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Unit-Consistent Aggregative Multidimensional Inequality Measures: A Characterization*

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Abstract

Inequality among people involves comparisons of social indicators such as income, health, education and so on. In recent years the number of studies both theoretical and empirical which take into account not only the individual's income but also these other attributes has significantly increased. As a consequence the development of measures capable of capturing multidimensional inequality and satisfying reasonable axioms becomes a useful and important exercise.
The aim of this paper is no other than this. More precisely, we consider the unit consistency axiom proposed by B. Zheng in the unidimensional framework. This axiom demands that the inequality rankings, rather than the inequality cardinal values as the traditional scale invariance principle requires, are not altered when income is measured in different monetary units. We propose a natural generalization of this axiom in the multidimensional setting and characterize the class of aggregative multidimensional inequality measures which are unit-consistent.

Keywords: multidimensional inequality indices, unit-consistency, aggregativity.
JEL Classification: D63

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1. INTRODUCTION

This work takes as a reference two recent papers concerning inequality measurement. The first one is Zheng (2007) who introduces a new unit consistency axiom in the unidimensional context and characterises families of inequality measures that fulfil this axiom. The other starting point is Tsui (1999) who derives the class of multidimensional relative inequality measures.

There are several answers to the question of how to distribute an additional amount of income among the whole population without changing the initial inequality level. Whereas the rightist view, according to Kolm's designation (1976), demands a proportional distribution and asks that the inequality measure be scale invariant, the leftist view requires that inequality remains unchanged when each individual in the population receives the same amount of the extra income, and as a consequence, they insist that the inequality measure should fulfil the translation invariance principle. The centralist view, in turn, argues for a combination of these two answers. Examples of measures which correspond to this point of view can be found, among others, in Kolm (1976), Bossert and Pfingsten (1990), Seidl and Pfingsten (1997), Chakravarty and Tyagarupananda ((1998), (2000)) and del Río and Ruiz-Castillo (2000).

Zheng (2007) accurately argues that all these invariance conditions, usually invoked as axioms to characterise most inequality measures impose value judgements in measuring inequality and there is no justification for any of them should be assumed to characterise an inequality measure. On the other hand, it is true that it makes no sense that inequality comparisons vary when income is measured in different monetary units. So Zheng introduces a new axiom, the unit consistency axiom which requires that the inequality rankings, rather than the inequality level, be not affected by the units in which incomes are expressed, such as dollars versus euros.

It is important to notice the differences between the scale invariance principle and the unit consistency axiom. It is clear that the scale invariance principle implies unit-consistency. However whereas the former is a cardinal condition, the later is an ordinal requirement. In other words, this axiom allows inequality values to vary when monetary units change, provided the inequality orderings are not altered. Zheng characterises the class of both decomposable (Zheng (2007)) and aggregative (Zheng (2005)) unit-consistent inequality measures. The families derived by Zheng are generalizations of the well-known Generalized Entropy family.

On the other hand after the seminal articles by Kolm (1977) and Atkinson and Bourguignon (1982) several researchers (Maasoumi (1986), Dardadoni (1995), Tsui ((1995), (1999)), List (1999), Weymark (2004), Gajdos and Weymark (2005), Savaglio (2006), Koshevoy and Mosler (2007), among others) are aware that in order to better answer the two questions posed by Sen (1997): “What is inequality?” and “Inequality of what?” it is necessary to take into account differences not only in income but also in other attributes related to health and education. Consequently, it makes sense to extend axioms regarded as suitable in measuring income inequality to the multidimensional context and develop multidimensional inequality measures which are able to summarize inequalities as regards different attributes. From an axiomatic non-welfarist approach Tsui (1999) is a prominent example in this field. He proposes a correlation-increasing majorization criterion and characterises the class of *Multidimensional Generalized Entropy measures* deriving a generalization of the Generalized Entropy family.

In this paper we propose a straightforward extension of the unit-consistency axiom to the multidimensional setting and characterise the class of multidimensional aggregative inequality measures which are unit-consistent. The derived family is actually a generalization of the families characterised by Tsui (1999) and Zheng (2005), and hence a generalization of the Generalized Entropy family.

The paper is structured as follows. The section below presents the notation and the definitions used in the paper. In Section 3 we introduce the generalization of the unit-consistency axiom to the multidimensional framework and present our characterisation results which are proved in the Appendix. Finally, Section 4 offers some concluding remarks. Most of the proofs of our paper follow both Zheng ((2005), (2007)) and Tsui (1999) papers and the relevant results by Shorrocks (1984) as well.

2. Notation and Basic Axioms of Multidimensional Inequality Indices

We consider a population consisting of $\mathrm{n} \geq 2$ individuals endowed with a bundle of $\mathrm{k} \geq 1$ attributes, such as income, health, education and so on. A multidimensional distribution is represented by a $n \times k$ -matrix $X = \left(x_{ij}\right)$, where $x_{ij} > 0$ represents *i*th individual's amount of *j*th attribute. The *i*th row of X is denoted by $\underline{x}_i$, the *j*th column is denoted by $\overline{x}^j$, $\mu_j\left(X\right)$ represents the mean value of the *j*th attribute and $\underline{\mu}\left(X\right) = \left(\mu_1\left(X\right),...,\mu_k\left(X\right)\right)$ is the vector of the means of attributes. The set of all the $n \times k$ -matrices over the positive real elements is denoted $M_{++}\left(\mathrm{n,k}\right)$ and *D* is the set of all such matrices.

A multidimensional inequality index is defined as a function $I : D \rightarrow \mathbb{R}$. In this paper, we assume that I possesses the four following properties, which are straightforward generalizations of their familiar one-dimensional equivalents:

i) *Continuity*: I is a continuous function in any individual's attributes.

ii) *Anonymity*: $I\left(X\right) = I\left(PX\right)$ for any $X \in M_{++}\left(\mathrm{n,k}\right)$ and for all $n \times n$ permutation matrices *P*.

iii) *Normalization*: $I(X)=0$ if all the rows of the matrix X are identical, i.e., all the individuals have exactly the same bundle of attributes.

iv) *Replication Invariance*: $I(Y)=I(X)$ if Y is obtained from X by a replication.

The above axioms are insufficient to guarantee that function I be able to capture the essence of multidimensional inequality and to establish whether one multidimensional distribution is more unequal than another. The well-known Pigou-Dalton transfer principle is the basic axiom to order unidimensional distributions in terms of inequality. It should be noted that, in the univariate context, this principle has a number of equivalent formulations (Hardy, Littlewood and Pólya (1934, 1952), Marshall and Olkin (1979)). The two following criteria, which are generalizations of two different mathematical formulations of this principle to the multivariate framework proposed by Kolm (1977) [1], are used in this paper:

Definition: *A Pigou-Dalton matrix* is an $n\times n$-matrix $T=\lambda E+(1-\lambda)P$, $0\leq\lambda\leq 1$, where E is the $n\times n$ identity matrix and P is a $n\times n$ permutation matrix which transforms other matrices by interchanging two rows.

v) *Uniform Pigou-Dalton Majorization (UPD):* A multidimensional inequality measure I is said to satisfy UPD if $I(TX)<I(X)$ for all $X\in M_{++}(\text{n,k})$ and for all $n\times n$ matrix which is a finite product of Pigou-Dalton matrices which are not permutation matrices of the rows of X.

vi) *Uniform Majorization (UM):* A multidimensional inequality measure I is said to satisfy UM if $I(BX)<I(X)$ for any $X\in M_{++}(\text{n,k})$ and for all $n\times n$ bistochastic matrix B that is not a permutation matrix of the rows of X.

[1] Apart from Kolm (1977) other generalizations of the Pigou-Dalton transfer principle to the multidimensional setting can be found in Marshall and Olkin (1979), Koshevoy and Mosler (2007), Fleurbaey and Trannoy (2003) and Savaglio (2006).

These criteria establish that multidimensional inequality should be a function of the uniform inequality of a multivariate distribution of attributes across people. On the other hand, Atkinson and Bourguignon (1982) and Walzer (1983) point out that a multidimensional inequality measure should also be sensitive to the cross-correlation between inequalities in different dimensions. This idea is captured by Tsui (1999) who introduces a new majorization criterion based on the concept of arrangement increasing transfers defined by Boland and Proschan (1988):

Definition. A distribution *Y* may be derived from a distribution *X* by *a correlation increasing transfer* if $X \neq Y$, *X* is not a permutation of *Y*, and there exist row indices p and q such that: i) $\underline{y}_p = \left(\min\left\{x_{p1}, x_{q1}\right\}, ..., \min\left\{x_{pk}, x_{qk}\right\}\right)$, ii) $\underline{y}_q = \left(\max\left\{x_{p1}, x_{q1}\right\}, ..., \max\left\{x_{pk}, x_{qk}\right\}\right)$ and iii) $\underline{y}_m = \underline{x}_m \ \ \forall m \neq p, q$.

Tsui (1999) formally introduces the Correlation Increasing Principle as follows:

vii) *Correlation Increasing Principle (CIM)*: A multidimensional inequality measure *I* is said to satisfy CIM if $I(X) < I(Y)$ whenever *Y* may be derived from *X* by a permutation of rows and a finite sequence of correlation increasing transfers.

CIM has an intuitive interpretation. We may imagine the situation in which the first individual in the society receives the lowest amount of each attribute; the second individual is endowed with the second lowest amount, up to the individual n which receives the greatest amount of each attribute. CIM ensures that this distribution is the most unequal in the sense that any other distribution matrix of the same amount of attributes is more equal than it[2].

[2] Bourguignon and Chakravarty (2003) make some objections to this axiom arguing that CIM is not sensitive to individual preferences and somehow implies that the attributes are substitutable. In turn Tsui (1999) and particularly Tsui (2002) highlight what CIM really means in the context of both inequality and poverty.

If the population in which we want to measure inequality is split into groups according to characteristics such as age, gender, race or area of residence, it seems desirable to demand some properties which allow us to relate inequality in each group to overall inequality. A minimal requirement is to demand that if inequality in one group increases, the overall inequality should also increase. This property proposed by Shorrocks (1984) in the unidimensional framework is generalized for multidimensional distributions in the following way:

viii) *Aggregative Principle*: A multidimensional inequality measure I is said to be aggregative if there exists a function A such that $I(X) = A\left(I(X_1), \underline{\mu}(X_1), \mathrm{n}_1, I(X_2), \underline{\mu}(X_2), \mathrm{n}_2\right)$, for all $X_1, X_2 \in D$ and A is a continuous and strictly increasing function in the index values $I(X_1)$ and $I(X_2)$.

This property is also known as Decomposability in some papers. Tsui (1999) proves that UPD and UM are equivalent for any multidimensional aggregative inequality index.

In the literature on inequality indices, invariance properties are often invoked.

ix) *Scale Invariance Principle*: A multidimensional inequality measure I is said to be scale invariant if $I(X) = I(X\Lambda)$, for all $\Lambda \in M_{++}(\mathrm{k,k}) / \ \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_k)$.

Relative inequality indices are those that are scale invariant.

x) *Translation Invariance Principle*: A multidimensional inequality measure I is said to be translation invariant if $I(X) = I(X + A)$, for all matrix A with identical rows $\underline{a} = (a_1, a_2, ..., a_k)$ and $a_j \geq 0$.

Absolute inequality indices are those that are translation invariant.

3. MULTIDIMENSIONAL UNIT-CONSISTENT MEASURES.

The above section ends with two possible answers as to how to distribute a given amount of attributes among all the individuals without altering inequality level. As already mentioned, in the unidimensional framework Zheng (2007) has analysed in depth the value judgements involved in the different ways in which this problem is faced and has proposed a new axiom of unit-consistency which requires that the inequality ranking between two distributions should not be affected by the unit in which income is expressed.

This axiom has a straightforward generalization to the multidimensional framework allowing several attributes to be measured in different units without changing the inequality rankings of the multidimensional distributions. Actually properties of this kind have already proposed in the literature as regards the social welfare functions which underlie the multidimensional relative indices (Tsui (1995) and Gajdos and Weymark (2005), for instance).

The natural generalization of the unit-consistency axiom to the multidimensional framework is the following:

xi) *Unit-Consistency Axiom*: A multidimensional inequality measure I is said to be unit-consistent if for any two multidimensional distributions $X, Y \in M_{++}(\text{n,k})$ such that $I(X) < I(Y)$ then $I(X\Lambda) < I(Y\Lambda)$ for any $\Lambda \in M_{++}(\text{k,k}) / \ \Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_k)$.

Similarly to the unidimensional context, also in the multidimensional one the scale invariance principle implies unit-consistency, and hence, every relative multidimensional inequality measure is unit-consistent. Unfortunately, none of the rest of the multidimensional indices traditionally used in the literature fulfils this property (it is straightforward to prove that the unit-consistency axiom is not met by, among others, the members which are not relative in the Maasoumi (1986) and Bourguignon (1999) families, the multidimensional generalization of the absolute Akinson-Kolm-Sen index proposed by Tsui (1995) and the multidimensional generalizations of the absolute Gini indices by Gajdos and Weymark (2005)).

Before formally characterising the aggregative multidimensional inequality measures which are unit-consistent, it is useful to identify the functional implication of the unit-consistence axiom for a general multidimensional index of inequality. All the proofs are presented in the Appendix.

Proposition 1*: A multidimensional inequality index $I : D \to \mathbb{R}$ is unit-consistent if and only if for any multidimensional distribution $X \in M_{++}(\mathrm{n,k})$ and for any diagonal matrix $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_k)$ with $\lambda_j > 0$, there exists a continuous function $f : \mathbb{R}^{\mathrm{k}}_{++} \times \mathbb{R} \to \mathbb{R}$ increasing in the last argument such that*

$$I(X\Lambda) = f(\lambda_1, \lambda_2, ..., \lambda_k; I(X)) \qquad (1)$$

This result reveals that in fact if any changes in the attribute units have no influence on inequality rankings, both the unit change matrix Λ, and the inequality value $I(X)$ must enter into $I(X\Lambda)$ independently.

The main objective of this section is to characterise the entire class of unit-consistent aggregative multidimensional inequality measures. The main results of our work are the two following theorems.

Theorem 2*: A multidimensional inequality measure $I : D \to \mathbb{R}$ satisfies UM (UPD), the Aggregative Principle and the Unit-Consistency Axiom if and only if there exists a continuous increasing transformation $F : \mathbb{R} \to \mathbb{R}_+$, with $F(0) = 0$, such that for any $X \in M_{++}(\mathrm{n,k})$ either:*

$$F\left(I(X)\right)=\frac{\rho}{\mathrm{n}\prod_{j=1}^{k}\mu_j^{\alpha_j-\tau}}\sum_{i=1}^{\mathrm{n}}\left[\prod_{j=1}^{\mathrm{k}}\left(x_{ij}\right)^{\alpha_j}-\prod_{j=1}^{\mathrm{k}}\left(\mu_j\right)^{\alpha_j}\right] \tag{2}$$

where $\tau\in\mathbb{R}$ *and the parameters* α_j *and* ρ *have to be chosen such that the function* $\phi\left(\underline{x}_i\right)=\rho\prod_{1\le j\le k}\left(x_{ij}\right)^{\alpha_j}$ *is strictly convex for all i.*

or

$$F\left(I(X)\right)=\frac{1}{\mathrm{n}\prod_{j=1}^{k}\mu_j^{-\tau}}\sum_{i=1}^{\mathrm{n}}\left(\frac{x_{im}}{\mu_m}\right)\left[\sum_{j=1}^{\mathrm{k}}a_{mj}\log\left(\frac{x_{ij}}{\mu_j}\right)\right] \tag{3}$$

where $\tau\in\mathbb{R}$ *and* $\mathrm{m}\in\{1,2,...,\mathrm{k}\}$ *and the parameters* a_{mj} *have to be chosen such that the function* $\phi\left(\underline{x}_i\right)=\sum_{j=1}^{k}\frac{x_{im}a_{mj}}{u_m}\log(x_{ij})$ *is strictly convex.*

or

$$F\left(I(X)\right)=\frac{1}{\mathrm{n}\prod_{j=1}^{\mathrm{k}}\mu_j^{-\tau}}\sum_{i=1}^{\mathrm{n}}\sum_{j=1}^{\mathrm{k}}\delta_j\log\left(\frac{\mu_j}{x_{ij}}\right) \tag{4}$$

where $\tau\in\mathbb{R}$ *and* $\delta_j>0$ *for all j.*

As already mentioned CIM is a compelling axiom to order rank matrix distributions in terms of inequality. If this property is also assumed then only the first of these expressions remains with additional conditions upon the coefficients.

Theorem 3*: A multidimensional inequality measure* $I:D\to\mathbb{R}$ *satisfies UM (UPD), CIM, the Aggregative Principle and the Unit-Consistency Axiom if and only if there exists a continuous increasing transformation* $F:\mathbb{R}\to\mathbb{R}_+$*, with* $F(0)=0$*, such that for any* $X\in M_{++}(\mathrm{n,k})$

$$F\left(I(X)\right) = \frac{\rho}{\mathrm{n}\prod_{j=1}^{\mathrm{k}} \mu_j^{\alpha_j - \tau}} \sum_{i=1}^{\mathrm{n}} \left[\prod_{j=1}^{\mathrm{k}} \left(x_{ij}\right)^{\alpha_j} - \prod_{j=1}^{\mathrm{k}} \left(\mu_j\right)^{\alpha_j} \right] \qquad (5)$$

where $\tau \in \mathbb{R}$, $\rho > 0$, $\alpha_j < 0, j = 1, 2, ..., k$.

Some remarks about the families derived in Theorem 2 and Theorem 3

i) Assuming the most usual majorization criteria we have derived the family of unit-consistent aggregative multidimensional inequality measures. As already mentioned unit-consistency is a minimal requirement in the sense that it only demands that inequality orderings are not altered when the units in which attributes are measured change. On the other hand, if the population is split into groups, the aggregative principle is also a minimal requirement which only demands that overall inequality should increase if one group inequality increases. Then in empirical applications it makes sense to choose measures from these families.

ii) If only one attribute is taken into consideration the families characterised above coincides, up to a constant, with the families identified by Zheng (2005). Moreover, if $\tau = 0$ the family identified in Theorem 2 coincides, up to a constant, with the Generalized Entropy family and, interestingly enough, the subfamily fulfilling CIM, Theorem 3, corresponds to the tail of this family which meets the transfer sensitive principle according to Schorrocks and Foster (1987).

iii) When we take the transformation F equal to the identity in Theorems 2 and 3, we find what can be considered “canonical forms” of these unit-consistent measures. As shown in the proofs, these forms fulfil a decomposition property, a sort of generalization of the additive decomposition in the unidimensional framework: for these measures overall inequality can be expressed as the sum of the inequality level of a hypothetical distribution in which each person's attributes are replaced by the corresponding means of their group and a weighted sum of the group inequality levels.

iv) For these canonical forms it holds that $I\left(X\Lambda\right)=\left(\prod_{1\leq i\leq k}\lambda_i\right)^{\tau}I\left(X\right)$. As a consequence they are relative measures if and only if $\tau=0$. These cases correspond exactly with the two families which Tsui (1999) characterises in Theorems 3 and 4. In other words, the families obtained in this paper are extensions of the two respective classes derived by Tsui (1999). In addition, remaining $\tau=0$ and taking a suitable increasing function F in Theorem 2 we obtain the multidimensional generalization of the relative Akinson-Kolm-Sen index (Tsui (1995).

v) On the other hand, when $\tau>0$ inequality increases when any attribute is increased for all people in the same proportion. These measures represent points of view designated as "variable views" according to Amiel and Cowell (1997) since the value judgements represented by these measures can vary from the intermediate to the extreme leftists depending on different distributions. In contrast, an extreme rightist view holds when $\tau<0$, since in these same situations inequality decreases.

vi) As regards absolute measures, it can be proved that none of the members identified in Theorem 2 fulfils the Translation Invariance Principle, even if only one of the attributes is affected by an absolute change[3]. In other words, in empirical applications if researchers consider dimensions for which it makes sense relative changes without changing inequality rankings, they should be aware that in these cases it is not possible to take into consideration also categorical variables for which absolute changes are bound to alter inequality values.

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[3] In fact it can be proved that given $A\in M_{+}\left(\text{n,k}\right)$ with identical rows $\underline{a}=\left(a_1,a_2,...,a_k\right)$ and $a_j\geq 0$, then $\left.\partial I(X+A)\big/\partial a_l\right|_{a_l=0}=0$ if and only if $\tau\prod_{1\leq j\leq k}\mu_j^{\alpha_j}=\frac{1}{n}\sum_{1\leq i\leq n}\prod_{1\leq j\leq k}x_{ij}^{\alpha_j}\frac{\mu_l\alpha_l-x_{il}\left(\alpha_l-\tau\right)}{x_{il}}$, but this is impossible since the right side term, taking into account that $\alpha_j<0$ tends to infinite when x_{il} tends to 0 whereas the left side term is a constant.

4. Concluding Remarks

There is a well-known result in the unidimensional setting established by Shorrocks (1984): the members of the Generalized Entropy family can be considered canonical forms of all aggregative relative inequality measures.

Tsui (1999) generalizes this result to the multidimensional setting deriving canonical forms of all multidimensional relative aggregative inequality measures. In turn, Zheng (2005) does the same replacing the scale invariance principle by the unit consistency axiom. In this paper we merge these two generalizations to identify the canonical forms of all the multidimensional unit-consistent aggregative inequality measures. As already mentioned the families we derive are generalizations of both Tsui and Zheng families, and consequently of the Generalized Entropy family.

In recent years several researchers are becoming aware that inequality is not just about differences in income and therefore other attributes related to health or education should also be taken into consideration in measuring inequality. Many efforts have been made in this field from both a normative and an axiomatic point of view.

In empirical applications concerned with the measure of inequality in a population classified into groups, both the aggregative principle and the unit-consistency are minimal requirements for an inequality measure. The families identified in this paper meet both properties and allow us to adopt different value judgements in measuring inequality. We hope that our paper will also be a contribution to this field.

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APPENDIX

Proof of Proposition 1: For any $X \in \mathrm{M}_{++}(\mathrm{n,k})$ and for any $\Lambda \in M_{++}(\mathrm{k,k})$ / $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_k)$, we define $J(X) = I(X\Lambda)$. The unit-consistency axiom implies that if $I(X) = I(Y)$ then $I(X\Lambda) = I(Y\Lambda)$, i.e., $J(X) = J(Y)$. Moreover, it also implies that if

$I(X)<I(Y)$ then $J(X)<J(Y)$. As a result it follows that $J(X)$ is an increasing function in $I(X)$. Hence, there exists an increasing function $f_{\lambda_1,\lambda_2,\ldots,\lambda_k}:\mathbb{R}\to\mathbb{R}$ such that

$$J(X)=f_{\lambda_1,\lambda_2,\ldots,\lambda_k}\left(I(X)\right) \qquad (6)$$

Since both $J(X)$ and $I(X)$ are continuous functions of *X*, it follows that $f_{\lambda_1,\lambda_2,\ldots,\lambda_k}(.)$ is also a continuous function. Defining $f:\mathbb{R}^{k}_{++}\times\mathbb{R}\to\mathbb{R}$ by $f(\lambda_1,\lambda_2,\ldots,\lambda_k;.)=f_{\lambda_1,\lambda_2,\ldots,\lambda_k}(.)$ we have

$$I(X\Lambda)=J(X)=f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(X)\right) \qquad (7)$$

where $f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(X)\right)$ is also a continuous function in the first arguments $\lambda_1,\lambda_2,\ldots,\lambda_k$. Indeed, for any j=1,…,k, infinitesimal changes in λ_j produce simultaneous infinitesimal changes in the $\lambda_j x_{ij}$s. Therefore, since *I* is a continuous function, they also produce small changes in $I(X\Lambda)$, and, as a consequence, f is continuous in λ_j, which completes the proof of the necessity. The sufficiency of the proposition is straightforward.

Q.E.D

In order to prove theorem 2 and consequently the particular situation considered in theorem 3, we follow two steps. Firstly we get a characterization theorem for a subfamily which meets a sort of decomposition property which demands that overall inequality can be expressed as the sum of the inequality level of a hypothetical distribution in which each person's attributes are replaced by the corresponding means of their group and a weighted sum of the group inequality levels. Then, following the equivalent unidimensional, we show that every aggregative measure can be expressed as an increasing transformation of one member of this family.

Let's begin with a previous definition and some results.

***Decomposition Property**:* If any population is classified in G non-empty subgroups $X=\left(X_1,X_2,...,X_G\right)$, the inequality index I is said to meet the decomposition property if the following relationship between the total inequality value $I\left(X\right)$ and the subgroup inequality values $I\left(X_g\right)$ holds:

$$I\left(X\right)=I\left(X_1,X_2,...,X_G\right)=\sum_{g=1}^{G} w_g\left(\underline{\mu}\left(X_g\right),\mathrm{n}\left(X_g\right)\right)I\left(X_g\right)+I\left(A_1\Lambda_1,...,A_G\Lambda_G\right)$$

where w_g is the weight attached to subgroup *g*, $A_g\in M_{++}\left(\mathrm{n}\left(\mathrm{X_g}\right),\mathrm{k}\right)$ of 1´s and $\Lambda_g\in M_{++}\left(\mathrm{k,k}\right)/\Lambda_g=diag\left(\mu_1\left(X_g\right),\mu_2\left(X_g\right),...,\mu_k\left(X_g\right)\right)$ for *g*=1,…,G.

***Lemma 5**: If a multidimensional inequality measure $I:D\to\mathbb{R}$ satisfies UM (UPD), Decomposition and the Unit-Consistency Axiom, then*

$$I\left(X\Lambda\right)=\left(\prod_{1\le j\le k}\lambda_j\right)^{\tau}I\left(X\right) \quad (8)$$

for any $X\in M_{++}\left(\mathrm{n,k}\right)$ and $\Lambda\in M_{++}\left(\mathrm{k,k}\right)/\Lambda=diag\left(\lambda_1,\lambda_2,...,\lambda_k\right)$, and some constant $\tau\in\mathbb{R}$.

Moreover I is a homogenous function of degree $k\tau$.

Proof: (Following Shorrocks (1984) and Zheng (2007)). For any multidimensional distribution $X\in M_{++}\left(\mathrm{n,k}\right)$ let $w\left(X\right)=\left(\underline{\mu}\left(X\right),n\left(X\right)\right)=\left(\underline{\mu},\mathrm{n}\right)\in\mathbb{R}_{++}^{k+1}$ be a "parameter-vector" for the distribution X.

The set of $\mathrm{X}\in D$ with a common parameter-vector *w*, constitutes the set $S\left(w\right)=\left\{X\in D/w\left(X\right)=w\right\}$. For each *w*, *S(w)* is a connected, open subset of *D* containing more than one element. Hence, by continuity, normalization and UM (UPD)

$$I\left(S(w)\right)=\left\{I(X)/X\in S(w)\right\}=\left[0,\xi(w)\right)$$

where $\xi(w)$ is strictly positive and may be finite and infinite.

Define $\Omega=\left\{w(X)/X\in D\right\}$. For each $w=\left(\underline{\mu},n\right)\in\Omega$ let X and Y be any two distributions with a common parameter vector *w*. By definition, $\underline{\mu}(X)=\underline{\mu}(Y)=\underline{\mu}$ and $n(X)=n(Y)=\mathrm{n}$. Now consider a new distribution $Z=(X,Y)$. Since *I* is a decomposable measure, we have

$$I(Z)=w_1\left(\underline{\mu},\mathrm{n}\right)I(X)+w_2\left(\underline{\mu},\mathrm{n}\right)I(Y) \tag{9}$$

where $w_1\left(\underline{\mu},\mathrm{n}\right)$ and $w_2\left(\underline{\mu},\mathrm{n}\right)$ are the weights for distributions *X* and *Y* respectively. The between-group inequality term in (9) is equal to 0 since *I* satisfies the normalization principle. Note also that $\underline{\mu}(Z)=\underline{\mu}$ and $n(Z)=2\mathrm{n}$.

Now multiplying the distributions *X*, *Y* and *Z* by any $\Lambda\in M_{++}(\mathrm{k,k})/$ $\Lambda=diag\left(\lambda_1,\lambda_2,\ldots,\lambda_k\right)$ we have

$$I(Z\Lambda)=w_1\left(\underline{\mu}\Lambda,\mathrm{n}\right)I(X\Lambda)+w_2\left(\underline{\mu}\Lambda,\mathrm{n}\right)I(Y\Lambda) \tag{10}$$

Assuming that *I* is unit-consistent and taking into account the proposition 1 there exists a continuous function which is increasing in the last argument, $f:\mathbb{R}^{\mathrm{k}}_{++}\times\mathbb{R}\to\mathbb{R}$ such that

$$\begin{aligned}&f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(Z)\right)=\\&w_1\left(\underline{\mu}\Lambda,\mathrm{n}\right)f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(X)\right)+w_2\left(\underline{\mu}\Lambda,\mathrm{n}\right)f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(Y)\right)\end{aligned} \tag{11}$$

Substituting (9) into (11) we further have

$$f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;w_1\left(\underline{\mu},\mathrm{n}\right)I(X)+w_2\left(\underline{\mu},\mathrm{n}\right)I(Y)\right)=$$

$$=w_1\left(\underline{\mu}\Lambda,\mathrm{n}\right)f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(X)\right)+w_2\left(\underline{\mu}\Lambda,\mathrm{n}\right)f\left(\lambda_1,\lambda_2,\ldots,\lambda_k;I(Y)\right) \tag{12}$$

Denoting $f(\lambda_1,\lambda_2,...,\lambda_k;.)=\tilde{f}(.)$, $I(X)=K$, $I(Y)=L$, $w_g(\underline{\mu},\mathrm{n})=w_g$ and $w_g(\underline{\mu},\mathrm{n})=\tilde{w}_g$ for g=1,2, equation (12) can be rewritten

$$\tilde{f}(w_1K+w_2L)=\tilde{w}_1\tilde{f}(K)+\tilde{w}_2\tilde{f}(L) \tag{13}$$

for all $K, L\in[0,\xi(w))$. The solution to this functional equation (Aczél (1966), p.66) is

$$w_1=\tilde{w}_1,\ w_2=\tilde{w}_2 \text{ and}$$

$$\tilde{f}(K)=\alpha K \text{ for some constant } \alpha\neq 0. \tag{14}$$

That is

$$I(X\Lambda)=f(\lambda_1,\lambda_2,...,\lambda_k;I(X))=\tilde{f}(I(X))=\alpha(\lambda_1,\lambda_2,...,\lambda_k)I(X)$$

Simplifying we write

$$I(X\Lambda)=\alpha(\Lambda)I(X) \tag{15}$$

for any $\mathrm{X}\in\mathrm{M}_{++}(\mathrm{n,k})$ and $\Lambda\in M_{++}(\mathrm{k,k})/\ \Lambda=diag(\lambda_1,\lambda_2,...,\lambda_k)$ and some positive function $\alpha(.)$.

The proof is completed by noting that for any two matrices $\Lambda,\mathrm{H}\in M_{++}(\mathrm{k,k})/$ $\Lambda=diag(\lambda_1,\lambda_2,...,\lambda_k)$ and $\mathrm{H}=diag(h_1,h_2,...,h_k)$ from (15) we have

$$\alpha(\Lambda\mathrm{H})=\alpha(\Lambda)\alpha(\mathrm{H})$$

and the solution to this functional equation (Aczél ((1966), p.350)) is

$$\alpha(\Lambda)=\alpha(\lambda_1,\lambda_2,...,\lambda_k)=|\det(\Lambda)|^{\tau}=(\lambda_1.\lambda_2...\lambda_k)^{\tau}$$

where τ is an arbitrary real constant and det(Λ) is the determinant of Λ, concluding that

$$I(X\Lambda)=(\lambda_1.\lambda_2...\lambda_k)^{\tau}I(X)$$

for any $\mathrm{X}\in\mathrm{M}_{++}(\mathrm{n,k})$ and $\tau\in\mathbb{R}$.

Let´s see that I is a homogenous function of degree $k\tau$. For all $t \in \mathbb{R}_{++}$ $I(tX) = I(XT) = \alpha(t,t,...,t) I(X) = |\det(T)|^{\tau} I(X) = t^{k\tau} I(X)$

where $T \in M_{++}(k,k)$ / $T = diag(t,t,...,t)$.

Q.E.D.

Lemma 6*: A multidimensional inequality measure $I: D \to \mathbb{R}$ satisfies UM (UPD), Decomposability and the Unit-Consistency Axiom if and only it is a positive multiple of the form*

$$I(X) = \frac{\rho}{n\prod_{j=1}^{k}\mu_j^{\alpha_j - \tau}} \sum_{i=1}^{n}\left[\prod_{j=1}^{k}(x_{ij})^{\alpha_j} - \prod_{j=1}^{k}(\mu_j)^{\alpha_j}\right] \tag{16}$$

where $\tau \in \mathbb{R}$ and the parameters α_j and ρ have to be chosen such that the function $\phi(\underline{x}_i) = \rho \prod_{j \le 1 \le k}(x_{ij})^{\alpha_j}$ is strictly convex.

or

$$I(X) = \frac{1}{n\prod_{j=1}^{k}\mu_j^{-\tau}} \sum_{i=1}^{n}\left(\frac{x_{im}}{\mu_m}\right)\left[\sum_{j=1}^{k} a_{mj}\log\left(\frac{x_{ij}}{\mu_j}\right)\right] \tag{17}$$

where $\tau \in \mathbb{R}$ and $m \in \{1,2,...,k\}$ and the parameters a_{mj} have to be chosen such that the function $\phi(\underline{x}_i) = \sum_{j=1}^{k}\frac{x_{im}a_{mj}}{u_m}\log(x_{ij})$ is strictly convex.

or

$$I(X) = \frac{1}{n\prod_{j=1}^{k}\mu_j^{-\tau}} \sum_{i=1}^{n}\sum_{j=1}^{k}\delta_j \log\left(\frac{\mu_j}{x_{ij}}\right) \tag{18}$$

where $\tau \in \mathbb{R}$ and $\delta_j > 0$ for all j.

Proof: If I satisfies UM (UPD), continuity, normalization, the aggregative principle and the replication invariance principle, Tsui ((1999), Theorem 1) establishes that there exist continuous functions ϕ and F such that, for every $X \in M_{++}(\mathrm{n,k})$ with mean vector $\underline{\mu}(X) = (\mu_1(X),...,\mu_k(X))$ we get

$$F(I(X),\underline{\mu}) = \frac{1}{\mathrm{n}}\sum_{i=1}^{n}\left(\phi(\underline{x}_i) - \phi(\underline{\mu})\right) \quad (19)$$

where F is strictly increasing in $I(X)$, $F(0,\underline{\mu}) = 0$ and ϕ is strictly convex, which specifies the structure of aggregative multidimensional inequality measures.

Now consider the same distributions X, Y and $Z = (X,Y)$, as they were considered in the proof of lemma 5, that is, $\underline{\mu}(X) = \underline{\mu}(Y) = \underline{\mu}$ and $n(X) = n(Y) = \mathrm{n}$. Since all decomposable multidimensional inequality measure is also aggregative applying (19) and the decomposability of I we have

$$\begin{aligned}&F(I(Z),\underline{\mu}) = F\left(w_1(\underline{\mu},\mathrm{n})I(X) + w_2(\underline{\mu},\mathrm{n})I(Y),\underline{\mu}\right) = \\ &0.5F(I(X),\underline{\mu}) + 0.5F(I(Y),\underline{\mu})\end{aligned} \quad (20)$$

Denote $F(.,\underline{\mu}) = \widetilde{F}(.)$, $I(X) = K$, $I(Y) = L$, $w_g(\underline{\mu},\mathrm{n}) = w_g$ for g=1,2. Then we can rewrite (20) as follows

$$\widetilde{F}(w_1K + w_2L) = 0.5\widetilde{F}(K) + 0.5\widetilde{F}(L) \quad (21)$$

for all K, $L \in [0,\xi(w))$. Resorting to Aczél ((1966), p.66) once again, the solution to (21) also satisfies

$$\widetilde{F}(K + L) = \widetilde{F}(K) + \widetilde{F}(L) \quad (22)$$

whose nontrivial solution is

$$\widetilde{F}(K) = \lambda K \text{ for some constant } \lambda \neq 0 \quad (23)$$

Replacing in (23) $\widetilde{F}(.)$ with $\mathrm{F}(.,\underline{\mu})$, K with $I(X)$ and using (19) we have

$$I\left(X\right)=\frac{1}{\mathrm{n}\lambda\left(\underline{\mu}\right)}\sum_{i=1}^{\mathrm{n}}\left(\phi\left(\underline{x}_i\right)-\phi\left(\underline{\mu}\right)\right) \tag{24}$$

for some continuous function $\lambda(.)$.

By the lemma 5 since *I* satisfies UM (UPD), decomposability and unit-consistency, then *I* is a homogenous function of degree $\mathrm{k}\tau$.

Let's define

$$G\left(X\right)=I\left(X\right)\prod_{j=1}^{\mathrm{k}}\mu_j^{-\tau}=\frac{I\left(X\right)}{\prod_{j=1}^{\mathrm{k}}\mu_j^{\tau}} \tag{25}$$

with $\tau\in\mathbb{R}$.

Since *I* is a decomposable measure, it is easy to see that $G\left(X\right)$ is also decomposable and therefore aggregative. Moreover $G\left(X\right)$ is homogenous of degree zero, that is, $G\left(X\right)$ satisfies the scale-invariance principle, since for any $\mathrm{t}\in\mathbb{R}_{++}$, taking into account that *I* is homogeneous of degree $\mathrm{k}\tau$, we get

$$G\left(tX\right)=G\left(XT\right)=I\left(XT\right)/\prod_{j=1}^{\mathrm{k}}\left(t\mu_j\right)^{\tau}=\frac{t^{\mathrm{k}\tau}I(X)}{t^{\mathrm{k}\tau}\prod_{j=1}^{\mathrm{k}}\mu_j^{\tau}}=G\left(X\right),$$

where $T\in M_{++}\left(\mathrm{k,k}\right)$ / $T=diag\left(t,t,...,t\right)$.

Applying the first functional expression in Tsui ((1999), Theorem 3) to $G\left(X\right)$ there exists a transformation *F* such that, for any $X\in M_{++}\left(\mathrm{n,k}\right)$) with mean vector $\underline{\mu}$ we get

$$F\left(G\left(X\right)\right)=F\left(I\left(X\right)/\prod_{j=1}^{\mathrm{k}}\mu_j^{\tau}\right)=\frac{\rho}{\mathrm{n}}\sum_{i=1}^{\mathrm{n}}\left[\prod_{j=1}^{\mathrm{k}}\left(\frac{x_{ij}}{\mu_j}\right)^{\alpha_j}-1\right]=$$

$$=\frac{\rho}{\mathrm{n}\prod_{j=1}^{\mathrm{k}}\mu_j^{\alpha_j}}\sum_{i=1}^{\mathrm{n}}\left(\prod_{j=1}^{\mathrm{k}}x_{ij}^{\alpha_j}-\prod_{j=1}^{\mathrm{k}}\mu_j^{\alpha_j}\right) \tag{26}$$

where $\sum_{\sigma\in\zeta_j} sig(\sigma)\prod_{1\le i\le j}\rho\alpha_{i\sigma(i)}>0$; $\alpha_{i\sigma(i)}=\alpha_i\alpha_{\sigma(i)}$ if $i\neq\sigma(i)$; $\alpha_{i\sigma(i)}=\alpha_i(\alpha_i\text{-}1)$ if $i=\sigma(i)$, ζ_j denotes the set of permutations of $\{1,2,...,j\}$ $\forall j\in K$, sgn(σ)=+1 if the permutation is even and sgn(σ)=-1 if the permutation is odd.

The proof of Theorem 3 by Tsui shows that these conditions upon the coefficients are in fact equivalent to demand that the function $\phi(\underline{x}_i)=\rho\prod_{j\le 1\le k}(x_{ij})^{\alpha_j}$ be strictly convex.

Now let's consider the same distributions *X*, *Y* and $Z=(X,Y)$, as they were considered in the proof of the lemma 5, that is, $\underline{\mu}(X)=\underline{\mu}(Y)=\underline{\mu}$ and $n(X)=n(Y)=\mathrm{n}$. Applying equations (25), (26) and the decomposability of *G* we have

$$F\left(w_1(\underline{\mu},\mathrm{n})\frac{I(X)}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}+w_2(\underline{\mu},\mathrm{n})\frac{I(Y)}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right)=0.5F\left(\frac{I(X)}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right)+0.5F\left(\frac{I(Y)}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right) \quad (27)$$

Denoting $I(X)=K$, $I(Y)=L$, $w_g(\underline{\mu},\mathrm{n})=w_g$ for *g*=1,2, equation (27) becomes

$$F\left(w_1\frac{K}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}+w_2\frac{L}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right)=0.5F\left(\frac{K}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right)+0.5F\left(\frac{L}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right) \quad (28)$$

for all *K*, $L\in[0,\xi(w))$. Resorting to Aczél ((1966), p.66) once again, we know that the solution to (28) also satisfies

$$F\left(\frac{K}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}+\frac{L}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right)=F\left(\frac{K}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right)+F\left(\frac{L}{\prod_{j=i}^{\mathrm{k}}\mu_j^\tau}\right) \quad (29)$$

whose nontrivial solution is

$$F\left(\frac{K}{\prod_{j=i}^{k}\mu_j^{\tau}}\right)=\rho\frac{K}{\prod_{j=i}^{k}\mu_j^{\tau}} \text{ for some constant } \rho\neq 0 \qquad (30)$$

Substituting onto (26) and replacing *K* with $I(X)$ we have that *I* is a positive multiple of the form

$$I(X)=\frac{\rho}{n\prod_{j=1}^{k}\mu_j^{\alpha_j - \tau}}\sum_{i=1}^{n}\left[\prod_{j=1}^{k}\left(x_{ij}\right)^{\alpha_j} - \prod_{j=1}^{k}\left(\mu_j\right)^{\alpha_j}\right]$$

where $\tau\in\mathbb{R}$ and the parameters α_j and ρ have to be chosen such that the function $\phi(\underline{x}_i)=\rho\prod_{j\leq 1\leq k}\left(x_{ij}\right)^{\alpha_j}$ is strictly convex.

In a similar way we can derive the other functional forms (17) and (18) considering the other two functional expression in Tsui ((1999), Theorem 3), which completes the proof of the necessity of the lemma.

As regards the sufficiency of the lemma, it is easy to see that the functional forms (16), (17) and (18) are decomposable with weights respectively

$$w_g=\frac{n_g}{n}\prod_{j=1}^{k}\left(\frac{\mu_j\left(X_g\right)}{\mu_j}\right)^{\alpha_j - \tau},\quad w_g=\frac{n_g}{n}\frac{\mu_m^g}{\mu_m}\prod_{j=1}^{k}\left(\frac{\mu_j\left(X_g\right)}{\mu_j}\right)^{-\tau},\quad w_g=\frac{n_g}{n}\prod_{j=1}^{k}\left(\frac{\mu_j\left(X_g\right)}{\mu_j}\right)^{-\tau}$$

for all *g*=1,…,G

It is also straightforward to prove that these three forms satisfy UM (UPD), continuity and normalization.

The sufficiency of the lemma is completed proving that these three functional forms are unit-consistent.

We are going to prove that the first functional form is unit-consistent, in the same way we can conclude for the other functional forms.

For any $X \in M_{++}(\text{n,k})$ and $\Lambda \in M_{++}(\text{k,k}) /$ $\Lambda = diag(\lambda_1, \lambda_2, ..., \lambda_k)$.

$$I(X\Lambda) = \frac{\rho}{\text{n}\prod_{j=1}^{\text{K}}(\lambda_j \mu_j)^{-\tau}} \sum_{i=1}^{\text{n}} \left[\prod_{j=1}^{\text{k}} \left(\frac{\lambda_j x_{ij}}{\lambda_j \mu_j} \right)^{\alpha_j} -1 \right] = \frac{\rho}{\text{n}(\lambda_1 \lambda_2 ... \lambda_k)^{-\tau} \prod_{j=1}^{\text{K}} \mu_j^{-\tau}} \sum_{i=1}^{\text{n}} \left[\prod_{j=1}^{\text{k}} \left(\frac{x_{ij}}{\mu_j} \right)^{\alpha_j} -1 \right] =$$

$$= (\lambda_1 . \lambda_2 ... \lambda_k)^{\tau} I(X)$$

Thus there exists a continuous function *f* which is increasing in the last argument, such that

$$I(X\Lambda) = f(\lambda_1, \lambda_2, ..., \lambda_k; I(X))$$

After proposition 1 *I* is unit-consistent.

Q.E.D.

Lemma 7*: A multidimensional inequality measure* $I : D \to \mathbb{R}$ *satisfies UM (UPD), CIM, Decomposability and the Unit-Consistency Axiom if and only it is a positive multiple of the form*

$$I(X) = \frac{\rho}{\text{n}\prod_{j=1}^{\text{k}} \mu_j^{\alpha_j - \tau}} \sum_{i=1}^{\text{n}} \left[\prod_{j=1}^{\text{k}} (x_{ij})^{\alpha_j} - \prod_{j=1}^{\text{k}} (\mu_j)^{\alpha_j} \right] \tag{31}$$

where $\tau \in \mathbb{R}$, $\rho > 0$, $\alpha_j < 0, j = 1,2,...,k$.

Proof. The proof is straightforward following Tsui ((1999), Theorem 4) it can be proved that the last two functional forms given by equations (17) and (18) of the lemma 6 are incompatible with the correlation increasing axiom.

Morever the correlation increasing axiom requires that ϕ defined in the same as in the previous proof should be not only strictly convex but also strictly L-superadditive. Hence we can clarify the restrictions on the parameters, which reduce to $\rho > 0$, $\alpha_j < 0, j = 1,2,...,k$.

Q.E.D.

*Proof of Theorem 2***:** One can easily adapt the results in Shorrocks (1984) to show that for any continuous aggregative multidimensional inequality index J there exists a decomposable multidimensional inequality index *I* and a continuous strictly increasing function $G:\mathbb{R}\to\mathbb{R}$ with $G(0)=0$ such that

$$I(X)=G(J(X))$$

Moreover, if J is unit-consistent the same holds for *I*. Indeed, if $I(X)<I(Y)$ i.e. $G(J(X))<G(J(Y))$ since G is a strictly increasing function then $J(X)<J(Y)$. As a consequence, for any $X\in \mathrm{M}_{++}(\mathrm{n,k})$ and $\Lambda\in M_{++}(\mathrm{k,k})/\ \Lambda=diag(\lambda_1,\lambda_2,...,\lambda_k)$, we have $J(X\Lambda)<J(Y\Lambda)$ and then $G(J(X\Lambda))<G(J(Y\Lambda))$, i.e., $I(X\Lambda)<I(Y\Lambda)$, concluding that *I* a unit-consistent multidimensional inequality index.

Denoting $F=G^{-1}$, we have that if *J* satisfies UM (UPD), the aggregative principle and the unit-consistency axiom, there exists a continuous function F such that, for every $X\in M_{++}(\mathrm{n,k})$

$$J(X)=F(I(X))$$

where F is strictly increasing and I is a decomposable and unit-consistent multidimensional inequality index. Therefore *I* belongs to the class characterized in lemma 6.

The sufficiency of this theorem is straightforward.

Q.E.D

Proof of Theorem 3: If *J* CIM, since F is a strictly increasing function, then *I* also satisfies CIM. Therefore *I* is a multidimensional inequality index which belongs to the class characterized in the lemma 7. This proves the necessity of the theorem. Once again the sufficiency of this theorem is straightforward.

Q.E.D