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Saddlepath Learning*

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ABSTRACT

Saddlepath learning occurs when agents know the form but not the coefficients of the saddlepath relationship defining rational expectations equilibrium. Under saddlepath learning, we obtain a completely general relationship between determinacy and e-stability, and generalise Minimum State Variable results previously derived only under full information. When the system is determinate, we show that a learning process based on the saddlepath is always e-stable. When the system is indeterminate, we find there is a unique MSV solution that is iteratively e-stable. However, in this case there is a sunspot solution that is learnable as well. We conclude by demonstrating that our results hold for any information set.

JEL Classification: C60, E00

Key Words: e-stability, determinacy, learning, saddlepath stability.

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1 Introduction

The saddlepath stability characterisation of rational expectations developed by Sargent and Wallace (1973) represented a major breakthrough in economic theory in the way it encapsulated the discipline rational expectations impose on the equilibrium dynamics of an economic model. In short, the saddle path describes the combinations of predetermined and non-predetermined variables that imply expectations consistent with the equilibrium dynamics of a model. The famous paper by Blanchard and Kahn (1980) then shows how the saddlepath representation can be obtained as a simple eigenvalue-eigenvector decomposition of the state space representation of the linearised model. The beauty of this is that the saddle path describes the evolution of rational expectations as an outcome of the model, rather than expectations having their own free parameters.

The logical cohesion of rational expectations represents the gold standard against which researchers interested in learning and bounded rationality should measure their efforts. It is a formidable benchmark as departures from fully rational expectations inevitably lead into the 'wilderness' of Sargent (1993) where it is not obvious how discipline should be imposed on the beliefs of agents. The problem is that agents can depart from rationality in an infinite number of ways. Attempts have been made to impose discipline by requiring departures from rationality to be small in a loosely-defined sense. For example, much of the adaptive learning literature follows Marcet and Sargent (1989) and Evans and Honkapohja (2000) by requiring that departures from rationality should be small enough so that in the long run the economy converges to rational expectations equilibrium.

In this paper we are inspired by the rational expectations literature to use the saddlepath relationship to impose discipline on boundedly rational learning. Our idea is that agents are rational in the sense of being able to correctly identify the form of the saddle path, but depart from full rationality by not knowing the precise coefficients of the saddle path. To give a concrete example, if agents are saddlepath learning in the Ramsey model then they know they need to learn the coefficients of the saddlepath relationship determining consumption as a function of the capital stock. In the terminology of Evans and Honkapohja (2000), saddlepath learning equates to assuming that agents learn adaptively using the saddlepath as their perceived law of motion for the economy.

Saddlepath learning is a refinement of the minimum state variable (MSV) approach developed in a series of papers by McCallum (1983, 1998, 1999), which disciplines bounded rationality by requiring agents to use the correct set of variables as the perceived law of motion of the economy. The additional requirement we impose is that agents are correctly able to decompose endogenous variables into states that are predetermined and states that are non-predetermined. Our refinement offers two main advantages over the MSV approach. Firstly, it is closer to rational expectations because from the outset it correctly imposes zeros on some of the coefficients of the standard learning regression. In the MSV approach agents must learn those zeros over time. Secondly, it has greater mathematical tractability that gives additional insight into existing results and enables us to derive new theoretical results.

The tractability of saddlepath learning allows us to derive new results on the relationship between e-stability and determinacy in linear rational expectations models when the current values of endogenous variables are in the information set of agents. The biggest news is that we are able to provide sufficient conditions for e-stability of saddlepath learning solutions when the rational expectations equilibrium is indeterminate. We establish that indeterminacy implies existence of a unique iteratively e-stable saddlepath learning solution. This result improves upon McCallum (2007), who was unable to obtain general results for the relationship between e-stability and indeterminacy with the MSV solution. For the case when the rational expectations equilibrium is determinate, we find it is much simpler to prove that determinacy implies e-stability under saddlepath than MSV learning. The tractability of the saddlepath learning framework also means we can extend this result and show that determinacy implies that the saddlepath learning process is iteratively e-stable and unique.

The second contribution of the paper uses saddlepath learning as a way to frame the recent debate on whether information delays, or more generally imperfect information, can overturn the result that determinacy implies e-stability under MSV learning. An important paper in this literature is Bullard and Eusepi (2009), who ask what economic assumptions drive the differences in necessary and sufficient conditions for determinacy and e-stability. They conclude that informational delays break equivalency connections. Such a result was anticipated by McCallum (2007):

> "It should be stated clearly at the outset that all results presented here are based on the assumption that current values of endogenous variables are included in individuals' information sets; if instead only lagged endogenous variables can be observed in the learning process then different E-stability and learnability results would be relevant. Analysis of a few particular problems in monetary economics involving the latter specification has been conducted in a well-known paper by Bullard and Mitra (2002) while recent papers by Adam (2003) and Adam et al (2006) have emphasized that differing assumptions about information sets relevant for learning can lead to different conclusions."

We argue that the emphasis of these papers on informational delays and information sets is not sufficiently nuanced. What matters most is not the information set *per se* but the way that agents use their information set to learn. In McCallum (2007) agents have information on the current values of endogenous variables and learn using a perceived law of motion that nests the saddlepath relationship. In Bullard and Eusepi (2009) agents only have information on the lagged values of endogenous variables, and learn with a law of motion that does not nest the saddlepath. In addition to the lagging of the information, we therefore see that Bullard and Eusepi (2009) change from saddlepath to non-saddlepath learning. This is crucial to their finding that informational delays break the equivalency between determinacy and e-stability. We show this by proving that equivalence returns once we assume that agents learn a saddlepath relationship under lagged information. Our result is therefore that determinacy implies e-stability for a wider class of models than previously suggested in the literature. It is only if agents are not learning the saddlepath that determinacy and e-stability are disconnected.

The paper is organised as follows: In Section 2 we define saddlepath learning in full information models where agents know the current values of endogenous variables. We derive propositions showing the equivalence of determinacy and e-stability conditions, and provide sufficient conditions for e-stability under indeterminacy. Section 3 shows how these results can be extended to models with lagged information under appropriate assumptions. The most general case of saddlepath learning under imperfect information is presented in Section 4. A final Section 5 concludes.

2 Learning with full information

In full information models the current values of endogenous variables are in the information set of agents. We work with linearised versions of these models that can be written in the Blanchard and Kahn (1980) form:

$$\begin{pmatrix} z_t \\ E_t x_{t+1} \end{pmatrix} = \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} z_{t-1} \\ x_t \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \varepsilon_t, \tag{1}$$

where E_t denotes expectations formed using information available at time t, z_t is a vector of predetermined variables, x_t is a vector of non-predetermined variables and ε_t is a vector of i.i.d. disturbances. The matrices $G_{11}, G_{12}, G_{21}, G_{22}$ are conformable with the dimensions of z_t and x_t. McCallum (2007) uses a slightly different formulation, but in Appendix A we show that it has an equivalent Blanchard and Kahn (1980) form so there is no loss of generality in our results.

2.1 Solution under rational expectations

In rational expectation equilibrium, agents form rational expectations from the current values of predetermined variables by using the saddlepath relationship:

$$E_t x_{t+1} = -N z_t. \tag{2}$$

The matrix N satisfies:

$$\begin{pmatrix} N & I \end{pmatrix} \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} = \Lambda \begin{pmatrix} N & I \end{pmatrix}, \tag{3}$$

from the eigenvalue-eigenvector decomposition of the $\mathbf{G}$ matrix, with Λ containing the unstable eigenvalues if the system is determinate. The matrix $G_{11} - G_{12}N$ represents the reduced-form dynamics and contains the stable eigenvalues if the system is determinate.

2.2 Saddlepath learning

In a learning setting, we move away from rational expectations and require agents to form expectations on the basis of past experience. There are many ways to do this, so it is important that deviations from rational expectations are small in the sense of not leading to large expectations errors. Throughout this paper, we impose the required discipline by assuming that agents know the form of the saddlepath relationship (2) but not the individual coefficients in the matrix N. The idea is that agents are rational enough to identify the form of the saddlepath relationship, but still have to learn its coefficients. All we require is that agents are able to decompose variables into those that are predetermined and those that are non-predetermined. Saddlepath learning is then equivalent to adaptive learning using the saddle path as the perceived law of motion of the economy.

The issue of learning under full information has only been pursued by a small number of authors, e.g. McCallum (2007), Giannitsarou (2006) and Gaspar et al (2006). The key question is how agents solve their joint estimation and optimisation problem as information is revealed within the current period. For example, an agent deciding consumption needs to know how to react to newly-released retail surveys or manufacturing output data. Gaspar et al (2006) suggest a way for agents to solve the estimation problem. At the beginning of period t, agents estimate the saddlepath relationship:

$$\tilde{E}_t x_{t+1} = -N_{t-1} z_t, \tag{4}$$

using information up to and including period $t-1$. In other words, $-N_{t-1}$ is the ordinary least-squares estimate of the linear regression of $\{x_j\}$ on $\{z_{j-1}\}$ for $j = 2, \ldots, t-1$. Then, during period t, agents use the estimated saddlepath relationship (4) to adjust expectations as information arrives about the state of the economy. Under this timing protocol, agents update their model at the beginning of each period and then apply their estimated model within the period. Gaspar et al (2006) motivate this by arguing that "it takes more time to re-estimate a forecasting model than to apply an existing model". We agree and think that restricting learning to the beginning of each period is a natural way of solving the estimation problem, without imposing the high computational costs that re-estimating the model at each point during the period would entail.

How agents solve their decision problem is not explicitly addressed by the existing papers on learning under full information. Rather, they implicitly assume that agents somehow make decisions so that endogenous variables and expectations are consistent with the structural form (1) by the end of the period.[1] One possibility is that agents adjust their decisions during the period as new information is revealed. Coupled with the Gaspar et al (2006) assumption that agents form expectations using the estimated saddlepath (4), this means that expectations $\tilde{E}_t x_t, \tilde{E}_t z_t, \tilde{E}_t x_{t+1}$ and endogenous variables x_t, z_t all adjust within the period. Crucially, by the end of the period everything has adjusted so

[1] This is of course equally true in rational expectations models with full information, which typically do not specify the mechanism by which decisions and rational expectations become consistent with the model at each period.

that endogenous variables and expectations are consistent with (i) structural form (1), (ii) estimated saddlepath (4) and (iii) $\tilde{E}_t z_t$ being an unbiased predictor of z_t. In full information models (iii) holds trivially because agents directly observe the current values of aggregate variables.

2.3 Determinate models

The law of motion for non-predetermined variables under learning is obtained by using the estimated saddlepath relationship (4) to substitute out for expectations in the structural form (1) and obtain:

$$x_t = -(G_{22} + N_{t-1}G_{12})^{-1}[(G_{21} + N_{t-1}G_{11})z_{t-1} + (H_2 + N_{t-1}H_1)\varepsilon_t]. \tag{5}$$

Using techniques from stochastic approximation theory, Evans and Honkapohja (2000) show that the coefficient estimates N_t converge to their rational expectations values N under least-squares learning if e-stability holds for the following differential equation:

$$\frac{dN}{dk} = (G_{22} + NG_{12})^{-1}(G_{21} + NG_{11}) - N, \tag{6}$$

where k is 'notional' time. Local e-stability requires the eigenvalues of the Jacobian of (6) to have negative real parts. The Jacobian is defined by examining deviations $\Delta vec(N)$ around the steady-state value of N:

$$\frac{d\Delta vec(N)}{dk} = \begin{pmatrix} (G_{22} + NG_{12})^{-1} \otimes G_{11}^T \\ -(G_{22} + NG_{12})^{-1} \otimes (G_{21} + NG_{11})^T (G_{22} + NG_{12})^{-T} G_{12}^T \\ -I \otimes I \end{pmatrix} \Delta vec(N).$$

In rational expectations equilibrium, the eigenvalue-eigenvector decomposition (3) implies that $(G_{22}+NG_{12})^{-1}(G_{21} + NG_{11}) = N$ and $(G_{22} + NG_{12})^{-1} = \Lambda^{-1}$, so the Jacobian simplifies to:

$$\frac{d\Delta vec(N)}{dk} = \left(\Lambda^{-1} \otimes (G_{11} - G_{12}N)^T - I \otimes I\right) \Delta vec(N). \tag{7}$$

It follows that:

Proposition 1 *If the Blanchard and Kahn (1980) conditions for determinacy are satisfied then the saddlepath learning process for N is e-stable.*

Proof. Determinacy requires that the number of stable eigenvalues of the $\mathbf{G}$ matrix in (1) to be equal to the number of predetermined variables. If this is the case then the eigenvalues of $\Lambda^{-1}\otimes(G_{11}-G_{12}N)^T - I\otimes I$ are of the form $\lambda_i/\lambda_j - 1$, where λ_i is a stable eigenvalue from $G_{11} - G_{12}N$ and λ_j is an unstable eigenvalue from Λ. The real parts of λ_i/λ_j then satisfy $\text{real}(\lambda_i/\lambda_j) \leq |\lambda_i/\lambda_j| < 1$ and the eigenvalues of the Jacobian have negative real parts. The process for learning N is e-stable.

Proposition 1 does not claim that the process for learning N is unique. The following Corollary provides sufficient conditions for uniqueness:

Corollary. If all the eigenvalues of Λ and $G_{11} - G_{12}N$ are real and positive then the rational expectations value of N is uniquely e-stable.

Proof: Any other N would be associated with a switch of stable and unstable eigenvalues between Λ and $G_{11} - G_{12}N$, in which case some values of λ_i/λ_j would be greater than one and not all eigenvalues of the Jacobian would have negative real parts.

The uniqueness corollary does not necessarily hold if eigenvalues are complex or if the eigenvalues have a mixture of positive and negative real parts, although convergence to an N not associated with the saddlepath would imply system instability and it is not clear that stochastic approximation theorems would apply in this case. To obtain unambiguous results on uniqueness we therefore turn to iterative e-stability. We think of iterative e-stability as freezing the estimated coefficients for a long period of time, and only re-estimating them at the beginning of the next period. If we do this many times we obtain discrete-time learning equations for N_k and $\Delta vec(N_k)$:

$$N_{k+1} = (G_{22} + N_k G_{12})^{-1}(G_{21} + N_k G_{11}), \tag{8}$$

$$\Delta vec(N_{k+1}) = \left(\Lambda^{-1} \otimes (G_{11} - G_{12}N)^T\right) \Delta vec(N_k). \tag{9}$$

This assumes there is convergence to N_k after a certain number of time periods, and when least squares learning is started afresh there is convergence to N_{k+1}, and so on. The question is whether this sequence converges.

Proposition 2 *If the Blanchard and Kahn (1980) conditions for determinacy are satisfied then the saddlepath learning process for N is iteratively e-stable and unique.*

Proof. The ordinary difference equation (9) converges provided the eigenvalues of $\Lambda^{-1} \otimes (G_{11} - G_{12}N)^T$ each have modulus less than unity. Since the system is determinate, the eigenvalues are of the form λ_i/λ_j, where λ_i is a stable eigenvalue from $G_{11} - G_{12}N$ and λ_j is an unstable eigenvalue from Λ. The eigenvalues λ_i/λ_j then satisfy $|\lambda_i/\lambda_j| < 1$ and the process for learning N is iteratively e-stable. Uniqueness follows because any other N would be associated with a switch of stable and unstable eigenvalues between Λ and $G_{11} - G_{12}N$, in which case some values of $|\lambda_i/\lambda_j|$ would be greater than one.

The proofs of Propositions 1 and 2 are intuitive under saddlepath learning. As agents can correctly differentiate between predetermined and non-predetermined variables, there is a natural correspondence between the eigenvalue-eigenvector decomposition for rational expectations equilibrium (3) and the Jacobian of the differential equation for learning (6). This correspondence is less apparent in the general set-up of McCallum (2007). The advantages of saddlepath learning are then two-fold. Firstly, saddlepath learning brings agents closer to rational expectations. In McCallum (2007), agents are less rational because they have to learn that lagged values of forward-looking variables are not useful for forecasting. Agents therefore face a redundant regressor problem, so expectations errors are larger and convergence to rational expectations is slower.[2] Secondly, saddlepath learning has greater transparency and is more amenable to mathematical analysis. In contrast, McCallum (2007) needs singular value decomposition to obtain a counterpart to Proposition 1 in the general set-up .

2.4 Indeterminate models

If the equilibrium of the model is indeterminate then the rational expectations solution has more stable eigenvalues than predetermined variables. This suggests an infinite number of possible stable solution paths, each dependent on the initial expectations of non-predetermined variables. Fortunately, in most cases only one of these paths survives the discipline imposed by saddlepath learning.

Proposition 3 *If the Blanchard and Kahn (1980) conditions for determinacy are not satisfied such that the system is indeterminate, then the saddlepath learning process for N is e-stable and unique provided that all eigenvalues of the indeterminate system are real and positive. The saddle path is associated with the largest eigenvalues of the system.*

Proof. The indeterminate system still satisfies the eigenvalue-eigenvector decomposition (3), although failure of the Blanchard and Kahn (1980) conditions means that some of the eigenvalues $\{\lambda_j\}$ in Λ are stable rather than unstable. The e-stability of saddlepath learning depends as before on the eigenvalues of the Jacobian (7) having negative real parts, so the condition for e-stability is $\text{real}(\lambda_i/\lambda_j) \leq |\lambda_i/\lambda_j| < 1$ as in Proposition 1. If all the eigenvalues of the system are real and positive, then this obtains uniquely when the saddle path is associated with the largest eigenvalues of the system. Collecting the largest eigenvalues as $\{\lambda_j\}$ into Λ means that $\lambda_j > \lambda_i$ and $|\lambda_i/\lambda_j| < 1$ for all i, j and the eigenvalues of the Jacobian have negative real parts as required. Associating the saddle path with any other eigenvalues would mean $|\lambda_i/\lambda_j| > 1$ for some i, j and the learning process for N would not be e-stable.

The e-stability of the saddle path associated with the largest eigenvalues of the system is also a feature of the minimum state variable (MSV) solution proposed by McCallum (2003) and McCallum (2007). Indeed, the restriction in Proposition 3 that all eigenvalues are real and positive mirrors the difficulty McCallum had in obtaining general results with the MSV solution. To derive general results we therefore switch to iterative e-stability.

[2] In empirical work, Slobodyan and Wouters (2009) avoid the redundant regressor problem by assuming that agents are saddlepath learning as we suggest, although it is unclear how they arrived at such an interpretation of McCallum (2007).

Proposition 4 *If the Blanchard and Kahn (1980) conditions for determinacy are not satisfied such that the system is indeterminate, then the saddlepath learning process for N is iteratively e-stable and unique. The saddle path is associated with the eigenvalues of the system that have the largest modulus.*

Proof. Iterative e-stability requires the modulus of the eigenvalues of $\Lambda^{-1} \otimes (G_{11} - G_{12}N)^T$ in (9) to be less than unity as in the proof of Proposition 2. This is uniquely achieved by associating the saddle path with the eigenvalues of largest modulus, in which case $|\lambda_j| > |\lambda_i|$ and $|\lambda_i/\lambda_j| < 1$ for all i, j and the iterative process (9) is stable. Associating the saddle path with any other eigenvalues would mean $|\lambda_i/\lambda_j| > 1$ for some i, j and the learning process for N would not be iteratively e-stable.[3]

Propositions 3 and 4 abstract from the possibility of sunspots by focusing exclusively on learning about the saddle path of the indeterminate system. To remedy this, suppose the system is indeterminate because one stable eigenvector $\acute{\lambda}$ is associated with the saddle path. In this case $\acute{\lambda}$ is the only stable eigenvalue of Λ. If v is the eigenvector corresponding to $\acute{\lambda}$ then:

$$\Lambda v = \acute{\lambda} v, \tag{10}$$

to within a multiplicative constant. If we associate a univariate stationary sunspot process:

$$\eta_t = \acute{\lambda}\eta_{t-1} + \zeta_t,$$

to the stable eigenvector, with ζ_t i.i.d. Gaussian distributed[4], then expectations in the indeterminate rational expectations equilibrium satisfy:

$$E_t x_{t+1} = -N z_t + v\acute{\lambda}\eta_t. \tag{11}$$

Combining (10) and (11) with the structural form (1) and the eigenvalue-eigenvector decomposition (3) gives:

$$x_t + N z_{t-1} + \Lambda^{-1}(H_2 + N H_1)\varepsilon_t = v\eta_t. \tag{12}$$

With saddlepath learning, agents use ordinary least squares to estimate a sunspots-augmented saddlepath relationship between x_t and z_{t-1}, η_t. The result is a sequence of estimates $\{N_k, v_k\}$ that converge in 'notional' time k if the sunspot system is e-stable. The characteristic equations for the convergence of $\{N_k\}$ are unchanged from (6) and (8) used in the absence of sunspots in Propositions 3 and 4, so the conditions for e-stability and iterative e-stability of the learning process for N are as before. The characteristic equation for the convergence of $\{v_k\}$ is:

$$\frac{dv}{dk} = (G_{22} + N G_{12})^{-1}\acute{\lambda} v - v. \tag{13}$$

[3] A minor caveat to this proof is that if there are two stable eigenvalues with identical modulus, and only one can be associated with the saddle path, then there is non-uniqueness and there could be convergence to two different saddle paths.

[4] Evans and McGough (2005) discuss common factor representations of this type.

In the limit $(G_{22} + NG_{12})^{-1} = \Lambda^{-1}$ and $\Lambda v = \acute{\lambda} v$ from (10) so dv/dk tends to 0 and v has a well-defined limit. It follows that the learning process for v converges if the learning process for N converges. We have:

Proposition 5 *If the system described in Proposition 3 has a stationary sunspot process then the augmented saddlepath learning process for N and v is also e-stable and unique. If the system described in Propositions 4 has a stationary sunspot process then the augmented saddlepath learning process for N and v is iteratively e-stable and unique.*

2.5 Models with constants

The results in Sections 2.3 and 2.4 are derived assuming no constants in the structural form (1) of the economy. If we allow for constants, and require agents to learn them, then the saddlepath relationship estimated by agents becomes:

$$\tilde{E}_t x_{t+1} = -N_{t-1} z_t + a_{t-1},$$

instead of (4). E-stability and iterative e-stability conditions must then be satisfied for the learning processes of both N and a. Our results replicate findings that are well-established in the literature on learning in models with constants, for example see Woodford (1990). Full results appear in Appendix B. Suffice to say here, for determinate models the learning process for a is always e-stable and iteratively e-stable so Propositions 1 and 2 continue to hold. For indeterminate models, the learning process for a is never e-stable when all the eigenvalues of the system are real and positive, so Proposition 3 no longer applies. The learning process for a is similarly never iteratively e-stable in indeterminate models and Proposition 4 fails in models with constants.

3 Learning with lagged information

The usual assumption in learning models is not that agents know the current values of endogenous variables. Instead, agents are assumed to have lagged information only relating to the previous period. This is sensible, but raises the question of how and when agents learn on the basis of that lagged information. Our preferred answer is inspired by the timing protocol we adopted for learning with full information. There, agents undertook learning at the beginning of period t using information about period $t-1$. By analogy, we suggest a timing protocol for lagged information models in which agents also learn at the beginning of period t using information about period $t-1$. Agents hence continue to estimate a saddlepath relationship at the beginning of period t, which with lagged information is between expected *future* values of non-predetermined variables and expected *current* values of predetermined variables. Bullard and Eusepi (2009) make a different assumption by requiring agents to estimate a relationship between expected *future* values of non-predetermined variables and the known *lagged* values of predetermined variables at the beginning of period t. As a consequence, there is no mechanism to ensure that a saddlepath relationship analogous to equation (4) holds under learning. Seen this way, it is not surprising that Bullard and Eusepi (2009) have difficulty obtaining general results for the relationship between determinacy and e-stability conditions in their models. They deviate from the full information benchmark twofold, first by restricting the information available to agents and second by changing from saddlepath to non-saddlepath learning.

In what follows we show that restricting the information set of agents does not in itself overturn our equivalence results. That Bullard and Eusepi (2009) find equivalence breaking down in their models must then be due to their assumption that agents are not saddlepath learning. Information delays *per se* are not enough. In this section we continue to work with models that can be written in the same Blanchard and Kahn (1990) form as the full information case, but impose restrictions on the parameters of the state space so our results are directly comparable to Bullard and Eusepi (2009). Specifically, they assume that predetermined variables z_t follow a purely exogenous process with known coefficients. The relevant structural form is then:

$$\begin{pmatrix} z_t \\ E_t x_{t+1} \end{pmatrix} = \begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} z_{t-1} \\ x_t \end{pmatrix} + \begin{pmatrix} H_1 \\ H_2 \end{pmatrix} \varepsilon_t, \tag{14}$$

with agents knowing G_{11}. The notation E_t is retained for expectations formed using the (now lagged) information available at time t. We return to the more general case where G_{12} is not necessarily zero in Section 4.

3.1 Solution under rational expectations

The saddlepath relationship in rational expectations equilibrium with lagged information is a consistency requirement on the expectations of future non-predetermined variables and the expectations of current predetermined variables:

$$E_t x_{t+1} = -N E_t z_t. \tag{15}$$

The matrix N satisfies the eigenvalue-eigenvector decomposition:

$$\begin{pmatrix} N & I \end{pmatrix} \begin{pmatrix} G_{11} & 0 \\ G_{21} & G_{22} \end{pmatrix} = \Lambda \begin{pmatrix} N & I \end{pmatrix} \tag{16}$$

Reduced-form equations describing how agents form rational expectations with lagged information are obtained by substituting the saddlepath relationship (15) into the structure of the economy (14), taking expectations, and simplifying using the eigenvalue-eigenvector decomposition (16):

$$\begin{aligned} E_t x_t &= -N z_{t-1}, \\ E_t z_t &= G_{11} z_{t-1}, \\ E_t x_{t+1} &= (G_{21} - N G_{22}) z_{t-1}. \end{aligned}$$

3.2 Saddlepath learning

Agents face a joint estimation and decision problem when learning with lagged information, as they did when learning with full information. Indeed, the e-stability and determinacy conditions for learning with lagged information are the same as those with full information if agents are saddlepath learning. To see why, we proceed analogously to the full information case by assuming agents solve their estimation problem by estimating a saddlepath relationship:

$$\tilde{E}_t x_{t+1} = -N_{t-1} \tilde{E}_t z_t, \tag{17}$$

at the beginning of period t. $-N_{t-1}$ is the ordinary least-squares estimate of the linear regression of $\{x_j\}$ on $\{z_{j-1}\}$ for $j = 2, \ldots, t-1$. It is subscript $t-1$ to stress that it is estimated using information up to and including period $t-1$ when information is lagged. Agents use the estimated saddlepath relationship (17) to ensure that expectations remain consistent during period t.

The way agents solve their decision problem can also be taken from the full information case, in that expectations $\tilde{E}_t x_t, \tilde{E}_t z_t, \tilde{E}_t x_{t+1}$ and endogenous variables x_t, z_t are all assumed to adjust within the period. As before, by the end of each period endogenous variables and expectations are consistent with (i) structural form (14), (ii) estimated saddlepath (17) and (iii) $\tilde{E}_t z_t$ being an unbiased predictor of z_t. Condition (iii) is no longer trivial when information is lagged, but holds because of the assumption that z_t follows a known exogenous process. This means that agents have unbiased expectations of the present - but not necessarily the future - when learning.

The structural form (14), saddlepath relationship (17), and the unbiasedness of $\tilde{E}_t z_t$ determine the law of motion for non-predetermined variables with learning and lagged information as:

$$x_t = -G_{22}^{-1}[(G_{21} + N_{t-1} G_{11}) z_{t-1} + (H_2 + N_{t-1} H_1)\varepsilon_t + N_{t-1}\eta_t], \tag{18}$$

where $\eta_t = \tilde{E}_t z_t - z_t$ is an error term with mean zero. It is obvious from a comparison of equation (18) to equation (5) with $G_{12} = 0$ that the least-squares estimates of N_{t-1} in the lagged information case will follow a similar path to the least-squares estimates of N_{t-1} in the full information case. The e-stability and determinacy conditions are the same for both cases. We summarise this as follows:

Proposition 6 *If the Blanchard and Kahn (1980) conditions for determinacy are satisfied then the saddlepath learning process for N is e-stable and uniquely iteratively e-stable under both full and lagged information.*

It is straightforward to confirm that Propositions 1 through 5 continue to hold for saddlepath learning with lagged information. We do not report these calculations, but take the opportunity to highlight that our earlier results on indeterminate models and sunspots also apply in the lagged information case. The isomorphism between e-stability and determinacy is not therefore dependent on whether agents have full or lagged information.

4 Learning with imperfect information

The equivalence of the full and lagged information results above is a portent of our most general result, namely that the relationship between e-stability and determinacy conditions is independent of the information set of agents. To prove this, we introduce a general model with *imperfect* information where agents observe linear combinations of the endogenous variables x_t and z_t subject to possible measurement errors. The model with imperfect information nests that with full information if agents observe x_t and z_t without measurement error. It nests the model with lagged information once lagged variables are included as observables amongst the predetermined variables z_t. To analyse the imperfect information case it is necessary to write the model in state space form. The structural form (1) is unsuitable as it stands because there are measurements in z_t on the left hand side and x_t on the right hand side. Instead, we expand the set of predetermined variables to $k_t = (\varepsilon_t^T \; z_{t-1}^T)^T$ and write the model as:

$$\begin{pmatrix} k_{t+1} \\ E_t x_{t+1} \end{pmatrix} = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} \begin{pmatrix} k_t \\ x_t \end{pmatrix} + \begin{pmatrix} B_1 \\ 0 \end{pmatrix} \varepsilon_{t+1}, \tag{19}$$

$$w_t = \begin{pmatrix} K_1 & K_2 \end{pmatrix} \begin{pmatrix} k_t \\ x_t \end{pmatrix} + v_t. \tag{20}$$

Equation (19) is the state transition equation of the state space form. Equation (20) is the measurement equation, with v_t the measurement errors. Note that $z_t = G_{11} z_{t-1} + G_{12} x_t + H_1 \varepsilon_t$ from the structural form (1), so z_t will be observable to agents for suitable choices of K_1 and K_2.

4.1 Solution under rational expectations

The saddlepath relationship in rational expectations equilibrium with imperfect information imposes consistency on expectations of future non-predetermined variables x_{t+1} and our expanded set of predetermined variables k_{t+1}:

$$E_t x_{t+1} = -N E_t k_{t+1}, \tag{21}$$

which is a generalisation of the saddlepath relationships with full and lagged information. The matrix N satisfies:

$$\begin{pmatrix} N & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \Lambda \begin{pmatrix} N & I \end{pmatrix}, \tag{22}$$

by an eigenvalue-eigenvector decomposition of the $\mathbf{A}$ matrix in (19). Λ is a square matrix containing the unstable eigenvalues if the system is determinate. In rational expectations equilibrium the saddlepath also imposes the consistency requirement $E_t x_t = -NE_t k_t$ on expectations of current variables. The law of motion for non-predetermined variables is:

$$x_t = -(A_{22} + NA_{12})^{-1}[(A_{21} + NA_{11})k_t + NB_1\varepsilon_{t+1} + N\zeta_{t+1}], \tag{23}$$

where $\zeta_{t+1} = E_t k_{t+1} - k_{t+1}$ is a rational expectations error.[5] Agents form rational expectations of current predetermined variables by applying the Kalman filter:

$$E_t k_t = E_{t-1} k_t + J(w_t - (K_1 - K_2 N)E_{t-1} k_t), \tag{24}$$

where the Kalman gain matrix J is defined by:

$$J = P(K_1 - K_2 A_{22}^{-1} A_{21})^T ((K_1 - K_2 N)P(K_1 - K_2 A_{22}^{-1} A_{21})^T + V)^{-1}, \tag{25}$$

and V is the variance-covariance matrix of the measurement errors v_t.[6] The matrix $P \equiv cov(k_t - E_{t-1} k_t)$ satisfies a Riccati equation:

$$\begin{aligned} P \;=\; & (A_{11} - A_{12} A_{22}^{-1} A_{21}) P (A_{11} - A_{12} A_{22}^{-1} A_{21})^T \\ & -(A_{11} - A_{12} A_{22}^{-1} A_{21}) P (K_1 - K_2 A_{22}^{-1} A_{21})^T \\ & \times((K_1 - K_2 A_{22}^{-1} A_{21}) P (K_1 - K_2 A_{22}^{-1} A_{21})^T + V)^{-1} \\ & \times P(K_1 - K_2 A_{22}^{-1} A_{21})(A_{11} - A_{12} A_{22}^{-1} A_{21})^T + B_1 cov(\varepsilon_t) B_1^T. \end{aligned} \tag{26}$$

The rational expectation of future predetermined variables is a linear mapping:

$$E_t k_{t+1} = (A_{11} - A_{12} N) E_t k_t,$$

of the rational expectation of current predetermined variables. Note that the matrix P in the Riccati equation is independent of the matrix N in the saddlepath relationship.

[5] ζ_{t+1} is a hybrid term that includes rational expectations errors caused by unpredictable shocks ε_{t+1} and by imperfect inferences on the current state of the economy. The rational expectations errors caused by shocks are $-B_1\varepsilon_{t+1}$, which cancels the $NB_1\varepsilon_{t+1}$ term in equation (23) and means there is correctly no role for $t+1$ shocks in determining x_t.

[6] See Appendix C and Pearlman et al. (1986) for a full derivation the Kalman gain matrix (25).

4.2 Saddlepath learning

Maintaining our analogy with the full information case, saddlepath learning with imperfect information requires agents to estimate a relationship:

$$\tilde{E}_t x_{t+1} = -N_{t-1}\tilde{E}_t k_{t+1}, \tag{27}$$

at the beginning of period t. $-N_{t-1}$ is the ordinary least-squares estimate of the linear regression of $\{E_j x_j\}$ on $\{E_j k_j\}$ for $j = 1, \ldots, t-1$. Expectations $\tilde{E}_t x_t, \tilde{E}_t k_t, \tilde{E}_t x_{t+1}, \tilde{E}_t k_{t+1}$ and endogenous variables x_t, z_t adjust as decisions are made, and by the end of the period expectations and endogenous variables are consistent with (i) structural form (19), (ii) estimated saddlepath (27) and (iii) $\tilde{E}_t k_{t+1}$ being an unbiased predictor of k_{t+1}. Once these conditions are satisfied, the law of motion for non-predetermined variables with learning and imperfect information is:

$$x_t = -(A_{22} + N_{t-1}A_{12})^{-1}[(A_{21} + N_{t-1}A_{11})k_t + N_{t-1}B_1\varepsilon_{t+1} + N_{t-1}\eta_{t+1}], \tag{28}$$

where $\eta_{t+1} = \tilde{E}_t k_{t+1} - k_{t+1}$ is an error term with mean zero.[7] Equation (28) has the same form as the law of motion for non-predetermined variables with full information (5), so again the least squares estimate of N_{t-1} in the imperfect information case follows a similar path to the least squares estimate of N_{t-1} in the full information case. The e-stability and determinacy conditions must therefore be the same in both cases:

Proposition 7 *If the Blanchard and Kahn (1980) conditions for determinacy are satisfied then the saddlepath learning process for N is e-stable and uniquely iteratively e-stable under any information set.*

Propositions 1-5 also hold for any information set when agents are saddlepath learning.

Condition (iii) that $\tilde{E}_t k_{t+1}$ is an unbiased predictor of k_{t+1} requires further justification when learning with imperfect information. Since $k_{t+1} = (\varepsilon_{t+1}^T \; z_t^T)^T$, it implies that agents recognise ε_{t+1} as a shock with mean zero, and are able to make unbiased inferences about the current state of predetermined variables z_t. Recognising ε_{t+1} as a shock is satisfied by assumption, but making unbiased inferences depends on how agents are assumed to infer the current state of the economy. In rational expectations equilibrium agents use equation (24) to infer the current state, so one possibility is to assume that agents have sufficient information to construct an unbiased Kalman filter with which to make inferences. Whilst the information needed to do so is arguably large, the calculations

[7] As with rational expectations, η_{t+1} is a hybrid term that captures expectations errors caused by shocks ε_{t+1} and imperfect inferences about the current state of the economy. Agents understand that ε_{t+1} is a shock with impact B_1, so $\eta_{t+1} = \tilde{E}_t(\varepsilon_{t+1}^T \; z_t^T)^T - (\varepsilon_{t+1}^T \; z_t^T)^T$ contains an element that cancels the $N_{t-1}B_1\varepsilon_{t+1}$ term in equation (28) as before. Fluctuations in x_t are then driven purely by expectations errors $\tilde{E}_t z_t - z_t$ and there is no dependence on $t+1$ dated shocks.

required are completely distinct from those needed to *solve* the rational expectations model and make decisions consistent with the saddlepath. As an extreme, it is not inconceivable that agents know the structural form (19) but are unable to solve the model forward to obtain the saddlepath relationship (21). Such agents can infer the current state of the economy but have to learn how to form expectations of the future.[8] Less extremely, agents could estimate the structure of the economy whilst learning the saddlepath relationship. The estimates of the parameters in the structural form converge by the properties of ordinary least squares estimators, and estimates P_t of the precision matrix converge to the rational expectations value P as the Riccati equation (26) is independent of N. This leaves the Kalman gain matrix J_t, which converges to the rational expectations value J as $P_t \to P$ and $N_t \to N$. We therefore conclude that $\tilde{E}_t k_{t+1}$ is an unbiased predictor of k_{t+1}, either because agents have sufficient information to construct an unbiased Kalman filter or as the outcome of a well-defined learning process.

5 Conclusions

The initial motivation for this paper was to discipline boundedly rational agents by requiring them to learn through a saddlepath relationship. Doing so ensures deviations from rational expectations are 'small', with agents able to correctly decompose endogenous variables into states that are predetermined and states that are non-predetermined. An unexpected by-product of saddlepath learning is greater mathematical tractability and transparency. Many of the terms that appear in the eigenvalue condition for determinacy also appear in the eigenvalue condition for e-stability of saddlepath learning. This is ultimately what gives us traction to clarify and improve upon existing results.

We show that the e-stability results of McCallum (2007) relate to learning about the saddlepath relationship. Furthermore, his inclusion of all variables in the least-squares regression leads to greater variability in the estimates and therefore slower convergence to rational expectations. For the indeterminacy case we have two new results. Firstly, there is convergence to a unique saddlepath under iterative e-stability. The saddlepath is associated with the eigenvalues of the system that have the largest modulus. Secondly, we show that the impact of sunspots on non-predetermined variables is also learnable. Thus when a model is indeterminate we cannot ignore sunspots under rational expectations or learning. All these results hold for any information set, be it full, lagged or imperfect, as long as agents are saddlepath learning. This generality suggests a much closer connection between determinacy and e-stability than found by Bullard and Eusepi (2009) for non-saddlepath learning.

[8] The Bullard and Eusepi (2009) assumption that predetermined variables follow a known exogenous process meant that agents were in exactly this position in the lagged information models of Section 3.

A Conversion of McCallum form to Blanchard and Kahn form

We show that the setup used by McCallum (2007) and others, namely:

$$A_0 E_t y_{t+1} + A_1 y_t = A_2 y_{t-1} + B\varepsilon_t, \tag{A.1}$$

where A_0 is not of full rank, can be rewritten in a state-space form suitable for applying the Blanchard-Kahn conditions. We assume that it is possible to solve the system for $E_t y_{t+1}$ in terms of y_t, y_{t-1}[9] and write the conversion to state space form as an algorithm. y_t is of dimension n.

1. Obtain the singular value decomposition for matrix A_0: $A_0 = UDV^T$, where U, V are unitary matrices. Assuming that only the first m values of the diagonal matrix D are non-zero, we can rewrite this as $A_0 = U_1 D_1 V_1^T$, where U_1 are the first m columns of U, D_1 is the first $m \times m$ block of D and V_1^T are the first m rows of V^T.

2. Multiply (A.1) by $D_1^{-1}U_1^T$, which yields:

$$V_1^T E_t y_{t+1} + D_1^{-1}U_1^T A_1 y_t = D_1^{-1}U_1^T A_2 y_{t-1} + D_1^{-1}U_1^T B\varepsilon_t. \tag{A.2}$$

   Now define $x_t = V_1^T y_t$, $s_t = V_2^T y_t$, and use the fact that $I = VV^T = V_1V_1^T + V_2V_2^T$ to rewrite this as:

$$E_t x_{t+1} + D_1^{-1}U_1^T A_1(V_1 x_t + V_2 s_t) = D_1^{-1}U_1^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + D_1^{-1}U_1^T B\varepsilon_t. \tag{A.3}$$

3. Multiply (A.1) by U_2^T, where U_2 are the last $n-m$ columns of U (with $U_2^T U_1 = 0$), which yields:

$$U_2^T A_1 y_t = U_2^T A_2 y_{t-1} + U_2^T Bu\varepsilon_t. \tag{A.4}$$

   This can be rewritten as:

$$U_2^T A_1(V_1 x_t + V_2 s_t) = U_2^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T B\varepsilon_t. \tag{A.5}$$

4. $U_2^T A_1 V_2$ is invertible by Assumption 1, which means that we can rewrite (A.3) as:

$$s_t + (U_2^T A_1 V_1)^{-1} U_2^T A_1 V_1 x_t = (U_2^T A_1 V_1)^{-1}(U_2^T A_2(V_1 x_{t-1} + V_2 s_{t-1}) + U_2^T B\varepsilon_t), \tag{A.6}$$

   and hence the whole system as:

$$\begin{bmatrix} I & 0 & 0 \\ 0 & I & 0 \\ F & 0 & I \end{bmatrix} \begin{bmatrix} s_t \\ x_t \\ x_{t+1,t} \end{bmatrix} = \begin{bmatrix} G_{11} & G_{12} & -G_{13} \\ 0 & 0 & I \\ G_{31} & G_{32} & -G_{33} \end{bmatrix} \begin{bmatrix} s_{t-1} \\ x_{t-1} \\ x_t \end{bmatrix} + \begin{bmatrix} H_1 \\ 0 \\ H_3 \end{bmatrix} \varepsilon_t,$$

[9] This rules out post-recursive expectations which are generated by systems of the form $a_t = \rho a_{t-1} + \varepsilon_t$, $b_t = E_t a_{t+1}$. The latter would be written for our purposes as $a_t = \rho a_{t-1} + \varepsilon_t$, $b_t = \rho a_t$.

where:

$$\begin{array}{lll} G_{11} = (U_2^T A_1 V_1)^{-1} U_2^T A_2 V_2 & G_{12} = (U_2^T A_1 V_1)^{-1} U_2^T A_2 V_1 & G_{13} = (U_2^T A_1 V_1)^{-1} U_2^T A_1 V_2 \\ G_{31} = D_1^{-1} U_1^T A_2 V_2 & G_{32} = D_1^{-1} U_1^T A_2 V_1 & G_{33} = D_1^{-1} U_1^T A_1 V_2 \\ H_1 = (U_2^T A_1 V_1)^{-1} U_2^T B & H_3 = D_1^{-1} U_1^T B & F = D_1^{-1} U_1^T A_1 V_2 \end{array}$$

Multiplying (A.5) by the inverse of the far left matrix in (A.5) yields the required state-space setup.

For the purist, this is not quite good enough, because the states $s_t,\ x_t$ are now linear combinations of the underlying variables $y_{1t},\ ...,\ y_{nt}$. Consider therefore the first terms of equations (A.3) and (A.6), where the latter are expressed again in terms of y_t. If the number of non-zero columns of V_1^T is equal to m then we can rewrite the forward-looking equation (A.3) purely in terms of those m elements of y_t that are multiplied by those non-zero columns. These elements of y_t are then the forward-looking looking variables. However, suppose the number of non-zero columns of V_1^T is greater than m e.g. a system like $E_t y_{1,t+1} + E_t y_{2,t+1} = y_{1t},\ y_{1t} - y_{2t} = y_{2,t-1} + \varepsilon_t$. In principle we can advance (A.6) by one period, take expectations, and then solve for $E_t y_{t+1}$ in terms of y_t, y_{t-1}. Then substitute some of these expectations into (A.3) so that one is left with only m forward-looking variables. But which ones to substitute out, and can we be sure that such a representation exists? The latter question is equivalent to asking whether there exists a set of m columns of V_1^T and a set of $n-m$ columns of V_2^T that are non-aligned with one another in $\Psi = [V_1\ V_2]^T$, and which are each of full rank. We prove this by contradiction. Denote the determinants of each such corresponding set i by d^{1i} and d^{2i}. Suppose that for all such corresponding sets either or both of d^{1i} and d^{2i} is 0, so that their product is 0. But the (non-zero) determinant of Ψ is the sum of all of these products $d^{1i}d^{2i}$ weighted by either 1 or -1. Hence not all of these can be zero. Thus we deduce that there are mutually exclusive subsets y^1 and y^2 of y such that $E_t y^1_{t+1} = f_1(y_t, y_{t-1}, \varepsilon_t),\ y^2_t = f_2(y^1_t, y_{t-1}, \varepsilon_t)$.

For the case of partial information the logic is identical, except that when we advance by one period and take expectations of (A.6), we introduce terms in $E_t y_t$, so that the forward-looking equations take the form $E_t y^1_{t+1} = g_1(y_t, E_t y_t, y_{t-1}, \varepsilon_t)$. This extension is covered for the rational expectations solution in Pearlman et al (1986), and is a trivial extension to our analysis of learning in the main text.

B Learning about constants

Here we summarize e-stability results on learning about the constants, which are well-known in the literature e.g. Woodford (1990), but cast them in our general framework. In addition, we extend the results to iterative e-stability. Suppose (1) has constants $(c_1\ c_2)^T$ on the RHS, and assume that expectations in (4) are augmented by the constants a_{t-1}. It then follows that there is an additional term $(G_{22} + N_{t-1}G_{12})^{-1}(a_{t-1} - N_{t-1}c_1 - c_2)$ on the RHS of (5) Recalling that under rational expectations we have $G_{22} + NG_{12} = \Lambda_U$, it is easy to show that under updated least-squares learning, a_t converges to its RE value provided that e-stability holds for:

$$\frac{da}{dk} = (\Lambda_U^{-1} - I)a.$$

This is the case when there is determinacy; if the eigenvalues of Λ_U are greater than 1, then their inverses will have real part less than 1. When there is indeterminacy, some of the eigenvalues of Λ_U may be less than 1; if these are real, then e-stability only holds if they are negative. If they are complex, then E-stability holds if the real part of their inverses are less than 1. For iterative E-stability, the requirement is stability of:

$$a_{k+1} = \Lambda_U^{-1} a_k.$$

This can only ever be the case when the system is determinate. For indeterminacy, at least one of the eigenvalues of Λ_U will have an inverse that lies outside the unit circle.

C Kalman gain matrix J with imperfect information

Assume that there is a saddlepath relationship:

$$E_t x_{t+1} + NE_t k_{t+1} = 0 \text{ where } \begin{pmatrix} N & I \end{pmatrix} \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix} = \Lambda \begin{pmatrix} N & I \end{pmatrix}, \tag{C.1}$$

and Λ is a square matrix with unstable eigenvalues only. It is clear that this is consistent with $E_t x_{t+j} + NE_t k_{t+j} = 0$ for all $j > 0$. In addition, by taking expectations of the equations involving k_{t+1} and $E_t x_{t+1}$ using information at time t, and applying the saddlepath relationship, this yields:

$$E_t x_t + NE_t k_t = 0. \tag{C.2}$$

Now focus on the equation involving $E_t x_{t+1}$. Taking expectations of this equation using information at time t and subtracting from the original equation yields:

$$A_{21}(k_t - E_t k_t) + A_{22}(x_t - E_t x_t) = 0. \tag{C.3}$$

From (C.2) and (C.3) we deduce that:

$$x_t = -A_{22}^{-1} A_{21} k_t + (A_{22}^{-1} A_{21} - N)E_t k_t. \tag{C.4}$$

Now assume that the Kalman filtering update for $E_t k_t$ is given by:

$$E_t k_t = E_{t-1} k_t + J(w_t - ME_{t-1} k_t) \text{ where } M = K_1 - K_2 N. \tag{C.5}$$

It then follows that we can rewrite the measurement equation:

$$w_t = K_1 k_t + K_2 x_t + v_t = (K_1 - K_2 A_{22}^{-1} A_{21}) k_t + K_2 (A_{22}^{-1} A_{21} - N)(E_{t-1} k_t + J(w_t - M E_{t-1} k_t)) + v_t. \quad \text{(C.6)}$$

Defining $D = K_1 - K_2 A_{22}^{-1} A_{21}$ and the innovations process $\tilde{k}_t = k_t - E_{t-1} k_t$, (C.6) can be rewritten as:

$$(I + (D - M)J)(w_t - M E_{t-1} k_t) = D \tilde{k}_t + v_t. \quad \text{(C.7)}$$

Now define $cov(\tilde{k}_t) = P$. Recalling that $k_t = \tilde{k}_t + E_{t-1} k_t$, it follows that we can use (C.7) to obtain the best estimate of k_t using the measurement w_t, which implies that $J = PD^T (DPD^T + V)^{-1} (I + (D - M)J)$ by comparing with (C.5), where $V = cov(v_t)$. Simple algebra then shows that:

$$J = PD^T (MPD^T + V)^{-1}. \quad (29)$$

Then taking expectations of the k_{t+1} equation gives the updating equation for $E_t k_{t+1}$ in terms of $E_t k_t$, and subtracting this from the k_{t+1} equation yields an equation for $\tilde{k}_{t+1}$ in terms of $k_t - E_t k_t$ and $x_t - E_t x_t$. Substituting for the latter from (C.3), for $E_t k_t$ from (C.5), and for $(w_t - M E_{t-1} k_t)$ from (C.7), it follows that the equation for $\tilde{k}_t$ is given by:

$$\tilde{k}_{t+1} = A \tilde{k}_t - APD^T (DPD^T + V)^{-1} (D \tilde{k}_t + v_t) + B_1 \varepsilon_{t+1},$$

with covariance matrix P therefore given by:

$$P = APA^T - APD^T (DPD^T + V)^{-1} PDA^T + U \qquad A = A_{11} - A_{12} A_{22}^{-1} A_{21} \qquad U = B_1 cov(\varepsilon_t) B_1^T.$$

This is the Ricatti equation commonly used in Kalman filtering, and we note that it is independent of the saddlepath matrix N.

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