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Abstract

Income is a poor indicator of welfare and should be use only as a as component of welfare. Wealth provides another dimension of well-being. We use the Kullback-Leibler Information Criterion (KLIC, hereafter) as a measure of the discrepancy between two attributes. This will allow economists and researchers to understand the kind of relationship that the attributes hold. The applications deals with an unusual analysis of income factors in which we treat them as different attributes. This allows us to estimate the substitution effects among income factors.

Keywords: entropy, cross-entropy, KLIC, multi-attribute, multifactor, welfare matrix, Theil index, Gini index
JEL Classification: D33

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1 Introduction

Welfare distribution has widely been discussed as the distribution of income. This means that individuals are not distinguished by anything other than income, because their other differences are either not known or are considered as irrelevant, or a mixture of both.[1] There are several reasons to believe that income is a poor indicator of welfare and should be used only as a as component of welfare. Wealth provides another dimension of well-being. Two people who have the same income may not be equally well off if one person has more wealth. If one person owns his home, for example, and the other person doesn't, then the former is better off. Many other attributes that compose an individuals well-being such as consumption, education, health or other attributes, are relevant to the individuals well being. In the literature there has been an increasing interest in the field of multidimensional inequality. First by Kolm [7] followed

[1] See Kolm [7].

by Atkinson and Bourguinon [2] who explored the foundation of multidimensional inequality. In the recent years there has been expanding literature on the topic[2].

We believe that the sources of income are dependent. Furthermore, we believe that the sources of income compensate each other as a response to inequality and/or polarisation. In other words, when one income source diminishes another increases to compensate for either inequality or polarisation. Therefore, it is important to measure the degree of mutual information between the sources of income as an approximation of early stages of polarisation.

The Kullback-Leibler Information Criterion (KLIC, hereafter) is a measure of the discrepancy between two probability distributions. In our case between two sources of income. The KLIC ($KL(v_j, v_i)$) is also referred to as the cross-entropy of v_j relative to v_i. In the discrete case it is defined by:

$$KL(v_j, v_i) = \sum_k v_j \ln(v_j/v_i) \qquad for \quad i \neq j \tag{1}$$

where:

v_j, v_i denote the finite or countable vectors of probability weights implied by the discrete distributions.

$i = (1, 2, ...m)$

$j = (1, 2, ...m)$

$m = 6$: sources of income

v_1 :Probability of recieving earned income: y_1

v_2 :Probability of receiving self employed income: y_2

v_3 :Probability of receiving property income: y_3

v_4 :Probability of receiving income derived from cooperatives: y_4

v_5 :Probability of receiving employers contributions: y_5

v_6 :Probability of receiving other sources of income: y_6

The distribution in the second position of the argument list in $KL(v_j, v_i)$, here v_i, is called the reference distribution. We refer to the first distribution in the argument list, here v_j as the subject distribution. In terms of our research the KLIC $KL(v_j, v_i)$ is the cross-entropy of income source j relative to income source i. In other words, the KLIC between two distributions is the response of income source j relative to the subject distribution of income source i.

Properties

- KL is not symmetric in v_j and v_i and thus in general $KL(v_j, v_i) \neq KL(v_i, v_j)$
- $KL(v_j, v_i) > 0$ for every $v_j > 0$ and
- $KL(v_j, v_i) = 0$ if $v_j = v_i$.
- The greater the value of $KL(v_j, v_i)$, the greater the discrepancy between the distributions v_j and v_j.

[2] See Maasoumi [9], Tsiu [17].

From the fact that the KLIC is not symmetric, note that it is not a true measure of the distance between v_j and v_i and it is referred to instead as a pseudo-distance function. For our purposes it is very important to notice this characteristic since one could easily misinterpret the results. Therefore, this characteristic should be borne in mind and when reporting results, all pairs of KLIC must be interpreted the resulting matrix will not be symmetric.

The KLIC will not only exhibit discrepancy between income factors but also measure how income factors compensate each other.

In terms of our hypothesis, this means that if there is no increased inequality or polarisation then the values for the KLIC will remain the same:

$$KL_t(v_j, v_i) = KL_{t+1}(v_j, v_i)$$

It is only in the presence of inequality or polarisation that the income factors respond to these changes and the KLIC will differ:

$$KL_t(v_j, v_i) \neq KL_{t+1}(v_j, v_i)$$

These attributes makes the KLIC a very interesting measure of polarisation. The results are presented on the following matrix for each year that has been analysed.

Table 1: Kullback-Leibler Information Criterion Matrix

$$\left\{ \begin{array}{cccccc} 0 & KL(v_2, v_1) & KL(v_3, v_1) & KL(v_4, v_1) & KL(v_5, v_1) & KL(v_6, v_1) \\ KL(v_1, v_2) & 0 & KL(v_3, v_2) & KL(v_4, v_2) & KL(v_5, v_2) & KL(v_6, v_2) \\ KL(v_1, v_3) & KL(v_{2,} v_3) & 0 & KL(v_4, v_3) & KL(v_5, v_3) & KL(v_6, v_3) \\ KL(v_1, v_4) & KL(v_2, v_4) & KL(v_3, v_4) & 0 & KL(v_5, v_4) & KL(v_6, v_4) \\ KL(v_1, v_5) & KL(v_2, v_5) & KL(v_3, v_5) & KL(v_4, v_5) & 0 & KL(v_6, v_5) \\ KL(v_1, v_6) & KL(v_2, v_6) & KL(v_3, v_6) & KL(v_4, v_6) & KL(v_5, v_6) & 0 \end{array} \right\}$$

Note that the main diagonal is equal to zero, and that the upper and lower diagonals are different. Therefore the matrix is not symmetric. Thus the Kullback-Leibler Information Criterion (KLIC, hereafter) which is a measure of divergence or distance between distributions. We aim to incorporate our application and other properties of the KLIC to the literature of multidimensional indices. The KLIC is a measure of the discrepancy between two probability distributions, in our case the distributions of two attributes. This will allow economists and researchers to understand the kind of relationship that the attributes hold. By this we mean that attributes may be complement or substitute goods between them.

In this paper we discuss the main definitions of multidimensional inequality, and the way the KLIC fits into the literature. Only two main applications were derived, multifactor analysis for the Mexican case and multidimensional inequality in Mexico. The first application deals with an usual analysis of income factors in which we treat them as different attributes. This allows us to estimate the substitution effects among income factors. The second application deals with the estimation of multidimensional inequality in Mexico using two attributes: monetary and non monetary income. Both applications seemed to have very intuitive results. Unfortunately our data base does not cover other variables apart from monetary and non monetary income and thus the impossibility of estimating multiattribute inequality. And although we do not have data for other attributes, different from income, the procedure of estimation would remain the same. It is kept for further research to obtain non income data and estimate multiattribute inequality using this procedure.

2 Background and Definitions

Let us start by defining the welfare matrix y as:

$$y = \left\{ \begin{array}{cccccccc} y_{11} & y_{12} & y_{13} & . & . & . & y_{1m} \\ y_{21} & . & . & . & . & . & y_{2m} \\ y_{31} & . & . & . & . & . & y_{3m} \\ . & . & . & . & . & . & . \\ y_{n1} & y_{n2} & . & . & . & . & y_{nm} \end{array} \right\} \tag{2}$$

where:

y_{ij} be the amount of attribute m received by individual or household i.

$i = 1, ..., n$: the number of individuals or households and,

$j = 1, ..., m$: the attributes.

Thus, the Welfare Matrix y (2) is a matrix of size $n \times m$. The Welfare Matrix is associated with the Probability Welfare Matrix v (3) in that it is a matrix of size $n \times m$ and each element v_{ij} is interpreted as the probability of individual i of receiving j type of income[3]

$$v = \left\{ \begin{array}{cccccccc} v_{11} & v_{12} & v_{13} & . & . & . & v_{1m} \\ v_{21} & . & . & . & . & . & v_{2m} \\ v_{31} & . & . & . & . & . & v_{3m} \\ . & . & . & . & . & . & . \\ v_{n1} & v_{n2} & . & . & . & . & v_{nm} \end{array} \right\} \tag{3}$$

The two Theil indices which belong to the Generalized Entropy Family (GE hereafter) also used in Chapter 3 are:

$$I_0(S) = \sum_{i=1}^{N} S_i^* \ln(S_i^*/p_i) \tag{4}$$

$$I_1(S) = \sum_{i=1}^{N} p_i \ln p_i/S_i^* \tag{5}$$

where:

$S_i^* = S_i / \sum_j S_j$

[3] $_i \, x_{im} = 1$

$p_i = 1/n$ is the units population share.

The GE are homogenous (in a single attribute), symmetric, and consistent with the Lorenz criterion with respect to S. Useful decomposability properties of $I(\cdot)$.

According to Maasouni the multidimensional inequality index for one of Theil measure is:

$$D_{-1}(v_{if}, S^*; Cf) = \sum_f C_f(\sum_i v_{if} \ln \frac{v_{if}}{S_i^*}) \tag{6}$$

where:
$S_i^* = \sum_f \delta_f v_{if}/K$
$K = \sum_f \delta_f T_f$
$C_f : \delta_f T_f / \sum_k \delta_k T_k$
$T_f := \sum_i v_{if}$
$\delta_f = \alpha_f / \sum_f v_f$

$$D_0(S^*, x_{if}; \delta_f) = \sum_f \delta_f(\sum_i S_i^* v_{if} \ln \frac{S_i^*}{v_{if}}) \tag{7}$$

where:
$S_i^* = \prod_{j=1}^{f} v_{if}^{\delta f}/K$
$K = \sum_i (\prod_f x_{if}^{\delta f})$

Both (6) and (7) are weighted averages of the KLIC. It follows that multidimensional inequality is no more than the weighted average of attribute inequalities and that this reflects the substitution effects between sources of income. Theil's second measure (7) satisfies "general homogeneity" property and Symmetry.

Two main applications stand out. The first is linked with our last chapter and some of the discussion of income sources. We can now estimate the substitution effects between income sources. The second application is regarding the multi-attribute case. Estimating multidimensional inequality incorporating other attributes such as monetary and non monetary income for the Mexican Case.

In both cases the application will use Multidimensional two Theil Index (7) because it satisfies the properties of "general homogeneity" property and symmetry.

3 Multifactor Inequality in Mexico and Income Sources Substitution Effects

The Probability Welfare Matrix v (3) that is a matrix of size $n \times m$ and each element v_{ij} is interpreted as the probability of individual i of receiving j type of income[4]

$$v = \left\{ \begin{array}{cccccccc} v_{11} & v_{12} & v_{13} & . & . & . & v_{1m} \\ v_{21} & . & . & . & . & . & v_{2m} \\ v_{31} & . & . & . & . & . & v_{3m} \\ . & . & . & . & . & . & . \\ v_{n1} & v_{n2} & . & . & . & . & v_{nm} \end{array} \right\} \tag{8}$$

where:

$i = (1, 2, ...n)$
$f = (1, 2, ...m)$
n : The number of households.
$m = 6$: Sources of income
$y_{i,1}$:Earned income,
$y_{i,2}$:Self employed income,
$y_{i,3}$:Property income,
$y_{i,4}$:Income derived from cooperatives,
$y_{i,5}$:Employers contributions and,
$y_{i,6}$:Other sources of income

$$D_0(S^*, v_{im}; \delta_f) = \sum_f \delta_f (\sum_i S_i^* v_{im} \ln \frac{S_i^*}{v_{im}}) \tag{9}$$

where:

$S_i^* = \prod_{f=1}^{M} v_{im}/K$
$K = \sum_i (\prod_f v_{im})$
$\delta_f = \alpha_f / \sum_f \alpha_f$

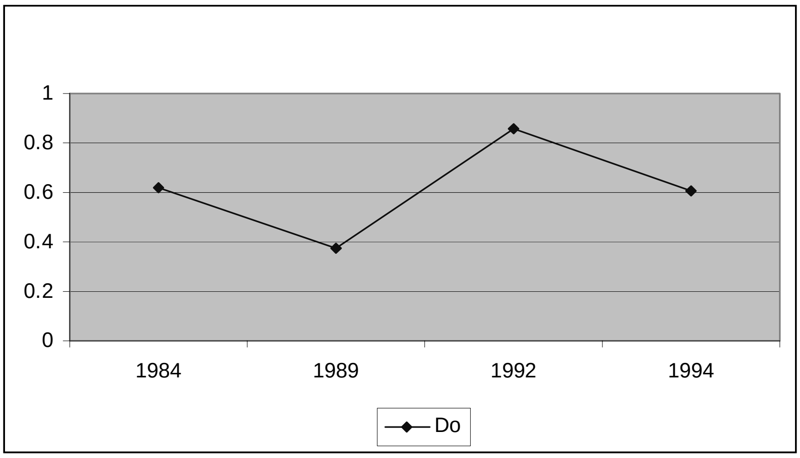
The results obtained were:

Table 2: Multifactor Inequality

	D_0
1984	0.618946669
1989	0.371157319
1992	0.858622255
1994	0.603259151

[4] $_i\ v_{im} = 1$

Figure 1: Multifactor Inequality in Mexico



The values of the Gini Index of Relative Inequality using the covariance approach (**??**) are shown in the following table:

Table 3: Gini Index

Year	**GINI Index**	**Rank**
1984	0.51224303	3
1989	0.50954235	2
1992	0.57664616	4
1994	0.48451204	1

Notice that the results are consistent in trend with those in *Table 3)* but they differ in magnitude. Let's recall Maasoumi's (1986) [9] interpretation of multidimensional inequality."Multidimensional inequality is no more than the weighted average of attribute inequalities. This reflects the substitution effects[5] of the income factors shown on *Table (7).*

The above table not only makes sense but is also consistent with the estimations on Chapter 3 *(See Table (3.1)*. This approach has allowed us to put together most of the results related to income factor analysis and to obtain a value of the substitution effects between factors. The substitution effects are, in our opinion, very valuable estimates because they express how much one source compensates for existing inequality and it also reflects early stages of polarisation.

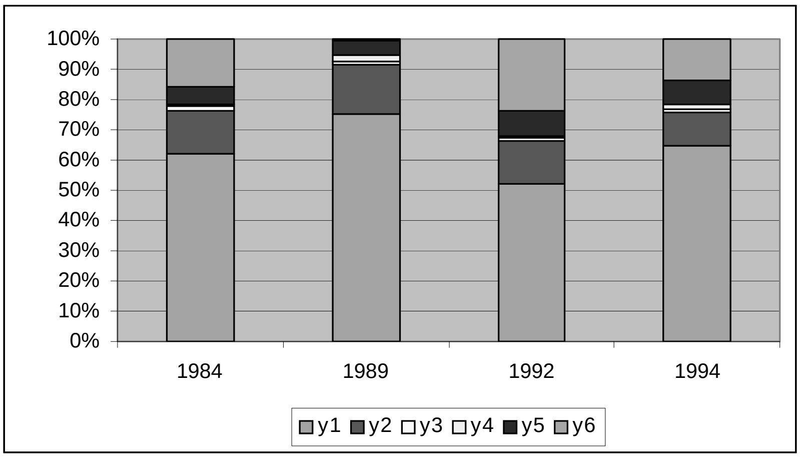
The next graph shows the substitution effects of the income factors using a multifactor analysis.

[5] Substitution Effect: when the price of a good increases it does so relative to all other goods. Although each good is unique it has substitutes - other goods that will serve almost as well. As the opportunity cost of a good rises, people will tend to buy less of it and more of its substitutes.

Table 4: Substitution Effects between Income Factors

	1984	1989	1992	1994
y_1	0.384594786	0.280250495	0.448665933	0.390663415
y_2	0.087517449	0.060059978	0.119302945	0.065346989
y_3	0.0106158	0.004008086	0.012182658	0.006274994
y_4	0.00103653	0.008084923	0.002560909	0.009824298
y_5	0.038089643	0.016918159	0.072060755	0.049327871
y_6	0.097092461	0.001835677	0.203849055	0.081821585

Figure 2: Multifactor, Substitution Effects



We can see that Earned income ($y_{i,2}$) and, Self employed income ($y_{i,2}$) are good substitutes and this is quite reasonable, since most of your income is derived from labour in one way or another. Another income factor with high substitution effect is ($y_{i,6}$) Other sources of income. This factor contains income from the sale of second-hand items, such as selling cars, heritage, lottery, loans, etc. The other values are smaller which means that the other income sources act as complementary goods rather than substitute goods. In the presence of income inequality and/or polarisation, agents will first compensate for it by increasing those income factors with higher substitution effect, rather than the ones that are complements.

3.1 Standard Theory: Decomposition of Inequality by Income Factors

Most index numbers in common usage exhibit some kind of decomposition property that enables the overall index value to be computed from sub-aggregates These sub-aggregates are typically based on grouping together observations which share common characteristics.

The problem can be formally stated if we denote the income of an individual i from source j by y_{ij} $(j = 1, 2, ...m)$ for m types of income. The distribution of factor j is represented by $(y_{1j}, ...y_{nj}) = y_j$. Assuming that the factor income categories are mutually exclusive, and that the total income is the sum of individual factor incomes, we have

$$y_i = \sum_{j=1}^{m} y_{ij}$$

Thus, the assessment of inequality contribution of factor j becomes the problem of determining the impact of y_j on y_i. If for any inequality index $I(y_i)$ we have some way of writing $I(y_i) = \sum_{j=1}^{n} \beta_j$, where β_j depends on income from source j, then β_j might be regarded as the contribution of factor j to aggregate inequality. Equivalently, the fractions $b_j = \frac{\beta_j}{I(y_i)}$ can be interpreted as proportional factor contributions.

We say that a function is a decomposition rule if it produces suitable values of b_j with the following properties:

$$\sum b_j = 1$$
$$\sum \beta_j = I(y_i)$$

For the Gini Index the decomposition rule is :

$$bj = \frac{\sum_{i=1}^{n} \left[i - \frac{n+1}{2}\right] \overline{y_j}}{\sum_{i=1}^{n} \left[i - \frac{n+1}{2}\right] \overline{y_i}} \tag{10}$$

where:

$$\sum b_j = 1$$
$$\sum \beta_j = GINI$$

The following results were obtained:

Table 5: Factor Contributions to Inequality

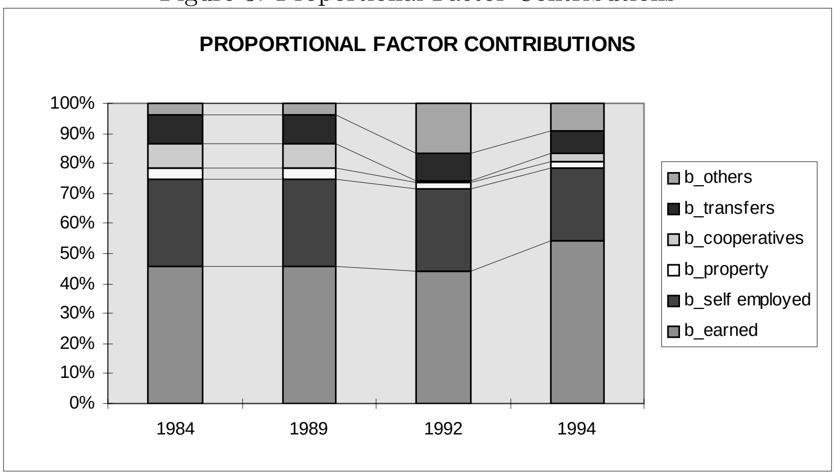
	1984	**1989**	**1992**	**1994**
β_{EARNED}	0.2365332	0.2291129571	0.22402225523	0.26451417461
$\beta_{SELF\_EMPLOYED}$	0.1339064	0.14888827467	0.1718982203	0.11521696311
$\beta_{PROPERTY}$	0.025268	0.02109505329	0.01228256321	0.00935108237
$\beta_{COOPERATIVES}$	0.0011633	0.04162961	0.00351754158	0.01366323953
$\beta_{TRANSFERS}$	0.0523992	0.04957847066	0.05708796984	0.03493331808
β_{OTHERS}	0.0629652	0.01926070083	0.10431529034	0.04278241313
$\sum \beta_i = Gini$	0.5122354	0.50956506654	0.5731238405	0.48046119084

Table 6: Proportional Factor Contributions to Inequality

	1984	**1989**	**1992**	**1994**
b_{EARNED}	0.4736	0.4677	0.4752	0.5253
$b_{SELF\_EMPLOYED}$	0.2613	0.2922	0.2981	0.2378
$b_{PROPERTY}$	0.0493	0.0414	0.0213	0.0190
$b_{COOPERATIVES}$	0.0023	0.0817	0.0061	0.0282
$b_{TRANSFERS}$	0.1023	0.0973	0.0990	0.0721
b_{OTHERS}	0.1229	0.0378	0.1809	0.0883
$\sum b_i$	1	1	1	1

It is not surprising that the highest factor of contribution is due to what can be described as payment for any kind of labour, including earned income and self-employed income, but there is one result that is worthy of more attention. This is the fact that the rest of the factor contributions reversed their positions in 1992.

Figure 3: Proportional Factor Contributions



4 Multidimensional Inequality in Mexico

For this application we will define welfare the matrix as W:

$$W = \left\{ \begin{array}{cc} w_{11} & w_{1,2} \\ w_{2,1} & w_{22} \\ w_{31} & w_{32} \\ . & . \\ . & . \\ . & . \\ w_{n1} & w_{n2} \end{array} \right\} \tag{11}$$

where:
$i = (1, 2, ...n)$
$f = (1, 2)$
n : The number of households.
$m = 2$ There are two attributes monetary income and non monetary income.

$$D_0(S^*, v_{im}; \delta_f) = \sum_f \delta_f (\sum_i S_i^* v_{im} \log \frac{S_i^*}{v_{im}}) \tag{12}$$

where:
$S_i^* = \prod_{f=1}^{M} v_{im}/K$
$K = \sum_i (\prod_f v_{im})$
$\delta_f = \alpha_f / \sum_f \alpha_f$
The results obtained were:

Table 7: Multidimensional Inequality

	D_0
1984	0.626453124
1989	0.603724627
1992	0.638633253
1994	0.59023655

These results are consistent with the Gini Coefficient *(See Table* **??***)* and with the previous results they differ in magnitude but they follow the same trend. The difference in magnitude should be a topic of further research.

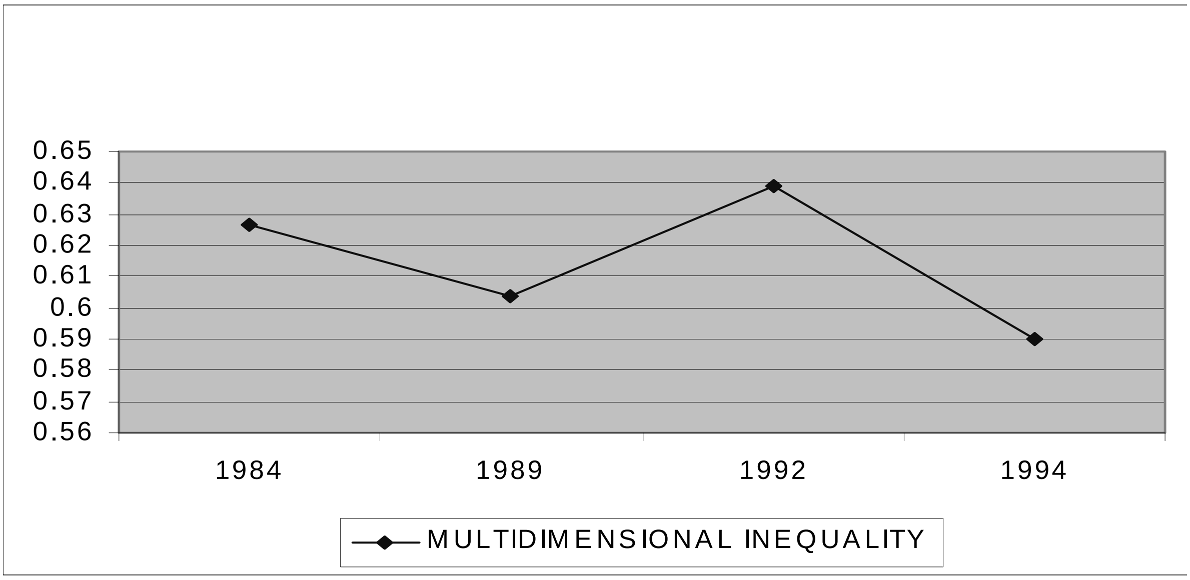
The substitution effects are shown on the following table:

Table 8: Substitution Effects between Monetary and Non-Monetary Income

	1984	1989	1992	1994
Monetary	0.451266179	0.430240301	0.497773814	0.372933906
Non-Monetary Income	0.175186945	0.173484326	0.140859439	0.217302643

The substitution effects between monetary and non-monetary income are consitent with intuition. Monetary income has a higher substitution effect.

Figure 4: Multidimensional Inequality



5 Conclusions

Two main applications of multidimensional inequality were drawn in this chapter. The first is regarding multifactor income inequality, maybe the most interesting application for the purposes of the research. This concluding chapter manages to put together consistently the different estimations for income factor contribution to overall inequality into one single index with the most useful interpretation of the substitution effects among factors.

There are several ways to express the degree of income inequality in a society, we have chosen to decompose income and the participation of each income factor into overall income inequality. This technique is relatively simple and revealing; it has allowed us to understand not only the impact of each income source to overall inequality but also it reveals the relationship between those income factors. When we first decompose inequality into income factors we did it believing there was a relationship between income factors and that those were not independent. *(See Table* **??***)* We used standard theory to describe the participation of each income source into overall inequality. Those estimations made us suspect about the relationship among income factors and the final results presented in this chapter showed that the relationship between income factors may be expressed in terms of their substitution effects. In practical terms it means that policy makers not only have, more information but a more powerful one. Policies designed to decrease income inequality may do so, by increasing by any means income derived from sources that act as complementary "goods". Those income sources that reflect payment to labour have higher substitution effects the other sources of income may be regarded as complementary "goods". The other application deals with multi-attribute inequality, applied to monetary and non monetary income. Monetary income is able to substitute non monetary income in a higher proportion than non monetary income. The applications of multi-attribute inequality have a wide range. Attributes such as education, health and any other attribute that contributes to the individual well being.

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