 | $\geqslant 45875$ | Section 3.3 |
| 63 | $\{[6]^1, [3]^{21}, [-1]^{27}, [-3]^{14}\}$ | $\{6, 4, 4; 1, 1, 3\}$ | $GH(2, 2)$ | 2 | 2 | [13] |
| 63 | $\{[8]^1, [\sqrt{8}]^{27}, [-1]^8, [-\sqrt{8}]^{27}\}$ | $\{8, 6, 1; 1, 1, 8\}$ | 7-cover $K_9$, $PG(2, 8)$ | 1 | 1 | [13] |
| 63 | $\{[10]^1, [5]^{12}, [1]^{14}, [-2]^{30}, [-4]^6\}$ | $\{10, 6, 4, 1; 1, 2, 6, 10\}$ | Conway-Smith | 1 | $\geqslant 1$ | |
| 64 | $\{[7]^1, [3]^{21}, [-1]^{35}, [-5]^7\}$ | $\{7, 6, 5; 1, 2, 3\}$ | Folded 7-Cube | 1 | 1 | [16] |
| 64 | $\{[9]^1, [5]^9, [1]^{27}, [-3]^{27}\}$ | $\{9, 6, 3; 1, 2, 3\}$ | $H(3, 4)$, Doob | 2 | $\geqslant 2$ | Section 4 |
| 64 | $\{[15]^1, [3]^{30}, [-1]^{15}, [-5]^{18}\}$ | $\{15, 12, 1; 1, 4, 15\}$ | $SRG \setminus$ spread | 94 | $\geqslant 94$ | [10] |
| 64 | $\{[8]^1, [\sqrt{8}]^{24}, [0]^{14}, [-\sqrt{8}]^{24}, [-8]^1\}$ | $\{8, 7, 6, 1; 1, 2, 7, 8\}$ | 4-cover $K_{8,8}$ | $\geqslant 1$ | $\geqslant 2$ | Proposition 7 |
| 64 | $\{[21]^1, [9]^7, [1]^{21}, [-3]^{35}\}$ | $\{21, 10, 3; 1, 6, 15\}$ | Halved 7-Cube | 1 | $\geqslant 1$ | |

(*continued on next page*)

Table 1 (*continued*)

| $v$ | Spectrum | Intersection array | Name DRG | $dr_\Sigma$ | $gr_\Sigma$ | Ref. |
|---|---|---|---|---|---|---|
| 64 | $\{[6]^1, [4]^6, [2]^{15}, [0]^{20}, [-2]^{15}, [-4]^6, [-6]^1\}$ | $\{6, 5, 4, 3, 2, 1; 1, 2, 3, 4, 5, 6\}$ | $H(6, 2)$ | 1 | $\geqslant 2$ | Section 4 |
| 65 | $\{[10]^1, [5]^{13}, [0]^{26}, [-3]^{25}\}$ | $\{10, 6, 4; 1, 2, 5\}$ | Locally Petersen | 1 | $\geqslant 1$ | |
| 68 | $\{[12]^1, [4]^{17}, [0]^{34}, [-5]^{16}\}$ | $\{12, 10, 3; 1, 3, 8\}$ | Doro | 1 | $\geqslant 1$ | |
| 70 | $\{[16]^1, [8]^7, [2]^{20}, [-2]^{28}, [-4]^{14}\}$ | $\{16, 9, 4, 1; 1, 4, 9, 16\}$ | $J(8, 4)$ | 1 | $\geqslant 3$ | Section 3.1 |
| 70 | $\{[4]^1, [3]^6, [2]^{14}, [1]^{14}, [-1]^{14}, [-2]^{14}, [-3]^6, [-4]^1\}$ | $\{4, 3, 3, 2, 2, 1, 1; 1, 1, 2, 2, 3, 3, 4\}$ | Doubled Odd(4) | 1 | $\geqslant 3$ | Proposition 5 |

### 3.1. Johnson graphs and Grassmann graphs

In this section we shall construct cospectral graphs for the Johnson graphs and their $q$-analogues, the Grassmann graphs. We shall first use the above lemma to construct cospectral graphs for the Johnson graphs with diameter at least three. This will generalize the construction for the diameter three case given in [13]. Recall that the Johnson graph $J(n,d)$ is defined on the $d$-tuples of a set $X$ of size $n$, where two $d$-tuples are adjacent if they differ in precisely one element. It is known that the Johnson graph $J(n,d)$ is determined as a distance-regular graph from its intersection array unless $n=8$, $d=2$, in which case there are 3 other strongly regular graphs: the Chang graphs (cf. [2, p. 258]).

Now consider the Johnson graph $J(n,d)$, with $n-3 \geqslant d \geqslant 3$, with vertex set $V$. Fix a set $Y$ of 4 elements of $X$. Let $D$ be the set of $d$-tuples that do not contain precisely three elements of $Y$. For each $(d-3)$-tuple $T$ on $X \setminus Y$, let $C_T$ be the set of (four) $d$-tuples containing $T$ and precisely three elements of $Y$.

Now $\{C_T\}$ is a regular partition of $V \setminus D$, with quotient graph $J(n-4, d-3)$. Moreover, for each vertex $s$ in $D$, and each $T$, we have that if $s$ intersects $Y$ in at most one element, then $s$ has no neighbors in $C_T$; if $s$ intersects $Y$ in two elements, then $s$ has either two or zero neighbors in $C_T$ (depending on whether $s$ contains $T$ or not, respectively); and if $s$ intersects $Y$ in four elements, then $s$ has either zero or four neighbors in $C_T$.

We thus have a switching partition as required in Lemma 3, and by switching we obtain a graph that is cospectral with $J(n,d)$.

We now claim that this cospectral graph is not the Johnson graph $J(n,d)$, in fact, that it is not distance-regular. Indeed, let $x$ be a vertex in $V \setminus D$, and let $y$ be a vertex in $D$ with one point in $Y$, which is also in $x$, and containing all $d-3$ points of $x$ in $X \setminus Y$ (these exist since $n \geqslant d+3$). Then $x$ and $y$ are not adjacent, and they have precisely two common neighbors (namely the two vertices intersecting $x$ and $y$ in their common point of $Y$, containing the point of $Y$ which is not contained in $x$, containing the $d-3$ points of $x$ in $X \setminus Y$, and one of the two points of $y \setminus x$ in $X \setminus Y$). Since the Johnson graph has $c_2 = 4$ (not 2), the cospectral graph is not distance-regular. We may thus conclude the following.

**Proposition 1.** *The Johnson graph $J(n,d), n-3 \geqslant d \geqslant 3$ has cospectral graphs that are not distance-regular.*

Next, we shall construct cospectral graphs for both the Grassmann graphs and the Johnson graphs, by constructing suitable partial linear spaces. Recall that the Grassmann graph $J_q(n,d)$ is the graph on the $d$-dimensional subspaces of an $n$-dimensional vector space over the finite field $GF(q)$ of $q$ elements. It has $\left[{n \atop d}\right] = \frac{(q^n-1)\cdots(q^{n-d+1}-1)}{(q^d-1)\cdots(q-1)}$ vertices (cf. [2, pp. 268–269]), and can be thought of as the $q$-analogue of the Johnson graph. Consider now the incidence structure $I_q(n,d)$ whose points are the $(d-1)$-dimensional subspaces, and whose lines are the $d$-dimensional subspaces of an $n$-dimensional vector space over $GF(q)$, where incidence is symmetrized containment. Then this is a partial linear space whose point graph is the Grassmann graph $J_q(n,d-1)$ and whose line graph is the Grassmann graph $J_q(n,d)$. If $N$ is the point-line incidence matrix of this partial linear space, then $NN^T - \left[{n-d+1 \atop 1}\right] I$ and $N^TN - \left[{d \atop 1}\right] I$ are the adjacency matrices of the point graph and the line graph, respectively. Since $NN^T$ and $N^TN$ have the same nonzero eigenvalues, it follows that the spectra of the point graph and the line graph are related. Later on, in the next section, we shall also use the fact that the spectrum of the incidence graph is related. For now, we remark that if we can adjust the partial linear space such that the point graph remains the same, while keeping the same number of lines, and the same line sizes, then the new line graph is cospectral with the old one, the Grassmann graph $J_q(n,d)$. This is exactly how we proceed.

Let $d \geqslant 3$, and let $n \geqslant 2d-1$. Let $X$ be an $n$-dimensional vector space over $GF(q)$, and fix a set of $t$, say, $(2d-2)$-dimensional subspaces $H_i$, $i=1,\ldots,t$ of $X$ such that the intersection of any two such subspaces has dimension at most $d-1$. Consider now the incidence structure $C_q(n,d)$ whose points are the $(d-1)$-dimensional subspaces of $X$, and which has the following two kinds of lines. The first kind of lines consists of the pairs $(S,i)$, where $S$ is a $(d-2)$-dimensional subspace of $H_i$, and $i=1,\ldots,t$, and such a line is incident to all $(d-1)$-dimensional subspaces of $H_i$ that contain $S$. The second kind of lines consists of the $d$-dimensional subspaces not contained in any of the $H_i$, $i=1,\ldots t$, and such a line is incident to all $(d-1)$-dimensional subspaces that are contained in it. (In fact, we replaced the $d$-dimensional subspaces of $H_i$ by their dual subspaces within $H_i$.) It now follows easily that $C_q(n,d)$ is a partial linear space whose point graph is $J_q(n,d-1)$, and which has the same number of lines, and the same line sizes as the original partial linear space $I_q(n,d)$. Thus the line graph of this partial linear space is cospectral with the Grassmann graph $J_q(n,d)$.

This cospectral graph has the following more explicit description. Pairs $(S,i)$ and $(T,j)$, where $S$ is a $(d-2)$-dimensional subspace of $H_i$ and $T$ is a $(d-2)$-dimensional subspace of $H_j$ (lines of the first kind) are adjacent if $i=j$ and $S$ and $T$ span a $(d-1)$-dimensional subspace, or if $H_i \cap H_j$ is a $(d-1)$-dimensional subspace containing both $S$ and $T$; a pair $(S,i)$, where $S$ is a $(d-2)$-dimensional subspace of $H_i$ and a $d$-dimensional subspace $T$ not contained in any of the $H_j$, $j=1,\ldots,t$ are adjacent if $S$ is contained in $T$, and moreover, $T$ intersects $H_i$ in a $(d-1)$-dimensional subspace; two $d$-dimensional subspaces not contained in $H$ are adjacent if they intersect in a $(d-1)$-dimensional subspace.

Suppose now that $n \geqslant 2d$. Then many (if not all) of the constructed cospectral graphs are not distance-regular. For example, consider the cases where there is a $d$-dimensional subspace $W$ that intersects $H_1$ (say) in a $(d-2)$-dimensional subspace $U$ and that is not contained in any of the $H_i$, $i=1,\ldots,t$. Note that the case $t=1$ is such a case. It follows that $(U,1)$ and $W$ are not adjacent, and moreover that they have at least $\left[{d \atop 1}\right]\left[{2 \atop 1}\right]$ common neighbors. Since $J_q(n,d)$ has intersection parameter $c_2 = \left[{2 \atop 1}\right]^2$, it thus follows that in this case the line graph of $C_q(n,d)$ is a non-distance-regular graph cospectral with $J_q(n,d)$. The exceptional case $n=2d-1$ does give distance-regular graphs, in fact this is the new family of such graphs as described in [8]. This case is also relevant for the next section.

**Proposition 2.** *The Grassmann graph $J_q(n,d)$, $n-3 \geqslant d \geqslant 3$, $q$ a prime power, has cospectral graphs that are not distance-regular.*

A similar construction as the above for the Grassmann graphs is also possible for the Johnson graphs, by letting $q=1$, and replacing $m$-dimensional subspaces by $m$-tuples; while the $q$-ary binomial coefficients $\left[{n \atop d}\right]$ reduce to the usual binomial coefficients $\binom{n}{d}$. The cospectral graphs so obtained from the Johnson graphs $J(n,3)$ for $t=1$ are the same as the ones obtained by switching. However, we expect that the cospectral graphs for the Johnson graphs with diameter larger than three are different from the cospectral graphs obtained by switching. We checked that this is indeed the case for $J(8,4)$.

Finally we remark that one can show that there are 4 partial linear spaces with point graph $J(6,2)$ and line graph cospectral with $J(6,3)$. All four are of the form $C_1(6,3)$, and these provide 4 of the 9 graphs cospectral with $J(6,3)$. More easily, one finds two partial linear spaces for $J(8,4)$. By computer we classified all partial linear spaces of the forms $C_1(7,3)$ and $C_1(8,3)$. There are 14 of the first form, and 270 of the second. We checked that these give rise to 14 mutually non-isomorphic graphs cospectral with $J(7,3)$, and that these are among the 100 graphs obtained by switching; and to 270 mutually non-isomorphic graphs cospectral with $J(8,3)$ (but we did not bother to check whether these were already obtained by switching).

### 3.2. Doubled Odd graphs and Doubled Grassmann graphs

Also the incidence graph of the above mentioned partial linear space $I_q(2d-1,d)$ (the exceptional case) is distance-regular; it is the so-called Doubled Grassmann graph $DO_q(d)$. For $q=1$, it reduces to the Doubled Odd graph $DO(d)$. We claim now that the incidence graph of the partial linear space $C_q(2d-1,d)$ is cospectral with $DO_q(d)$. Indeed, this follows from the fact that the spectrum of the incidence graph can be derived from the spectra of the point graph and the line graph. Moreover, it is not distance-regular, since $c_3$ is not well-defined. Indeed, if (the point) $P$ is a $(d-1)$-dimensional subspace not contained in $H_1$, then $P$ and the line $P\cap H_1$ are not adjacent (incident), but clearly all neighbors (incident points) of the latter line are at distance 2 from $P$. Thus $c_3(P,P\cap H_1)$ equals the valency of the graph, which is therefore not distance-regular.

**Proposition 3.** *The Doubled Grassmann graph $DO_q(d)$ on the $(d-1)$-dimensional and $d$-dimensional subspaces of a $(2d-1)$-dimensional vector space over $GF(q)$ has a cospectral graph that is not distance-regular, for all $d \geqslant 3$, and all prime powers $q$.*

As one might expect, the same arguments hold for the Doubled Odd graphs (the case $q=1$). For $d=3$, the constructed incidence graph is the only graph cospectral with the Desargues graph $DO(3)$, as was determined by Bussemaker and Cvetković [4] using a computer, and independently by Schwenk [20]. This also follows from the fact that $J(5,2)$ is uniquely determined by its spectrum, and the following characterization.

**Proposition 4.** *The Doubled Odd graph $DO(d)$, $d \geqslant 3$ has one non-distance-regular cospectral graph that has (at least) one of the halved graphs equal to the Johnson graph $J(2d-1,d-1)$.*

**Proof.** Let $G$ be a non-distance-regular graph cospectral with the Doubled Odd graph $DO(d)$. Since $c_2=1$ in the Doubled Odd graph, it follows from Lemma 1 that $G$ also has $c_2=1$, and hence $k_2=d(d-1)$ (the same as in the Doubled Odd graph). Suppose now that the corresponding partial linear space of points and lines (indeed, $G$ is also bipartite) has point graph (this is a halved graph of $G$) equal to the Johnson graph $J(2d-1,d-1)$. We can thus identify the points with the $(d-1)$-tuples from a set $X$ of size $2d-1$, such that points intersecting in $d-2$ elements are collinear.

Consider then the $d$ lines through a fixed point $p$. After removing $p$ from these lines we obtain a partition into $(d-1)$-cliques of the local graph of $J(2d-1,d-1)$ with respect to $p$. Since this local graph is a lattice graph of sides $d-1$ and $d$ there are essentially only two ways to make such a partition. The first gives the lines

$$\{p\}\cup\big\{p\cup\{y\}\setminus\{x\}\mid x\in p\big\},\quad y\notin p$$

(i.e., such lines contain all $(d-1)$-tuples from a $d$-tuple $(p \cup \{y\})$), whereas the second gives the lines

$$\{p\} \cup \big\{p \cup \{y_0\} \setminus \{x\} \mid x \in p\big\} \quad \text{and} \quad \{p\} \cup \big\{p \cup \{y\} \setminus \{x\} \mid y \notin p \cup \{y_0\}\big\}, \quad x \in p,$$

for some $y_0 \notin p$. For the readability of this proof, let us call the lines in the first set and the first line in the second set odd, and let us call the others even. In the Doubled Odd partial linear space $I_1(2d-1, d-1)$ all lines are odd, so (since $G$ is not the Doubled Odd graph) there is a $(d-1)$-tuple $p$, and a $y_0 \notin p$ such that there is one odd line "$p \cup \{y_0\}$" and $d-1$ even lines through $p$ (the above second set of lines). Now let $x' \in p$, $y' \notin p \cup \{y_0\}$. Then the point $q = p \cup \{y'\} \setminus \{x'\}$ is on the even line $\{q\} \cup \{q \cup \{y\} \setminus \{y'\} \mid y \notin q \cup \{y_0\}\} = \{p\} \cup \{p \cup \{y\} \setminus \{x'\} \mid y \notin p \cup \{y_0\}\}$ through $p$ (which manifestly is also an even line through $q$) and it follows that also $q$ is on $d-1$ even lines, and one odd line. Now $q$ and $q \cup \{y_0\} \setminus \{y'\}$ must be on this odd line, which must be of the form $\{q\} \cup \{q \cup \{y_0\} \setminus \{x\} \mid x \in q\}$, and then also all even lines through $q$ are uniquely determined. By repeatedly applying this argument, it follows that all lines through $(d-1)$-tuples not containing $y_0$ are uniquely determined; each such point $p$ is on $d-1$ even lines, and on one odd line of the form $\{p\} \cup \{p \cup \{y_0\} \setminus \{x\} \mid x \in p\}$. Note that each such odd line contains exactly one $(d-1)$-tuple not containing $y_0$. Each $(d-1)$-tuple $p_0$ containing $y_0$ is on the odd lines through $p_0 \cup \{y\} \setminus \{y_0\}$, $y \notin p_0 \cup \{y_0\}$, and these $d$ lines are clearly distinct. This means that all lines are determined, and hence that the partial linear space $C_1(2d-1, d-1)$ and the above constructed graph cospectral with the Doubled Odd graph must be obtained. □

Another construction of a graph cospectral with the Doubled Odd graph is as follows. It has as a halved graph the graph cospectral with the Johnson graph $J(2d+1, d)$ obtained by switching in the previous section. Let $d \geqslant 3$. Consider the $d$-tuples of a set $X$ of size $2d+1$ as points, and the $(d+1)$-tuples as lines. As in the construction of a graph cospectral with the Johnson graph, fix a set $Y$ of four elements of $X$. Each line $L$ not intersecting $Y$ in precisely three elements is incident to the points contained (as $d$-tuples) in $L$. Each line $L$ intersecting $Y$ in precisely three elements (here we "switch") is incident to the $d-2$ points contained in $L$ and containing three elements of $Y$, and the three points $L \setminus Y \cup \{x, y\}$, $x \in L \cap Y$, $y \in Y \setminus L$. It is now straightforward to check that the point graph of this partial linear space is the graph that was obtained through switching in $J(2d+1, d)$ in the previous section, and hence is cospectral with it. Thus it follows that this incidence graph is cospectral with the Doubled Odd graph $DO(d+1)$. It is clearly different from the first construction, and from the Doubled Odd graph itself. We thus may conclude the following.

**Proposition 5.** *The Doubled Odd graph $DO(d)$ has cospectral graphs that are not distance-regular. For $d = 3$ there is exactly one such cospectral graph, while for $d \geqslant 4$ there are at least two.*

A further remark here is that the line graph of the above "switched" partial linear space is cospectral with $J(2d+1, d)$, but cannot be isomorphic to it, according to Proposition 4. For $d = 3$ we found (by computer) that this line graph is isomorphic to the point graph. We do not know what happens for $d > 3$.

### *3.3. Taylor graphs*

For many of the Taylor graphs we can find cospectral graphs by switching. A Taylor graph is a distance-regular antipodal double cover of a complete graph. The local graph of a Taylor graph is a strongly regular $(v, k, \lambda, \mu)$ graph with $k = 2\mu$, and conversely, each such strongly regular graph is the local graph of a Taylor graph. We therefore speak of the Taylor graph over a strongly regular graph (the local graph).

In a Taylor graph over a strongly regular graph with parameters $(v = 4\mu + 1,\ k = 2\mu,\ \lambda = \mu - 1,\ \mu)$, let $C_1$ be the set of $k$ common neighbors of two adjacent vertices (that is, the set of neighbors of a vertex in the local graph), and let $D$ be the set of remaining vertices. Since each vertex in $D$ is adjacent to either none, half ($\mu$), or all of the vertices in $C_1$, this gives a switching partition $\Pi = \{D, C_1\}$. After switching (Lemma 3), a vertex in $C_1$ and its original antipode have $\mu - 1$ common neighbors, which implies that the new graph is not distance-regular, except when $\mu = 1$, where the Taylor graph is the Icosahedron, the Taylor graph over the 5-cycle. Indeed, this graph is determined by its spectrum. Among the examples for which we do get cospectral graphs are the Taylor graphs over the Paley graphs $P(4\mu + 1)$, $\mu > 1$. By computer we searched for switching partitions in the Taylor graphs over the strongly regular graphs $SRG(25, 12)$, and in the cospectral graphs so obtained (and repeating this), to obtain a total of 249,106 graphs cospectral with these Taylor graphs. For the Taylor graphs over $SRG(29, 14)$ we thus obtained 45,875 cospectral graphs.

Another construction of graphs cospectral with Taylor graphs needs the presence of cliques meeting the "Delsarte bound." It was given essentially already in [13]. One can show that if $\theta_1$ is the second largest eigenvalue of the Taylor graph, then a clique can have size at most $\theta_1 + 1$ (this corresponds to a so-called Delsarte clique in the local strongly regular graph). If $C_1$ is such a large clique, and $D$ is the set of remaining vertices, then also here this is a switching partition, and switching gives a graph that is not distance-regular. Some examples of Taylor graphs for which we can thus construct cospectral graphs are the Taylor graphs over the symplectic graphs, the orthogonal graphs, and the unitary (also called Hermitian) graphs (cf. [19]). Small examples are given by the Taylor graphs over $GQ(2, 2)$, $T(6)$, and the complement of the Schläfli graph. The Schläfli graph itself however does not have Delsarte cliques, so the method does not work for the corresponding Gosset graph, as was mistakenly reported in [13].

**Proposition 6.** *A Taylor graph over a strongly regular graph with parameters* $(v = 4\mu + 1,\ k = 2\mu,\ \lambda = \mu - 1,\ \mu)$, $\mu > 1$ *has cospectral graphs that are not distance-regular. A Taylor graph with second largest eigenvalue* $\theta_1$ *containing a clique of size* $\theta_1 + 1$ *also has cospectral graphs that are not distance-regular.*

### 3.4. Antipodal covers of complete bipartite graphs

There is a correspondence between bipartite regular graphs with five eigenvalues (indeed, distance-regular antipodal covers of complete bipartite graphs are such graphs) and so-called partial geometric designs (cf. [9]). Examples of the latter are transversal designs, and these form the key to the construction of graphs cospectral with distance-regular antipodal covers of complete bipartite graphs. The incidence structure between the two biparts of such a cover is a (square) resolvable transversal design (this is also called a symmetric net). A transversal design is a design of points and blocks, such that all blocks have the same size, each point is in the same number of blocks, and such that the points can be partitioned into groups, such that each block intersects each group in one point, and such that two points from different groups meet in a constant number, say $\mu$, of points. The bipartite incidence graph of such a transversal design is easily checked to be distance-regular with respect to all points (with $c_2 = \mu$). The transversal design is called resolvable if the blocks can be partitioned into parallel classes of points. If the design is square, this is equivalent to the property that the dual design is transversal, and hence that the incidence graph is also distance-regular with respect to the blocks.

Now let $G$ be an antipodal distance-regular $r$-cover of a complete bipartite graph $K_{n,n}$. The corresponding resolvable transversal design thus has $\mu = \frac{n}{r}$. Fix one of the fibres $F$ of the cover $G$; this fibre corresponds to one of the groups of points of the transversal design (without loss of generality). Delete all edges on vertices in $F$, and instead, connect each vertex in $F$ to all vertices in $\mu$ of the opposite fibres (of blocks), such that all such fibres (all blocks) are adjacent to one of the vertices in $F$. It is easily checked that the new incidence structure is still a transversal design, and thus that the new graph is cospectral with $G$. However, any two vertices $b_1$ and $b_2$ from the same block-fibre have $c_2(b_1, b_2) = 1$, since these blocks intersect only in the point in $F$. So unless $\mu = c_2 = 1$, the new graph is not distance-regular. Indeed, if $\mu = c_2 = 1$, then distance-regularity is determined by the spectrum: we have the incidence graph of an affine plane minus a parallel class of lines.

**Proposition 7.** *A distance-regular antipodal $r$-cover of $K_{n,n}$ has cospectral graphs that are not distance-regular, unless $n = r$, in which case any cospectral graph is the distance-regular incidence graph of an affine plane minus a parallel class of lines.*

By using the correspondence between bipartite graphs with five eigenvalues and partial geometric designs we were able to compute all graphs cospectral with the unique distance-regular antipodal 3-cover of $K_{6,6}$ (cf. [2, p. 399]). It turned out that there are 40 graphs of which 8 correspond to transversal designs. In a similar way all 327 graphs cospectral with the antipodal 2-cover of $K_{8,8}$ (a Hadamard graph) were already determined in [9]. For completeness we remark that Mavron and Tonchev [18] determined all (four) distance-regular 3-covers of $K_{9,9}$.

## 4. Hamming graphs: A challenge for the reader

A construction of graphs cospectral with the Hamming graph $H(q,q)$ is obtained by considering its cliques of size $q$ (lines). By dualizing, i.e., by taking the $q$-cliques as vertices, which are adjacent if they intersect, we obtain a graph cospectral with $H(q,q)$. This dual graph is not distance-regular for $q > 2$ (cf. [14]). Moreover, by taking the product with $H(n-q,q)$, we obtain a graph cospectral with $H(n,q)$, which hence is not determined by the spectrum if $n \geqslant q \geqslant 3$.

For $q = 2$, it is known that $H(4,2)$ has one cospectral graph, the Hoffman graph [15]. Thus $H(n,2)$, $n \geqslant 4$ is not determined by the spectrum either. However, each of $H(2,2)$ and $H(3,2)$ is determined by its spectrum.

Note that $H(2,q)$ (a strongly regular graph) is determined by the spectrum except when $q = 4$, where there is also the (strongly regular) Shrikhande graph. This implies that $H(n,4)$, $n \geqslant 2$ has cospectral (but distance-regular) graphs, the so-called Doob graphs.

What remains is the following challenging question: Does $H(n,q)$, $q > n \geqslant 3$ have cospectral graphs that are not distance-regular? The smallest of these graphs is $H(3,4)$, which we expect to have cospectral graphs that are not distance-regular. However, we doubt that these can be constructed by the usual switching methods, since this also seems to be impossible in the case of $H(3,3)$. This graph has three cospectral non-distance-regular graphs, none of which could be obtained through switching in $H(3,3)$ or any of the other ones, by using Lemma 3 with $m = 1$ and $C_1$ of size 4 or 6; or $m = 2$ and $C_1$ and $C_2$ of size 4. An attempt to construct a graph cospectral with $H(3,4)$ via the dual graph (on the 48 4-cliques as described above) was not successful. Indeed, by computer we determined that no graph cospectral with this dual (besides the dual itself) exists which has constant $\lambda = a_1 = 1$ (which is necessary to dualize back, and obtain a graph cospectral with $H(3,4)$).

One final observation here is that the matrix

$$\begin{bmatrix} 0 & 3(q-1) & 0 & 0 & 0 \\ 1 & q-2 & q-1 & q-1 & 0 \\ 0 & 3 & \frac{3}{2}(q-3) & \frac{3}{2}(q-1) & 0 \\ 0 & 1 & \frac{1}{2}(q-1) & \frac{1}{2}(q+1) & 2(q-2) \\ 0 & 0 & 0 & 6 & 3(q-3) \end{bmatrix},$$

has eigenvalues $3q-3, 2q-3, q-3, -3$, and $-3$, i.e., the eigenvalues of $H(3,q)$. For $q = 3$ the dual of $H(3,3)$ has a partition with above quotient matrix (the sizes of the parts in the partition are $1, 3(q-1), (q-1)^2, 3(q-1)^2$, and $(q-2)(q-1)^2$). Perhaps also for other (odd) $q$ there are graphs cospectral with $H(3,q)$ with such a partition.

## 5. Small distance-regular graphs

In Table 1 we list all feasible (i.e., satisfying all known conditions) intersection arrays for distance-regular graphs on at most 70 vertices, except for the complete graphs, strongly regular graphs, incidence graphs of symmetric designs, and the polygons. (For these exceptions distance-regularity follows from the spectrum.) For each listed intersection array we also list the corresponding spectrum, the number $dr_\Sigma$ of corresponding distance-regular graphs, and the number $gr_\Sigma$ of graphs with the corresponding spectrum. The list of intersection arrays is obtained by inspection of the lists of three-class association schemes in [6], and the lists of distance-regular graphs in [2] and its corrections.

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