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Rankings of Income Distributions: A Note on Intermediate Inequality Indices*

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Abstract

The purpose of this paper is to analyze the advantages and disadvantages of several intermediate inequality measures, paying special attention to whether inequality rankings between income distributions are affected by the monetary units in which incomes are expressed.

Keywords: Income distribution; Intermediate inequality indices; Unit-consistency.
JEL Classification: D63.

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1. Introduction

There is a wide consensus in the literature about the properties an inequality measure has to satisfy when using it to compare income distributions having the same mean. Basically, it is necessary to invoke the symmetry axiom—which warranties anonymity—and the Pigou-Dalton principle of transfers—which requires a transfer of income from a richer to a poorer person to decrease inequality.[1] However, if we are interested in comparing two income distributions that have different means, we need to specify the type of mean-invariance property we want our inequality indices to satisfy. This implies the need to introduce another judgment value into the analysis, and there is no agreement among scholars with respect to this matter. Some opt to invoke the scale invariance axiom, so that the inequality of a distribution will be unaffected when all incomes increase (or decrease) by the same proportion. This is the approach followed by the relative inequality indices. Others prefer instead to call on the translation invariance axiom, under which inequality remains unaltered if all incomes are augmented (or diminished) in the same amount, thereby giving rise to the absolute inequality measures. However, as Kolm (1976) pointed out, some people may prefer an intermediate invariance approach between these two extreme views. He labeled such an inequality attitude as "centrist", against the "rightist" and "leftist" labels he used to term the aforementioned relative and absolute notions, respectively.

So far, the intermediate and absolute inequality indices have rarely been applied for ranking income distributions, since these measures are cardinally affected by the currency unit in which incomes are expressed. In a recent paper, Zheng (2007) invokes a new axiom, the unit consistency axiom, requiring that inequality rankings between income distributions remain unchanged when all incomes are multiplied by a scalar. In this new scenario, not only relative measures, but also absolute and intermediate measures that satisfy the unit consistency axiom, appear as plausible options in empirical research.

The purpose of this paper is to analyze the advantages and disadvantages of several intermediate inequality measures, paying special attention to the unit consistency

[1] Properties such as normalization, continuity, differentiability, and replication invariance are also commonly invoked, but they are of a more technical nature.

axiom. First, we demonstrate why one of the most referenced intermediate indices, that proposed by Bossert-Pfingsten (1990) (B-P hereafter), is not unit-consistent. This analysis allows us to show that the problem lies in the iso-inequality criteria behind that index, which helps us explain why the decomposable intermediate inequality measures à la B-P proposed by Chakravarty and Tyagarupananda (2000) do not satisfy unit consistency either, as shown by Zheng (2007). Second, we show that the intermediate measures proposed by Kolm (1976) do not cover the whole intermediate space and, also, that they may violate unit consistency.

2. Unit consistency and intermediate inequality measures

In order to ensure independence of the unit of measurement without imposing scale invariance, Zheng (2007) introduces the following property into inequality measures:[2]

Unit consistency. For any two distributions $x,\ y \in \Re^n_{++}$ and any inequality measure I ($I : \bigcup_{n\geq 2} \Re^n_{++} \to \Re_+$), if $I(x) < I(y)$, then $I(\theta x) < I(\theta y)$ for any $\theta \in \Re_{++}$.

Certainly, any relative inequality measure satisfies the above property since they are defined as those where $I(\theta x) = I(x)$ for any $\theta \in \Re_{++}$. However, we should keep in mind that there are other unit-consistent indices, apart from the scale invariant ones. In this vein, as shown by Zheng (2007), the variance and the "fair compromise" measure proposed by Krtscha (1994) are absolute and intermediate indices, respectively, that satisfy this property. In what follows, we analyze several intermediate inequality equivalence criteria by distinguishing between linear and non-linear invariances.

2.1. Linear invariance criteria

The μ-inequality concept proposed by B-P is the intermediate inequality measure most frequently mentioned in the literature. According to this invariance criterion, an intermediate inequality index should satisfy the following condition for a given $\mu \in [0,\ 1]$:

[2] Zoli (2003) also proposes an analogous property named "weak currency independence".

$$I_{\mu}\left[x\right]=I_{\mu}\left[x+\tau(\mu x+(1-\mu)1^{n})\right]$$

for any $n\geq 2, x\in\Re^{n}_{++}$, and $\tau\in\Re$, such that $x+\tau(\mu x+(1-\mu)1^{n})\in\Re^{n}_{++}$, where $1^{n}\equiv(1,......,1)$. As shown in Figure 1, for a given income distribution $x\in\Re^{2}_{++}$, the distributions which are μ-inequality equivalent to it are those located on the line defined by point x and vector $\mu x+(1-\mu)1^{n}$ (which represents an intermediate attitude between the relative ray given by x and total equity given by 1^{n}). In particular, $I_{\mu}(x)=I_{\mu}(y)$.

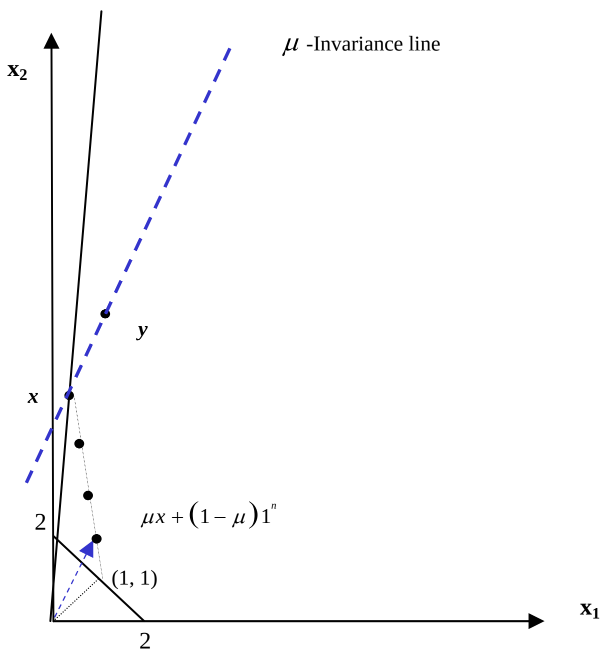

Figure 1. Invariance in B-P ($n=2$, μ=0.25)

By using a numerical example in a five-dimensional space, Zheng (2007) showed that the decomposable intermediate inequality measures à la B-P, characterized by Chakravarty and Tyagarupananda (2000), are not unit-consistent. By using the same distributions, we can prove that B-P's index also violates the above axiom. In fact, if $\mu=0.5$, $x=(1,2,3,4,5)$ and $y=(0.1,0.1,0.2,0.6,0.6)$, then $I_{0.5}(x)=0.068>0.015=I_{0.5}(y)$, but $I_{0.5}(10x)=0.122<0.148=I_{0.5}(10y)$, where $I_{\mu}(x)=(1+s)\left[1-\prod_{i=1}^{n}\left(\frac{y_i+s}{\overline{y}+s}\right)^{1/n}\right]$, $s\equiv\frac{1-\mu}{\mu}$ and $\overline{y}$ represents the mean of distribution y.

In Figure 2, we illustrate why this popular intermediate inequality equivalence criterion leads to measures that do not satisfy the unit consistency axiom.

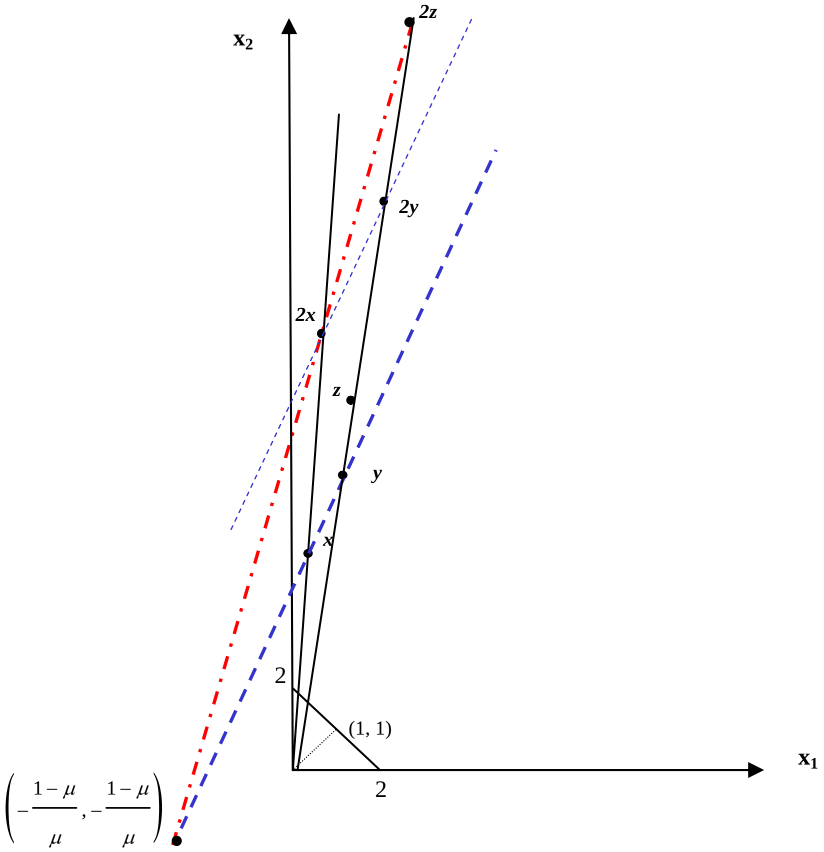

Figure 2. Unit consistency in B-P ($n = 2$, $\mu = 0.25$).

Thicker dash lines represent the two μ-invariance lines passing through points x and $2x$, that is, the set of distributions equivalent to x and $2x$, respectively. Vector y represents an income distribution that is equivalent to x, since it is located on the invariance line of the latter. It is easy to see that any distribution between y and z has a larger inequality level than x because of the Pigou-Dalton transfer axiom. However, distributions resulting from doubling their individual incomes (which are located between $2y$ and $2z$) would have instead a lower inequality level than distribution $2x$. Therefore, changes in the currency unit affect rankings between income distributions.

The above graphical analysis permits us to illustrate that the aforementioned five-dimensional example was not an isolated one. We have shown that, even in a twodimension space, for any given income distribution it is possible to find an interval of distributions that violate the axiom when comparing them with the former distribution. The explanation of this behavior relies on the notion of inequality equivalence proposed by B-P. The slope of the inequality invariance line given by direction $\mu x + (1-\mu)1^n$ does depend on the total income of distribution x. In fact, keeping the relative inequality as constant, the larger the total income, the larger this slope (the slope of the invariance line corresponding to $2x$ is larger than that of x, as shown in Figure 2).[3] This means, first, that μ may represent a different intermediate inequality attitude depending on the distribution in which the index is evaluated. Since the invariance lines are, therefore, not parallel, it is impossible to state that μ-inequality rankings are not affected by changes in the scale when comparing any two distributions. Thus, we have shown that the heart of this equivalence criterion is incompatible with the unit consistency axiom, so that any measure based on this notion violates this axiom.

Second, the μ-inequality concept approaches the "rightist" view of inequality when aggregate income rises.[4] This means that results obtained by using this intermediate concept can be quite close to those obtained with relative measures, which can be seen as unsuitable for a "centrist" measure. In fact, in Figure 3, which shows the $\mu-$iso-inequality contours corresponding to distribution $x = (20,80)$ for two parameter values, we see that the invariance line corresponding to $\mu = 0.5$ is roughly indistinguishable from the ray passing by x (which defines the iso-inequality line of relative measures). Moreover, to obtain an iso-inequality contour closer to the "leftist" view, it would be necessary to choose a parameter value extraordinary low (for example $\mu = 0.005$).[5]

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[3] Since the two invariance lines cut in the third quadrant (at point $\left(-\frac{1-\mu}{\mu}, -\frac{1-\mu}{\mu}\right)$, as shown by B-P), it is possible to construct inequality indices based on this invariance notion. If the two lines cut in the first quadrant, the index would not be well-defined.

[4] This tendency to the relative ray was initially pointed by Seidl and Pfingsten (1997) and Del Río and Ruiz-Castillo (2000).

[5] This explains why Atkinson and Bradolini (2004) find similar empirical results either by using B-P's index or relative indices, even when considering extremely low $\mu-$values $(\mu = 0.0027 \Leftrightarrow \xi = 365 \text{ dollars})$.

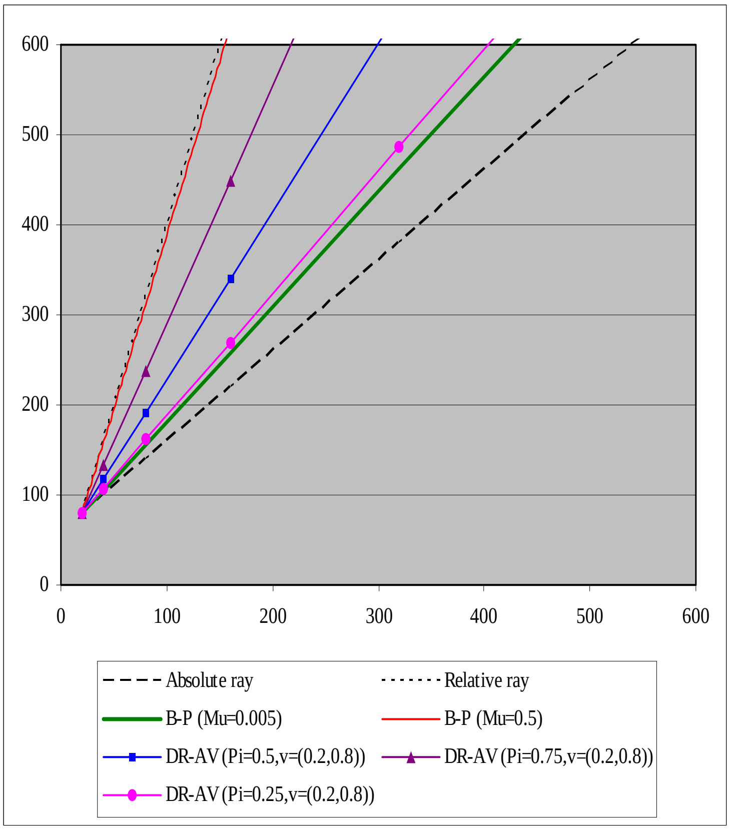

Figure 3. Iso-inequality contours corresponding to distribution $(20,80)$: Linear cases

If one is interested in defining a linear centrist measure as a convex combination between a relative and an absolute ray, one could fix not only parameter μ, but also the reference distribution that gives rise to the "rightist" and "leftist" views. In this regard, Del Río and Ruiz-Castillo (2000) (DR-RC hereafter) propose the (v, π)-inequality, where v is a vector belonging to the n-dimensional simplex, and $\pi \in [0,\ 1]$. The first component fixes the distribution of reference, while the second refers to the convex combination between the relative and absolute rays associated to v.[6] Once these two components are fixed, we can calculate the n-dimensional simplex vector $\alpha = \pi v + (1-\pi)\left(\frac{1}{n}\right)1^n$, which defines the direction of the inequality equivalence ray, and the set of income distributions $\Gamma^{'}(\alpha)$ for which α represents an intermediate attitude. This set can be expressed as follows:

[6] We have changed their original notation to make it clearer. In particular, we have switched vector x by simplex vector v, since only the income shares of the distribution of reference are required to obtain the invariance ray.

$\Gamma^{'}(\alpha)=\left\{x\in D:\ \pi_x v_x+(1-\pi_x)\frac{1^n}{n}=\alpha,\ \text{for some } \pi_x\in[0,\,1]\right\}$, where D is the set of all possible ordered income distributions, and v_x represents the vector of income shares associated to x (it therefore belongs to the n-dimensional simplex).

In this vein, an intermediate inequality index is (v, π)-invariant in the set of income distributions $\Gamma^{'}(\alpha)$ if for any $x\in\Gamma^{'}(\alpha)$ the following expression holds:

$$I_{(v,\pi)}(x)=I_{(v,\pi)}(y),\ \text{for any } y\in P_{(v,\pi)}(x),$$

where $P_{(v,\pi)}(x)=\left\{y\in D:\ y=x+\tau\left(\pi v+(1-\pi)\frac{1^n}{n}\right),\ \tau\in\Re\right\}$ represents the inequality invariance line. Note that this line is obtained as the convex combination π between the "leftist" and "rightist" views associated with vector v (and also with any other distribution that has the same income shares than v), which means that invariance lines are always parallel (see Figure 4).

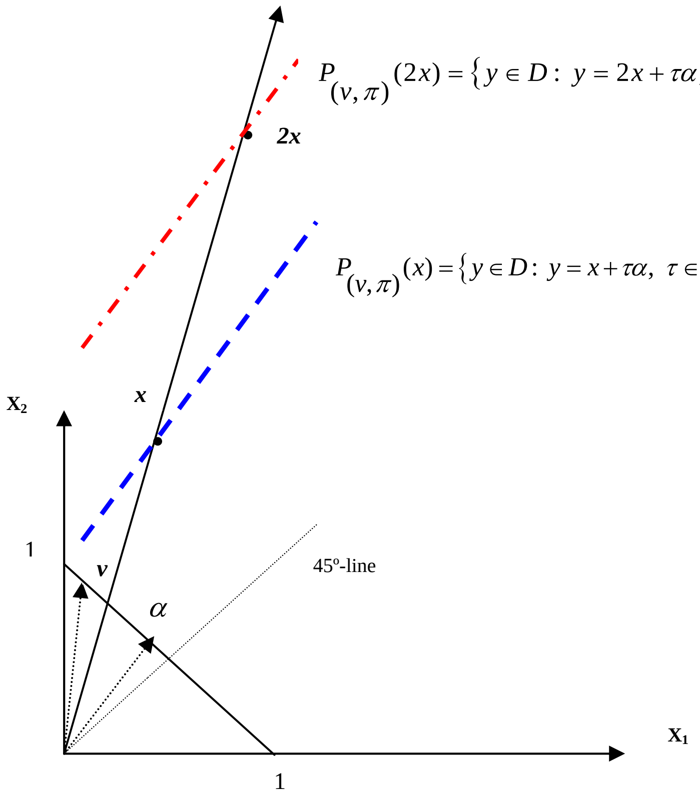

Figure 4. Invariance in DR-RC ($n=2,\ \pi=0.25$).

Therefore, the (v, π)-invariance notion does not have the problem shown in Figure 2, which can help explain why the family of indices based on this approach, proposed by Del Río and Alonso-Villar (2007) (DR-AV hereafter), does satisfy the unit consistency axiom. The invariance lines corresponding to three of these indices are shown in Figure 3, where $\pi \in \{0.25, 0.5, 0.75\}$ and $v = (0.2, 0.8)$. We can see that $\pi = 0.5$ leads to an iso-inequality contour which is "equidistant" from the "rightist" and "leftist" views of distribution $(20, 80)$, when choosing the vector of reference, v, as that given by the income shares of that distribution.

2.2. Non-linear invariance criteria

An alternative to the above intermediate notions is to assume that the iso-inequality contours are not straight lines. In this regard, Krtscha (1994) proposes an adaptive intermediate notion that gives rise to parabolas. According to his "fair compromise" notion, to keep inequality unaltered, any extra income should be allocated among individuals in the following way. The first extra dollar of income should be distributed so that 50 cents goes to the individuals in proportion to the initial income shares, and 50 cents goes in equal absolute amounts. The second extra dollar should be allocated in the same manner, starting now from the distribution reached after the first dollar allocation, and so on. This index [and the generalizations proposed by Zheng (2007)] does satisfy unit consistency, as shown by the latter. This "centrist" attitude is rather challenging since it approaches the absolute view rather soon when income increases, which makes it difficult for inequality to decrease when analyzing an economy over time. In Figure 5, we can see that, according to Krstcha's index, inequality would remain unaltered with respect to distribution $(20, 80)$ if the poorer reached an income of 400 and the richer of 590, which would imply income shares of 40% and 60%, respectively.

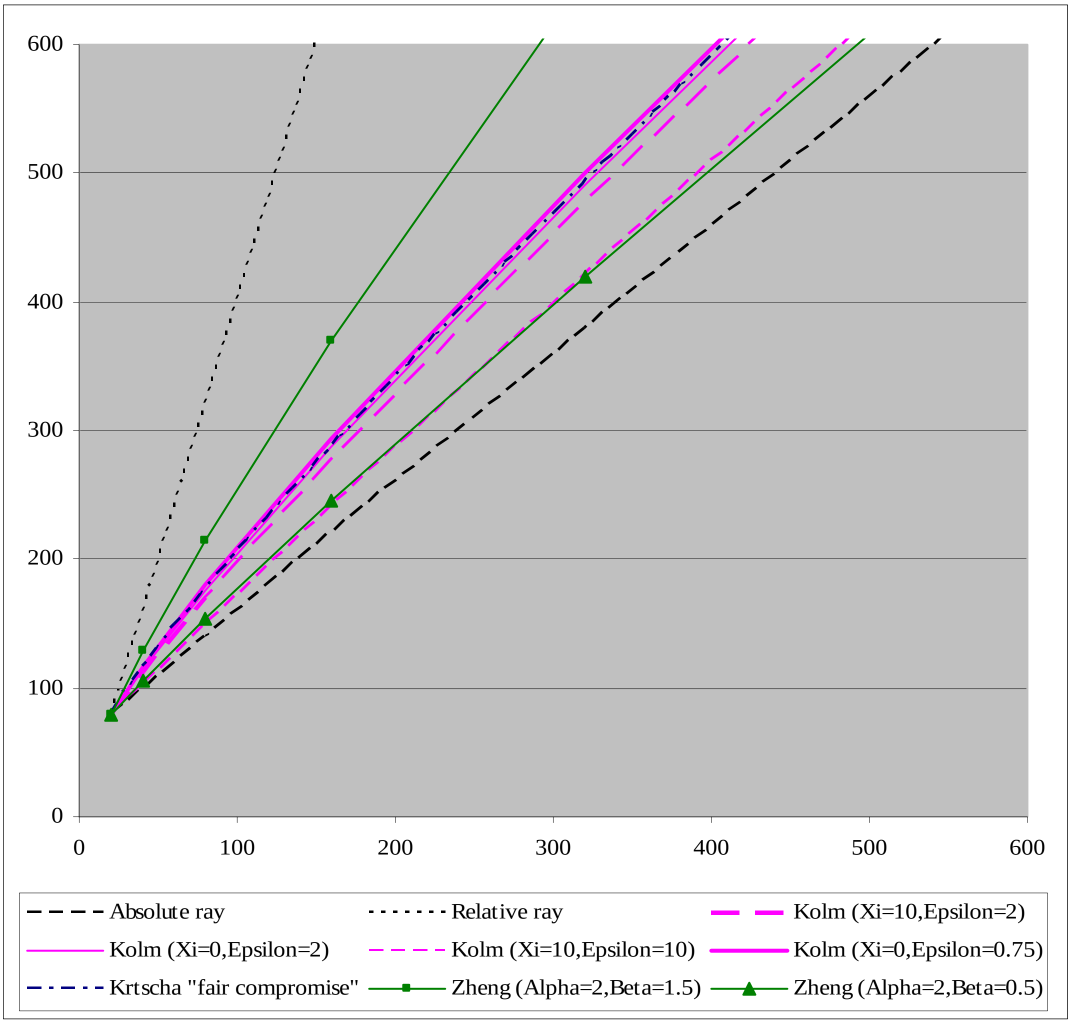

Figure 5. Iso-inequality contours corresponding to distribution $(20,80)$: Non-linear cases[7]

Kolm's (1976) "centrist" measures also lead to iso-inequality contours that are not straight lines. As opposed to Zheng's family of indices, Kolm does not cover the whole intermediate space since, as shown in Figure 5, "centrist" attitudes are close to the "leftist" view, while those near the "rightist" view are not permitted for any parameter value.[8] On the other hand, Kolm's "centrist" measures may violate the unit consistency axiom when $\xi \neq 0$ (if $\xi = 0$ the index is homogeneous of degree 1 and, therefore, it does satisfy the axiom). In this regard, if $\xi = 10$ and $\varepsilon = 10$, for distributions

[7] Kolm's family of indices has iso-inequality contours that monotonically approach the absolute ray as either ξ or ε increases (if $\varepsilon > 1$). However, when $\varepsilon \in [0,1]$, there is no monotonicity with respect to this parameter. In this example, the contour closer to the relative ray is that corresponding to $\xi = 0, \varepsilon = 0.75$.

[8] Recent empirical evidence obtained by Atkinson and Brandolin (2004, p. 13) seems to support this idea: "Kolm's centrist measure basically confirms the pattern shown by Kolm's absolute measure".

$x=(2,2,6,7,7)$ and $y=(2,2,3,8,8)$, it follows that $I_{(10,10)}(x)=1.63<1.66=I_{(10,10)}(y)$ while $I_{(10,10)}(2\,x)=4.13>3.94=I_{(10,10)}(2y)$, where $I_{(\xi,\varepsilon)}(x)=\bar{x}+\xi-\left[\frac{1}{n}\left(\sum_{i=1}^{n}\left(x_i+\xi\right)^{1-\varepsilon}\right)\right]^{\frac{1}{1-\varepsilon}}$. Therefore, $\left(\xi,\varepsilon\right)$-inequality rankings may be affected by currency units.

3. Final remarks

The unit consistency axiom, recently invoked by Zheng (2007), guarantees that inequality rankings between income distributions remain unaffected by the unit in which incomes are expressed. This axiom does not impose such strong value judgments on inequality measurement as the scale invariance condition and, therefore, intermediate indices satisfying it appear to be plausible options for empirical research. We have revised the centrist measures offered by the literature in order to check whether they are unit-consistent. We have shown that both the class of intermediate inequality indices proposed by Bossert-Pfingsten (1990) and those of Kolm (1976) are affected by the currency unit. Therefore, only the "fair compromise" index proposed by Krtscha (1994), the generalizations proposed by Zheng (2007), and the indices proposed by Del Río and Alonso-Villar (2007)—which, as opposed to the others, are ray invariant—are intermediate inequality measures satisfying unit consistency. One advantage of the first two indices is that they also are decomposable, which can be very helpful for some types of empirical analysis.

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