Minimum resolvable coverings with small parallel classes

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Abstract

For prime power q, we determine the minimum number of parallel classes in a resolvable 2-$(kq, k, 1)$ covering for almost all values of k.

In September 1995 Jan van Haastrecht gave a dinner on the occasion of his retirement. He had invited twenty colleagues. The host and the guests were to be seated at three tables each of which had seven places. The diner consisted of five courses. Jan wanted the guests to change places between the courses in such a way that everyone would meet everyone else (Jan included) at least once at some table. At that time no solution was known and so a few couples didn't meet.

A t-(v, k, λ) covering consists of a v-set V (of points) together with a collection of k-subsets of V (called blocks) such that every t-subset of V is a subset of at least λ blocks. A covering is called resolvable if the collection of blocks admits a partitioning into classes (called parallel classes), such that each class consists of v/k disjoint blocks. Thus the dinner problem just described can be rephrased as: does there exist a resolvable 2-$(21, 7, 1)$ covering with five parallel classes? We speak of a minimum resolvable t-(v, k, λ) covering if there exists no resolvable t-(v, k, λ) covering with fewer parallel classes. In this note we consider minimum resolvable 2-$(v, k, 1)$ coverings with a given cardinality $q = v/k$ of a parallel class. We shall show that if q is the order of an affine plane, the number of parallel classes in such a covering equals $q + 1$ if q divides k and $q + 2$ otherwise with just a small number of exceptions. In particular for $q \leq 3$ there are no exceptions, meaning that Van Haastrecht's problem has a solution.

Define $r(q,k)$ to be the number of parallel classes in a minimum resolvable 2-$(kq,k,1)$ covering. It is well-known (see [1]) that $r(q,k) \geq q+(q-1)/(k-1)$ with equality if and only if there exists a resolvable 2-$(kq,k,1)$ design. In particular $r(q,2) = 2q-1$ for all q and $r(q,q) = q+1$ if q is a prime power (or, more precisely, the order of an affine plane).

Lemma 1 *If m is a positive integer then $r(q,km) \leq r(q,k)$.*

Proof. Take a minimum resolvable 2-$(kq,k,1)$ covering and replace each of the kq points by an m-set. This gives a resolvable covering with block size km and the same number of classes as before. □

So $r(q,qm) = q+1$ if there exists an affine plane of order q. The next result characterizes the case $r(q,k) = q+1$.

Theorem 1 *The number of parallel classes in a minimum resolvable* 2-$(kq,k,1)$ *covering is at least $q+1$. Equality holds if and only if q divides k and q is the order of an affine plane.*

Proof. Put $r = r(q,k)$ and let x_i be the number of pairs that are covered exactly i times. Then clearly

$$x_0 = 0,\ \sum_{i=1}^{r} x_i = \binom{kq}{2},\ \sum_{i=1}^{r} i x_i = qr\binom{k}{2}.$$

Let x be the number of times we can choose an unordered pair of points and an unordered pair of blocks such that both points are in both blocks. On the one hand

$$x = \sum_{i=1}^{r} \binom{i}{2} x_i.$$

To find a second expression for x, define $\sigma_{b,b'}$ to be the number of points in the intersection of a block b and a block b'. Then

$$x = \frac{1}{2}\sum_{b \neq b'} \binom{\sigma_{b,b'}}{2}.$$

Fix a parallel class C and a block $b' \notin C$. It is straightforward that

$$\sum_{b \in C} \binom{\sigma_{b,b'}}{2} \geq q \binom{k/q}{2}$$

with equality if and only if $\sigma_{b,b'} = k/q$ for all $b \in C$. Hence

$$x \geq \frac{r(qr-q)}{2} \cdot q \binom{k/q}{2} = \frac{kr(r-1)(k-q)}{4}.$$

Combining our formulas for x_i we obtain

$$0 \geq \sum_{i=1}^{r} (i-1)(i-r)x_i = \sum_{i=1}^{r} (2\binom{i}{2} x_i - rix_i + rx_i) \geq$$

$$\frac{kr(r-1)(k-q)}{2} - qr^2 \binom{k}{2} + r \binom{kq}{2} = \frac{k^2 r(q-1)(q+1-r)}{2}.$$

Therefore $r \geq q+1$. Suppose $r = q+1$. Then we have equality everywhere, which yields that $x_i = 0$ if $2 \leq i \leq r-1$ and that any two blocks from different parallel classes meet in exactly k/q points. So the covering is just an affine plane wherein each point is replaced by a set of size k/q (as in Lemma 1). □

Theorem 2 *Let q be the order of an affine plane and let k be a positive integer such that*

$$\left\lceil \frac{k}{q} \right\rceil \leq \frac{2k}{2q-1}. \tag{1}$$

Then there exists a resolvable 2-$(kq, k, 1)$ covering with $q+2$ parallel classes.

Proof. Define $m = \lceil k/q \rceil$ and $n = mq - k$, then $0 \leq n < q$ and (1) becomes $m \geq 2n$. Take a parallel class C of an affine plane of order q and fix n lines in C. Replace (similar to the construction of Lemma 1) each point of these n lines by an $(m-1)$-set and all the other points by an m-set. Then we obtain a resolvable covering of kq points with $q+1$ classes and all blocks, not coming from a line of C have k points. However, from the blocks corresponding to C, n have $q(m-1)$ points (the small blocks) and $q-n$ have qm points (the large blocks). From this 'unbalanced' covering we will make one with all block sizes equal by replacing the parallel class corresponding to C by two new parallel classes with all block sizes equal to k in the following way. For each large block, we fix a set S of m points corresponding to one point of the original affine plane and, in addition, we take two disjoint subsets S_1 and S_2 of S, both of size n (we can do so because $m \geq 2n$). For $i = 1$ and $i = 2$ we make a new parallel class by moving the points of S_i from each large block to the small blocks such that all block sizes become k. It is easy to check that this indeed produces gives a resolvable 2-$(kq, k, 1)$ covering with $q + 2$ parallel classes. □

For a given prime power q, Theorem 1 and 2 give the value of $r(q, k)$ for almost all k. Indeed, for the inequality (1) to hold it suffices that $k \geq 2(q-1)^2$. In particular, for $q = 2$ there are no exceptions. Suppose $q = 3$. Only for $k = 2$, 4 and 7 condition (1) is not satisfied. We know $r(3, 2) = 5$ and, by Lemma 1 and Theorem 1, $r(3, 4) = 5$. To make a covering with 5 classes if $k = 7$ (the dinner problem of the introduction) we first give one for $k = 5$ (different from the one given by Theorem 2) in the table below.

1 2 3 A D	1 4 7 A B	1 5 9 A E	1 6 8 A F	1 4 5 A C
4 5 6 B F	2 5 8 C D	2 6 7 C F	2 4 9 D E	2 8 9 B E
7 8 9 C E	3 6 9 E F	3 4 8 B D	3 5 7 B C	3 6 7 D F

The point set is $\{1, \ldots, 9\} \cup \{A, \ldots, F\}$ and the rows within each box represent the three blocks of a parallel class. The letters form a resolvable 2-$(6, 2, 1)$ covering (a 1-factorization of the complete graph on six vertices) and the digits in the first four classes represent the affine plane of order three. Given these ingredients, it is an easy exercise to complete the covering by hand. Next, by letting each letter represent a pair of points, we obtain the desired covering with $k = 7$. In fact, replacing the digits by some m-set, and the letters by some m'-set, provides coverings with five classes and $q = 3$ for any k.

If $q = 4$ then (1) is satisfied for $k \notin \{2, 3, 5, 6, 9, 10, 13, 17\}$. We know $r(4, 2) = 7$ and Lampken and Mills [2] showed that $r(4, 3) = 7$. For the remaining six values of k we don't know the value of $r(4, k)$. However, it can be seen that $r(4, k) \leq 7$. For $k = 6$, 9 and 10 this follows from Lemma 1 and for $k = 5$, 13 and 17 it is a straightforward exercise.

If $q = 5$ we have $r(5, 2) = 9$, $r(5, 3) = 7$ (Kirkman's schoolgirls problem), $r(5, 4) = 7$ (see [1] or [3]) and so, by Lemma 1, $r(5, k) \leq 7$ if k is divisible by 3 or 4. Now Theorem 1 and 2 only leave $k \in \{7, 11, 13, 17, 22, 26, 31\}$ unsolved.

If q is not a prime power, little is known. An easy recursive construction gives $r(q_1q_2, k) \leq r(q_1, k)r(q_2, k)$, but this bound is rather rough. For example if $q = 6$ we get $r(6, k) \leq 12$. But it is known that $r(6, 2) = 11$, $r(6, 3) = 9$ (see [1] or [2]) and that $r(6, 4) = 8$ (see [1] or [3]). So, by Lemma 1, we can easily do better for most values of k. Moreover, Theorem 1 gives $r(6, k) \geq 8$ for all k, hence $r(6, k) = 8$ if k is divisible by 4. It seems more likely that $r(q, k) = q + 2$ if q is not the order of an affine plane, provided that k is big enough (and maybe $k \geq q$ suffices).

Acknowledgement. The author wants to thank Jan van Haastrecht for the posed problem and for the invitation to an excellent dinner.

References

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