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# Measuring Bipolarization, Inequality, Welfare and Poverty[*]

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**Abstract**

This paper analyzes the relationship between bipolarization and inequality, welfare and poverty measures. First, we clarify the similarities and differences between bipolarization and inequality measures. Second, it is shown that bipolarization is the difference between the welfare levels of the richer and poorer income groups when feelings of identification between individuals are based on their utility functions. In fact, bipolarization is interpreted as the welfare of the richer group that is wasted to compensate for income bipolarization. Third, a relationship between bipolarization measurement and the normalized poverty deficit index is established. These findings are applied to the polarization measures of Wolfson (1994), Esteban and Ray (1994) and Lasso de la Vega and Urrutia (2006).

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## 1 INTRODUCTION

The relationship between inequality, poverty and welfare measures has been the focus of a large body of research on distribution (see, for example, Lambert, 2001, and the references therein). However, relationships between these concepts and income polarization have hardly been analyzed. The similarities and differences between welfare, inequality and poverty measures are well known, but we know little about the meaning of income polarization in terms of welfare, poverty and inequality. In this paper, the measurement of bipolarization is linked to the other primary features of an income distribution.

Polarization measures have recently been proposed as useful characterizations of income distributions.[1] It is argued that polarization measures represent a major departure from standard measures of inequality. This is because polarization concentrates the income distribution on several focal or polar modes, whereas inequality relates to the overall dispersion of the distribution. However, many empirical comparisons of inequality and polarization measures suggest that they are closely related. For example, Ravallion and Chen (1997) found a high degree of correspondence based on a comparative analysis of the Gini coefficient and the Wolfson bipolarization index. This, and similar findings, suggest that further analysis of the differences and similarities between polarization and inequality measures is warranted. In the first part of this paper, we address this issue in the context of the polarization indexes of Esteban and Ray (1994), Wolfson (1994) and Lasso de la Vega and Urrutia (2006).

[1] See, among others, Esteban and Ray (1994), Wolfson (1994, 1997), Esteban *et al.* (1999), Wang and Tsui (2000), Gradín (2000, 2002), Zhang and Kanbur (2001), D'Ambrosio (2001), D'Ambrosio and Wolff (2001), Chakravarty and Majumder (2001), Rodríguez and Salas (2003), Duclos and Échevin (2004), Duclos *et al.* (2004), Prieto *et al.* (2004, 2005) and Lasso de la Vega and Urrutia (2006).

Since measures of polarization do not fulfill the principle of progressive transfers, they are not directly compatible with standard welfare analysis. Thus, can polarization measures be interpreted in terms of welfare? Can the welfare of society be related to the degree of income polarization? Chakravarty and Majumder (2001) was seminal study investigating this issue for the Wolfson bipolarization index. In the second part of this paper, an alternative model is proposed in which bipolarization (measured by the Wolfson, 1994, the Esteban and Ray, 1994 or the Lasso de la Vega and Urrutia, 2006 polarization indexes) is interpreted in terms of welfare. In particular, there is a relationship between bipolarization and welfare measures when envy between people is incorporated in the utility function. It is shown that bipolarization is the difference between the richer income group's welfare and the poorer income group's welfare when individuals' utilities depend not only on their own income but also on their group incomes. Consequently, bipolarization increases when the welfare of the richer income group rises and/or when the welfare of the poorer income group falls.

Nevertheless, a fundamental difference arises between the models for the Wolfson (1994), the Esteban and Ray (1994) and the Lasso de la Vega and Urrutia (2006) indexes. The Wolfson bipolarization measure requires two different sets of individual utility functions, one for each income group. However, the Esteban and Ray (1994) and the Lasso de la Vega and Urrutia (2006) measures require the same utility function for all individuals.

In addition, it is shown that the overall welfare level of a society is the richer group's welfare diminished by the level of income bipolarization. That is, bipolarization can be interpreted as the welfare of the richer income group that is sacrificed to compensate for bipolarization between poorer and richer groups.

A striking feature of underdeveloped countries is the large proportion of poor people. For example, in 2004, at least 70 percent of people in Sierra Leone were poor (World Bank, 2005). Thus, in underdeveloped countries, it may be worth measuring income bipolarization between poor people and the rest of society. Policymakers in these countries should be aware of the social conflict (measured by the bipolarization index) due to poverty. Thus, instead of dividing the income distribution into two income groups based on median or mean incomes, it might be more appropriate to use the poverty line to measure income bipolarization. In the third part of this paper, we show that bipolarization and poverty measures are closely related when the income value used to separate income groups represents the poverty line. In that case, bipolarization between the poor and others in the income distribution explicitly considers the value of a poverty index known as the normalized poverty deficit index. In fact, the normalized poverty deficit index is equal to the Esteban and Ray (1994) polarization measure, whether the poverty line is the mean income and the identification sensitivity parameter is unity.

The paper is organized as follows. Links between inequality and income bipolarization are clarified and extended in Section 2. In Section 3, the relationship between bipolarization and welfare is analyzed. Poverty and polarization measures are linked in Section 4. Section 5 concludes the paper.

## 2 BIPOLARIZATION AND INEQUALITY

First, we briefly describe three polarization measures that are used throughout this paper: the Wolfson (1994) bipolarization index, the Esteban and Ray (1994) polarization measure (ER) and the Lasso de la Vega and Urrutia (2006) polarization index (LU). Results on the relationship between bipolarization and inequality are then proposed.

Let $x = (x_1, x_2, \ldots, x_n) \in \Re^n_{++}$ be an $n$-dimensional vector of positive incomes, in increasing order, such that $x_1 \leq x_2 \leq \ldots \leq x_n$, and let $x_i$ be the income of the $i$th person. We assume that $n$ is even. Results for the Wolfson index are clearer when this condition is imposed. Nevertheless, differences with respect to the results for the Wolfson index when $n$ is odd are negligible if $n$ is large. Vector $x$ is normalized to the value of mean income, $\mu$. Median income is $m$.

Wolfson's index of bipolarization (Wolfson, 1994) was originally proposed for a population divided into two groups by the median value:

$$P^W(x) = 4\frac{\mu}{m}\left[(0.5 - L(0.5)) - \frac{G(x)}{2}\right] \tag{1}$$

where $G(x)$ is the Gini coefficient of the income distribution, and $L(0.5)$ is the Lorenz curve at the median percentile. The larger the term in brackets, the fewer individuals or households there are with mid-level incomes and, hence, the greater is bipolarization. Note that, in our context, $\mu = 1$.

Wang and Tsui (2000) generalize a new class of indexes based on the Wolfson index by using the *increased bipolarity* axiom (that progressive income transfers within groups increase bipolarization) and the *increased spread* axiom (that regressive income transfers between groups increase bipolarization). However, we do not use this class of indexes for reasons that will become apparent.

The Wolfson index of bipolarization has been reformulated in terms of the between-groups Gini coefficient $G^B(x)$ and the within-groups Gini coefficient $G^W(x)$ as follows (see Rodríguez and Salas, 2003):

$$P^W(x) = 2\frac{\mu}{m}\left[G_m^B(x) - G_m^W(x)\right] \tag{2}$$

where the income groups are separated by the median income value. Therefore, bipolarization and inequality are explicitly represented within the same framework; subtraction and addition of the within-groups dispersion corresponds to bipolarization and inequality, respectively.

ER assumes that each individual is subject to two forces: *identification* with members considered to belong to the same group, and *alienation* from those considered to belong to other groups.[2] Effective antagonism increases in identification and alienation in such a way that increased intra-group identification reinforces the effect of alienation. Polarization represents total effective antagonism. This broad starting point is combined with a set of four axioms. These axioms restrict the functional forms that can be incorporated into the general framework established in ER. The following measure satisfies the axioms:[3]

$$P^{ER}(x;\alpha) = \sum_i \sum_j \pi_i \pi_j \pi_i^{\alpha} \left|\mu_i - \mu_j\right| \tag{3}$$

[2] Duclos *et al.* (2004) also propose a polarization measure that relies on the identification–alienation framework. This measure is not considered here because it deals with a completely different domain: continuous distributions.

[3] In the original measure incomes are in logarithms and a positive scalar, *K*, is specified for normalization. For clarity of exposition, and without loss of generality, we do not consider these minor changes. This note also applies to the Lasso de la Vega and Urrutia (2006) polarization index (see below).

where $\pi_i$ and $\mu_i$ are the percentage of population and the mean income value, respectively, of income group $i$. The alienation term is $|\mu_i - \mu_j|$, the identification term is $\pi_i^{\alpha}$ and $\alpha \in [1,1.6]$ is a positive constant that represents the importance of group identification.

This formulation is restricted to distributions that are prearranged in groups. However, datasets usually take a large number of values so observations have to be regrouped into a small number of groups before the measure is applied. This regrouping eliminates the dispersion within groups. Lasso de la Vega and Urrutia (2006) have recently proposed a generalization of the ER measure that incorporates inequality within groups and overcomes the shortcomings of previous extensions (see Esteban *et al.*, 1999).

Lasso de la Vega and Urrutia (2006) propose the following polarization measure:

$$P^{LU}(x;\alpha,\beta) = \sum_i \sum_j \pi_i \pi_j \pi_i^{\alpha} (1 - G_i)^{\beta} |\mu_i - \mu_j| \quad (4)$$

where $\beta \geq 0$ is the degree of sensitivity towards group cohesion. The new identification term of each member of group $i$ is $\pi_i^{\alpha}(1 - G_i)^{\beta}$. Clearly, this measure reduces to the ER polarization index if $\beta = 0$.

In the bipolarization case, the following result is obtained for the LU polarization measure.

PROPOSITION 1: For the bipolarization case ($N = 2$), the LU polarization index becomes:

$$P_h^{LU}(x;\alpha,\beta) = \left(\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right) G_h^B(x) \tag{5}$$

where $h \in \Re_+$ is the income value that separates the income distribution into two different income groups. Thus, as in the Wolfson bipolarization case, bipolarization according to the LU measure is a function of the between-groups Gini coefficient and the within-groups dispersion.

*Proof*: If we consider two income groups separated by the *h* value:

$$P_h^{LU}(x;\alpha,\beta) = \pi_1 \pi_2 \left(\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right)(\mu_2 - \mu_1) \tag{6}$$

where the mean income values are

$$\mu_1 = \frac{L(\pi_1)}{\pi_1} \quad \text{and} \quad \mu_2 = \frac{1-L(\pi_1)}{1-\pi_1}. \tag{7}$$

$L(\pi_1)$ is the value of the Lorenz curve evaluated at $\pi_1$. If we take into consideration for two income groups that the vertical distance between the Lorenz curve value at $\pi_1$, $L(\pi_1)$, and the 45-degree line, is equal to the between-groups Gini coefficient, that is:

$$G_h^B(x) = \pi_1 - L(\pi_1) \tag{8}$$

and substituting (7) into (6), we obtain expression (5).

This index is only a function of the income dispersion between groups for the bipolarization case if $\beta$ = 0 (the ER case):

$$P_h^{LU}(x;\alpha,0) = \left(\pi_1^{\alpha} + \pi_2^{\alpha}\right) \mathrm{G}_{\mathrm{h}}^{\mathrm{B}}(x) \tag{9}$$

Note that LU in this case is simply the between-groups Gini coefficient when $\alpha$ = 1.

From (5), when $h = m$, the following expression is also derived:

$$P_m^{LU}(x;\alpha,\beta) = \left(\frac{1}{2}\right)^{\alpha} \left((1-G_1)^{\beta} + (1-G_2)^{\beta}\right) G_m^B(x) \tag{10}$$

Therefore, the relationship between the Wolfson and LU polarization measures when there are two income groups separated by the median income value and $\beta$ = 0 is the following:

$$P^W(x) = \frac{1}{m}\left[2^{\alpha} P_m^{LU}(x;\alpha,0) - 2\mathrm{G}_{\mathrm{m}}^{\mathrm{W}}(x)\right]. \tag{11}$$

Consequently, $P^W(x) = \frac{2}{m}\left[P_m^{LU}(x;1,0) - \mathrm{G}_{\mathrm{m}}^{\mathrm{W}}(F)\right]$ when $\alpha$ = 1.

Nevertheless, the ER and LU polarization indexes can be applied to any number of income groups. By contrast, the Wolfson polarization measure can only be applied to two income groups.

In the next section, we use abbreviated welfare functions that incorporate the Gini coefficient to interpret bipolarization in terms of welfare.

## 3 BIPOLARIZATION AND WELFARE

An interesting relationship between bipolarization and welfare measurements arises when envy between people in the same income group is incorporated into their utility functions.

We begin with the following general social welfare function:

$$v(x) = W(U_1(x_1, x),...,U_n(x_n, x)) \tag{12}$$

where $W$ is strictly increasing and $U_i(x_i, x)$ represents individual $i$'s preferences over her or his own income and the income distribution $x$. In principle, preferences are not individualistic. Let $h \in \Re_+$ be the income that separates the income distribution into two different groups: incomes below $h$ and incomes above $h$. The first group have preferences according to the utility functional $U_h(\cdot,y)$, where $y$ is the income distribution for people below $h$. The second group has preferences according to the utility function $U_{\bar{h}}(\cdot, t)$, where $t$ is the income distribution for people above $h$. People care about their own incomes and the income distribution of the group to which they belong. Consequently, $v$ must be partially symmetric; that is, society should be indifferent to permutations among incomes of each group but not necessarily between groups (see Cowell, 1980). Moreover, it is assumed that $W(\cdot)$ is additively separable. We can decompose overall welfare into contributions from the two different income groups as follows:

$$\begin{aligned} v(x) &= \pi_1 \sum_{i=1}^{h} \frac{1}{h} U_h(x_i, y) + \pi_2 \sum_{i=h+1}^{n} \frac{1}{(n-h)} U_{\bar{h}}(x_i, t) \\ &= \pi_1 W_1 + \pi_2 W_2 \end{aligned} \tag{13}$$

where $W_1$ and $W_2$ are the average welfare levels of each group. Therefore, $\pi_1 W_1$ and $\pi_2 W_2$ are the contributions of each group to overall welfare. We use this decomposition of social welfare to propose an interpretation of bipolarization in terms of welfare. First, however, we present two preliminary but necessary results.

It is well known that the rankings induced by a symmetric, increasing and individualistic abbreviated welfare function, and by $-G$, on two income distributions with the same mean income are not necessarily the same (Newbery, 1970). Nevertheless, the use of an abbreviated welfare function containing the Gini coefficient can be justified when the welfare function is non-individualistic (see, for example, Sheshinski, 1972, and Kakwani, 1980 and 1986).

Let $D(x_i;x_j)$ be the relative deprivation felt by an individual with income $x_i$ in relation to an individual with income $x_j$, as follows (see Runciman, 1966, Yitzhaki, 1979 and Hey and Lambert, 1980):

$$\begin{aligned} D(x_i;x_j) &= x_j - x_i \qquad \text{if } x_i \le x_j \\ D(x_i;x_j) &= 0 \qquad \text{if } x_i \ge x_j . \end{aligned} \tag{14}$$

Then, the deprivation felt by an individual with income $x_i$ is:

$$D(x_i) = \frac{1}{n}\sum_{j=1}^{n} D(x_i;x_j) . \tag{15}$$

Now, let $U^D(x_i,x)$ be the utility function of an individual with income $x_i$, as follows:

$$U^D(x_i,x) = ax_i - bD(x_i) \qquad a,\, b > 0. \tag{16}$$

The individual cares not only about his or her own income but also about the distribution to which he or she belongs. In particular, the higher the deprivation felt by the individual, the lower his or her utility.

The following result justifies the use of an abbreviated welfare function that incorporates the Gini coefficient when *W* is non-individualistic.

RESULT 1 (see Lambert, 2001, pp. 123-4).[4] Let $U^D(x_i, x) = ax_i - bD(x_i)$ be the utility function of an individual with income $x_i$. Then, $W^D(x) = \frac{1}{n}\sum_{i=1}^{n} U^D(x_i, x) = \mu_x(a - bG(x))$ for every income distribution *x*.

Subsequently, we use this result (for $a = b = 1$) to link overall bipolarization in society to welfare.

A parallel result arises when the concept of *relative satisfaction* is introduced (see Yitzhaki, 1979 and Hey and Lambert, 1980).

Let $S(x_i; x_j)$ be the relative satisfaction felt by an individual with income $x_i$ in relation to an individual with income $x_j$, as follows:

$$S(x_i; x_j) = x_i - x_j \qquad \text{if } x_i \geq x_j$$

$$S(x_i; x_j) = 0 \qquad \text{if } x_i \leq x_j \,. \tag{17}$$

The satisfaction felt by an individual with income $x_i$ is:

[4] A similar result, $W(x) = \mu_x[a - 0.5b(1 + G(x))]$, is obtained when the altruistic utility function, $U(x_i, x) = x_i[a - bF(x_i)]$ *a, b* > 0, is used, in which the arguments are the individual's own income level and the proportion of people who are worse off than that individual.

$$S(x_i) = \frac{1}{n}\sum_{j=1}^{n} S(x_i; x_j) \,. \tag{18}$$

Now let $U^S(x_i, x)$ be the utility function of the individual with income $x_i$, as follows:

$$U^S(x_i, x) = ax_i + bS(x_i) \quad a,\ b > 0. \tag{19}$$

In this case, envy is different: an individual with income $x_i$ is better off when more people have less income than does he or she. People care about status. Consequently, the greater the relative satisfaction felt by an individual with income $x_i$, the higher his or her utility.

The following result enables the use of an abbreviated welfare function (incorporating the Gini coefficient) when $W$ is non-individualistic in a different way than in the context of Result 1.

RESULT 2.[5] Let $S(x_i)$ be the relative satisfaction function and let $U^S(x_i, x) = ax_i + bS(x_i)$ ($a,\ b > 0$). Then, $W^S(x) = \frac{1}{n}\sum_{i=1}^{n} U^S(x_i, x) = \mu_x(a + bG(x))$.

*Proof*: Substituting equations (17), (18) and (19) into the welfare function yields:

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[5] A similar result, $W(x) = \mu_x\left[a + 0.5b(1 + G(x))\right]$, is obtained when the utility function, $U(x_i, x) = x_i\left[a + bF(x_i)\right]$ $a,\ b > 0$, is used, in which the arguments are the individual's own income level and the proportion of people who are worse off than that individual.

$$W^S(x) = a\mu + \frac{b}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(x_i - x_j\right). \tag{20}$$

Since $G(x) = \frac{1}{2n^2\mu}\sum_{i=1}^{n}\sum_{j=1}^{n}\left|x_i - x_j\right|$, we have:

$$\mu G(x) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{i}\left(x_i - x_j\right) \tag{21}$$

Substituting expression (21) into equation (20) completes the proof.

Therefore, the social welfare function is not inequality averse if people feel relative satisfaction. We use Result 2 (for $a = b = 1$) to link overall bipolarization in society to welfare.

In the following proposition, we provide an interpretation of the Wolfson bipolarization measure in terms of welfare.

PROPOSITION 2. Let $x \in \Re^n_{++}$ be an income distribution separated into two groups by the median income *m*, let $P^W(x)$ be the Wolfson bipolarization measure, let $W_1^S$ be the welfare level of the group below *m* and let $W_2^D$ be the welfare level of the group above *m*. Then, it follows that:

$$P^W(x) = \frac{1}{2m}\left[W_2^D - W_1^S\right]. \tag{22}$$

Bipolarization is represented by half the difference between the normalized welfare levels (based on the median income) of the richer and poorer income groups when income groups are separated by median income. Bipolarization increases when the welfare of the richer income group rises or when the welfare of the poorer income group falls, and *vice versa*.

*Proof*. The additive decomposition of the Gini coefficient based on population groups when income groups do not overlap is:[6]

$$G(x) = G^B(x) + G^W(x) = G^B(x) + \sum_k \pi_k r_k G_k \tag{23}$$

where $\pi_k$ is the proportion of the population in group $k$, $r_k$ is group $k$'s share of total income and $G_k$ is the Gini coefficient of group $k$.

It follows from equation (23) that:

$$G_m^W(x) = \pi_1\left(\frac{\pi_1\mu_1}{\mu}\right)G_1 + \pi_2\left(\frac{\pi_2\mu_2}{\mu}\right)G_2 \tag{24}$$

where $\mu_1$ and $G_1$ are, respectively, the mean income and the within-groups Gini coefficient for group 1, and $\mu_2$ and $G_2$ are, respectively, the mean income and the within-groups Gini coefficient for group 2.

Moreover, it follows from equations (7) and (8) that:

[6] See, for example, Bhattacharya and Mahalanobis (1967), Pyatt (1976), and Lambert and Aronson (1993).

$$G_h^B(x) = \pi_1 - \pi_1 \frac{\mu_1}{\mu}. \tag{25}$$

Therefore, substituting (24) and (25) into equation (2) yields:

$$P^W(x) = 2\frac{\mu}{m}\left[\pi_1 - \pi_1 \frac{\mu_1}{\mu} - \frac{(\pi_1)^2 \mu_1}{\mu} G_1 - \frac{{\pi_2}^2 \mu_2}{\mu} G_2\right]. \tag{26}$$

Since the mean income value, $\mu$, is $\pi_1\mu_1 + \pi_2\mu_2$, it follows that:

$$P^W(x) = \frac{2}{m}\left[(\pi_1)^2 \mu_1 + \pi_2\pi_1\mu_2 - \pi_1\mu_1 - (\pi_1)^2 \mu_1 G_1 - (\pi_2)^2 \mu_2 G_2\right]. \tag{27}$$

Substituting $\pi_1 = 0.5$ into the above yields:

$$P^W(x) = \frac{1}{2m}\left[\mu_2 - \mu_1 - \mu_1 G_1 - \mu_2 G_2\right]. \tag{28}$$

We need only consider Result 1, $W_2^D = \mu_2(1 - G_2)$, and Result 2, $W_1^S = \mu_1(1 + G_1)$, for $a = b = 1$, to obtain the result in (22).

A result derived from Proposition 2 is stated in the following corollary.

COROLLARY 1.[7] Let $P^W(x)$ be the Wolfson bipolarization measure and let $v(x)$ be the social welfare function. Then, if $h = m$, it follows that:

$$v(x) = W_2^D - mP^W(x)\,. \tag{29}$$

*Proof*. Given equations (13) and (22), the proof of this result is straightforward.

Overall social welfare is the richer group's welfare minus the level of bipolarization weighted by the median income value. That is, bipolarization can be interpreted as the welfare of the richer group that is sacrificed to compensate for income bipolarization between poorer and richer income groups.

Bipolarization is interpreted as a function of individual welfare levels that depends not only on individuals' own incomes but also on their envy of others in their own income groups. In particular, people in the income group above $h$ ($=m$) envy (feel relative deprivation towards) individuals with higher incomes, whereas people in the group below $m$ envy (feel relative satisfaction towards) individuals with lower incomes.

In this context, greater heterogeneity within the group implies greater envy within the group and, therefore, less internal unity. In fact, individuals from the same group envy one another since they move away from their group's mean income. At the same time, alienation is determined in this context by the difference between $\mu_2$ and $\mu_1$; that is, the income differential between the corresponding groups.

[7] Another result derived from Proposition 2 but with a less clear interpretation is that $v(x) = W_1^S + mP^W(x)$. Social welfare is the poorer group's welfare plus the level of polarization weighted by the median income.

A relevant question is whether it is reasonable to assume that individuals from the richer group feel deprived and that members from the poorer group feel affluent. People envy others in their own income groups. Therefore, richer people may feel deprived in comparison to those who have even more income than they do. Similarly, poorer people may feel more satisfied than those who have less income than they do. However, asymmetrical feelings of envy are counterintuitive. We must assume that mentalities, as well as incomes, differ: how individuals think depends on the groups to which they belong. Consequently, the Wolfson bipolarization measure implies an unreasonable welfare model.

Given this problem, and given that only the median income value can be used to split the income distribution into two groups, we have not attempted to identify a welfare model for Wang and Tsui's (2000) generalization of the Wolfson measure. Instead, we focus on the polarization measures of Lasso de la Vega and Urrutia (2006) and Esteban and Ray (1994).

The following proposition generalizes the relationship between bipolarization and welfare to the LU polarization index when β = 1.

PROPOSITION 3. Let $P_h^{LU}(x;\alpha;1)$ be the LU polarization measure for two income groups separated by *h* and β = 1, let $\overline{W}_1^D$ be the normalized welfare level of the group below *h* and let $\overline{W}_2^D$ be the normalized welfare level of the group above *h*. Then, it follows that:

$$P_h^{LU}\left(x;\alpha;1\right) = (\pi_2)^{\alpha+1}\overline{W}_2^D - (\pi_1)^{\alpha+1}\overline{W}_1^D \,. \tag{30}$$

where the normalization term is the corresponding level of welfare when there is not bipolarization in society. According to the LU polarization index, bipolarization is the weighted difference between the normalized welfare levels of richer and poorer income groups. However, in this case, throughout the income distribution, individuals have the same feelings of envy (deprivation).

*Proof*. Given equations (5) and (25), we have:

$$P_h^{LU}(x;\alpha,\beta) = \left(\pi_1^{\alpha+1}(1-G_1)^{\beta} + \pi_2^{\alpha}\pi_1(1-G_2)^{\beta}\right)(\mu-\mu_1) \tag{31}$$

that is,

$$P_h^{LU}(x;\alpha,\beta) = \pi_2^{\alpha}\pi_1(\mu-\mu_1)(1-G_2)^{\beta} - \pi_1^{\alpha+1}(\mu_1-\mu)(1-G_1)^{\beta}\ . \tag{32}$$

We know that $\pi_1\mu - \pi_1\mu_1 = \pi_2(\mu_2-\mu)$ so:

$$P_h^{LU}(x;\alpha,\beta) = \pi_2^{\alpha+1}(\mu_2-\mu)(1-G_2)^{\beta} - \pi_1^{\alpha+1}(\mu_1-\mu)(1-G_1)^{\beta}\ . \tag{33}$$

Moreover, $P_h^{LU}(x;\alpha;\beta) = 0$ if and only if $\mu_1 = \mu_2 = \mu$. Therefore, given Result 1, $W_1^0 = \mu(1-G_1)$ and $W_2^0 = \mu(1-G_2)$ can be interpreted as the welfare levels of the corresponding groups when there is no bipolarization in society. Hence, we obtain the following:

$$P_h^{LU}(x;\alpha;1) = (\pi_2)^{\alpha+1}\left(W_2^D - W_2^0\right) - (\pi_1)^{\alpha+1}\left(W_1^D - W_1^0\right). \tag{34}$$

To prove the result, we need only define the normalized welfare levels of groups 1 and 2 as $\overline{W}_1^D = W_1^D - W_1^0$ and $\overline{W}_2^D = W_2^D - W_2^0$ , respectively.

Note that if $h$ = $m$, then $P_m^{LU}(x;\alpha;1) = \left(\frac{1}{2}\right)^{\alpha+1}\left[\overline{W}_2^D - \overline{W}_1^D\right]$. Despite their apparent similarity, this result and that in (22) differ substantially.

A result derived from Proposition 3 is stated in the following corollary.

COROLLARY 2. Let $P_h^{LU}(x;\alpha;\beta)$ be the LU polarization measure for two income groups separated by $h$ and let $\overline{v}(x)$ be the normalized social welfare function. Then, if $\beta = 1$, it follows that:

$$\overline{v}(x) = \frac{1}{\pi_1^\alpha}\left[\left(\pi_1^\alpha \pi_2 + \pi_2^{\alpha+1}\right)\overline{W}_2^D - P_h^{LU}(x;\alpha,1)\right]. \qquad (35)$$

where the normalization term in $\overline{v}(x)$ is the level of social welfare when there is no bipolarization in society.

*Proof*. Equation (13) implies that $\overline{v}(x) = \pi_1\overline{W}_1 + \pi_2\overline{W}_2$. Hence, given (30), the result is proven.

Again, bipolarization can be interpreted as the welfare of the richer group that is sacrificed to compensate for income bipolarization. However, in this case, there are no asymmetrical feelings of envy. Moreover, any income, $h$, including median income, ca be used to split the income distribution into two groups. In fact, if $h = m$, then $\bar{v}(x) = \overline{W}_2^D - 2^{\alpha} P_m^{LU}(x;\alpha,1)$.

Furthermore, the results in Proposition 3 and Corollary 2 apply to the ER polarization index if $\beta$ is 0, rather than 1, in equation (33). However, these results for the ER measure imply that the simple individual utility function, $U(x_i) = x_i$, is used. Therefore, in this framework, people only care about their own incomes.

We can go further by assuming the altruistic utility function, $U(x_i) = x_i\left(a - \frac{b}{n\mu}\sum_{i=1}^{n} D(x_i)\right)^{\beta}$, with a > 0, b > 0 for all individuals. In this case, individuals care not only about their own incomes but also care about the normalized average level of deprivation in society. The nonnegative parameter, $\beta$, represents the importance of altruism. Given $\mu G(x) = \frac{1}{n}\sum_{i=1}^{n} D(x_i)$ (see Yitzhaki, 1979, and Hey and Lambert, 1980), assuming this utility function yields the welfare function $W(x) = \mu\left(a - bG(x)\right)^{\beta}$. Consequently, the results in Proposition 3 and Corollary 2 apply for all values of $\beta$ in the LU polarization index.

## 4 BIPOLARIZATION AND POVERTY

Let the poverty line, *z*, be the income level that divides the income distribution in two groups. In this case, bipolarization between poor people and others is explicitly based on a poverty index. To be specific, the LU, ER and Wolfson polarization measures are functions of the normalized poverty deficit index, which belongs to the Foster–Greer–Thorbecke family of poverty measures (see Foster *et al.*, 1984).

First, recall some concepts. Given the poverty line, $z$, the proportion of people below $z$, denoted by $\pi_1$, is the so-called headcount ratio, which is the proportion of people who are poor in $x$. This index only relates to the incidence of poverty. Another widely quoted poverty index is the income gap ratio, $I_z(x) = 1 - \frac{\mu_1}{z}$, in which $\mu_1$ is the mean income of the poor in $x$. The income gap ratio measures the intensity of poverty but conveys nothing about its incidence. The poverty deficit index, $D_z(x) = \sum_{i=1}^{s} \frac{z - x_i}{n}$, is the sum of income distances to the poverty line for poor people, where $x_s$ is the largest poor income. The family of poverty indices introduced by Foster *et al.* (1984) is $T_z^{FGT}(x;\gamma) = \frac{1}{n}\sum_{i=1}^{n} \Gamma(x_i)^{\gamma}$ where $\Gamma(x_i) = \max\left\{\frac{z - x_i}{z}, 0\right\}$ and $\gamma \geq 0$. This family of poverty measures is the normalized poverty deficit index or the product of the headcount and income gap ratios, $D_z(x)/z = \pi_1 I_z(x)$, when $\gamma = 1$. Hence, the normalized poverty deficit index is, by definition, sensitive not only to the incidence of poverty but also to its intensity. More sophisticated poverty measures that are also sensitive to the distribution of income among the poor are not considered in this paper. This is because, since they satisfy the principle of progressive transfers, it is difficult to obtain from them a clear-cut relationship between bipolarization and poverty measurement because bipolarization measures do not satisfy the principle. Below, we present some results on bipolarization and poverty.

PROPOSITION 4. Let $P_z^{LU}(x;\alpha,\beta)$ be the LU polarization index for two income groups separated by the $z$ income value and let $T_z^{FGT}(x;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures. Then, it follows that:

$$P_z^{LU}(x;\alpha,\beta) = \left[\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right]\left[T_z^{FGT}(x;1) + \frac{\pi_1\mu_1}{\mu}\left(\frac{\mu - z}{z}\right)\right]. \tag{36}$$

The LU polarization measure is a function of the normalized poverty deficit index and the poor's proportion of total income.

*Proof*: Consider equations (5) and (25):

$$P_z^{LU}(x;\alpha,\beta) = \left[\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right]\left[\pi_1 - \pi_1\frac{\mu_1}{\mu}\right]. \tag{37}$$

To complete the proof, we sum and subtract the term $\pi_1\frac{\mu_1}{z}$ and then use the following:

$$T_z^{FGT}(x;1) = \pi_1\left(1 - \frac{\mu_1}{z}\right). \tag{38}$$

Typically, the poverty line, $z$, is at a percentage of mean income. In this case, the following corollary might be applied.

COROLLARY 3. Let $P_z^{LU}(x;\alpha,\beta)$ be the LU polarization index for two income groups, let $T_z^{FGT}(x;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures and let $z = p\mu$, where $p\in[0,1]$, be the poverty line. Then, it follows that:

$$P_z^{LU}(x;\alpha,\beta) = \left[\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right]\left[T_z^{FGT}(x;1) + \frac{\pi_1\mu_1}{\mu}\left(\frac{1-p}{p}\right)\right]. \tag{39}$$

If the poverty line, $z$, is assumed to be 50 percent of $\mu$, equation (39) implies $P_z^{LU}(x;\alpha,\beta) = \left[\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right]\left[T_z^{FGT}(x;1) + \frac{\pi_1\mu_1}{\mu}\right]$. Moreover, in this case for $\alpha$ = 1 and $\beta$ = 0 the LU polarization index is simply the normalized poverty deficit index plus the poor's proportion of total income, that is, $P_z^{LU}(x;1,0) = T_z^{FGT}(x;1) + \frac{\pi_1\mu_1}{\mu}$.

However, if the poverty line is in fact the mean income the following corollaries might be applied.

COROLLARY 4. Let $P_z^{LU}(x;\alpha,\beta)$ be the LU polarization index for two income groups, let $T_z^{FGT}(x;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures and let $z = \mu$ be the poverty line. Then, it follows that:

$$P_\mu^{LU}(x;\alpha,\beta) = \left[\pi_1^{\alpha}(1-G_1)^{\beta} + \pi_2^{\alpha}(1-G_2)^{\beta}\right]T_\mu^{FGT}(x;1). \tag{40}$$

COROLLARY 5. Let $P_z^{LU}(x;1,0)$ be the LU polarization index for two income groups and $\alpha$ = 1 and $\beta$ = 0, let $T_z^{FGT}(x;\gamma)$ be the Foster–Greer–Thorbecke family of poverty measures and let $z = \mu$ be the poverty line. Then, it follows that:

$$P_\mu^{LU}(x;1,0) = T_\mu^{FGT}(x;1) = G_\mu^{B}(x). \tag{41}$$

The LU polarization measure (ER case) is the normalized poverty deficit index when the poverty line that separates the two income groups is at the mean value of income and when the identification sensitivity parameter, $\alpha$, is unity. In this case, bipolarization and poverty are exactly the same. Furthermore, LU is simply the between-groups Gini coefficient when $\alpha = 1$ and $\beta = 0$. Therefore, bipolarization measured by LU (or ER), poverty measured by the normalized poverty deficit index and inequality measured by the between-groups Gini coefficient are exactly the same.

To link the Wolfson bipolarization index to poverty measures, we use median income to represent the poverty line. Then, it follows that:

$$P^{W}(x) = \frac{2}{m}\left[T_m^{FGT}(x;1) + \frac{\mu_1}{2\mu}\left(\frac{\mu - m}{m}\right) - G_m^{W}(x)\right].$$

Again, bipolarization between poor people and others (measured by the Wolfson polarization index) depends explicitly on the normalized poverty deficit index and the poor's proportion of total income.

## 5 CONCLUDING REMARKS

In this paper, we have clarified the similarities and differences between inequality and polarization measures. In particular, the polarization measures of Esteban and Ray (1994), Wolfson (1994) and Lasso de la Vega and Urrutia (2006) have been related to the between-groups and within-groups Gini components in the context of two income groups. These relationships may therefore be useful for interpreting empirical results. For example, the result in Ravallion and Chen (1997) is easily interpreted by considering the relationship between the Wolfson bipolarization index and the between-groups Gini coefficient.

We also proposed two different models of bipolarization in terms of welfare. In both models, polarization measures are interpreted as the difference between the welfare levels of richer and poorer income groups. Moreover, we showed that social welfare is the richer group's welfare minus the level of income bipolarization. However, the Wolfson polarization measure requires the assumption of an asymmetry that is counterintuitive. That is, people with above-median incomes feel deprived relative to individuals with higher incomes, whereas people below the median are required to feel satisfied relative to lower-income individuals. In contrast, the welfare models for the LU and ER polarization measures assume that all people feel deprived relative to individuals with higher incomes. Consequently, it seems appropriate to use the LU and ER indexes to measure bipolarization, at least from a welfare point of view.

Finally, does reduced poverty lower social conflict? To answer this question, we must relate poverty and bipolarization when income bipolarization is considered a reasonable proxy for social conflict. In this context, the polarization measures of Wolfson (1994), Esteban and Ray (1994) and Lasso de la Vega and Urrutia (2006) are functions of the normalized poverty deficit index. Furthermore, the Esteban and Ray (1994) polarization measure and the normalized poverty deficit index are found equivalent when the identification sensitivity parameter is unity and when the poverty line is at the mean value of income.

## REFERENCES


Bhattacharya, N., Mahalanobis, B., 1967. Regional disparities in household consumption in India. *Journal of the American Statistical Association* 62, 143-61.

Chakravarty, S.R., Majumder, A., 2001. Inequality, polarization and welfare: theory and applications. *Australian Economic Papers* 40, 1-13.

Cowell, F.A., 1980. On the structure of additive inequality measures. *Review of Economic Studies* 47, 521-31.

D'Ambrosio, C., 2001. Household characteristics and the distribution of income in Italy: an application of social distance measures. *Review of Income and Wealth* 47, 43-64.

D'Ambrosio, C., Wolff, E., 2001. Is wealth becoming more polarized in the United States? *Working Paper 330*, Levy Economics Institute, Bard College.

Duclos, J.-Y., Échevin, D., 2005. Bi-polarization comparisons. *Economic Letters* 87, 249-58.

Duclos, J.-Y., Esteban, J., Ray, D., 2004. Polarization: concepts, measurement, estimation. *Econometrica* 72, 1737-72.

Esteban, J., Ray, D., 1994. On the measurement of polarization. *Econometrica* 62, 819-51.

Esteban, J., Gradín C., Ray, D., 1999. Extensions of a measure of polarization, with an application to the income distribution of five OECD countries. *Documentos de Traballo 9924*, Universidade de Vigo.

Foster, J.E., Greer J., Thorbecke, E., 1984. A class of decomposable poverty measures. *Econometrica* 52, 761-65.

Gradín C., 2000. Polarization by sub-populations in Spain, 1973-91. *Review of Income and Wealth* 46, 457-74.

Gradín, C., 2002. Polarization and inequality in Spain: 1973-91. *Journal of Income Distribution* 11, 34-52.

Hey, J.D., Lambert, P.J., 1980. Relative deprivation and the Gini coefficient: comment. *Quarterly Journal of Economics* 95, 567-73.

Kakwani, N.C., 1980. *Income, inequality and poverty: methods of estimation and policy applications.* Oxford: Oxford University Press.

Kakwani, N.C., 1986. *Analysing redistribution policies: a study using Australian data.* Cambridge: Cambridge University Press.

Lambert, P., 2001. *The distribution and redistribution of income.* Manchester: Manchester University Press.

Lambert, P., Aronson, J.R., 1993. Inequality decomposition analysis and the Gini coefficient revisited. *The Economic Journal* 103, 1221-27.

Lasso de la Vega, M.C., Urrutia, A.M., 2006. An alternative formulation of the Esteban-Gradín-Ray extended measure of polarization. *Journal of Income Distribution*, forthcoming.

Newbery, D., 1970. A theorem on the measurement of inequality. *Journal of Economic Theory*, vol. 2, pp. 264-66.

Prieto, J., Rodríguez, J.G., Salas, R., 2004. Is an inequality-neutral flat tax reform really neutral? *Documento de Trabajo E2004/43*, Fundación Centro de Estudios Andaluces, CentrA.

Prieto, J., Rodríguez, J.G., Salas, R., 2005. Interactions inequality-polarization: characterization results. *Papeles de Trabajo Nº15/05*, Instituto de Estudios Fiscales.

Pyatt, G., 1976. The interpretation and disaggregation of Gini coefficients. *The Economic Journal* 97, 459-67.

Ravallion, M., Chen, S., 1997. What can new survey data tell us about recent changes in distribution and poverty. *World Bank Economic Review* 11, 357-82.

Rodríguez, J.G., Salas, R., 2003. Extended bi-polarization and inequality measures. *Research on Economic Inequality* 9, 69-83.

Runciman, W.G., 1966. *Relative deprivation and social justice.* London: Routledge and Kegan Paul/Penguin Books.

Sheshinski, E., 1972. Relation between a social welfare function and the Gini index of income inequality. *Journal of Economic Theory* 4, 98-100.

Wang, Y.Q., Tsui, K.Y., 2000. Polarization orderings and new classes of polarization indices. *Journal of Public Economic Theory* 2, 349-63.

Wolfson, M.C., 1994. When inequalities diverge. *American Economic Review* 84, 353-58.

Wolfson, M.C., 1997. Divergent inequalities: theory and empirical results. *Review of Income and Wealth* 43, 401- 21.

The World Bank, 2005. *World development indicators*, Electronic version at http://www.worldbank.org/data/wdi2005/wditext/Section2.htm.

Yitzhaki, S., 1979. Relative deprivation and the Gini coefficient. *Quarterly Journal of Economics* 93, 321-24.

Zhang, X., Kanbur, R., 2001. What difference do polarization measures make? An application to China. *Journal of Development Studies* 37, 85-98.