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Abstract

In this paper we investigate the problem of measuring social mobility when the social status of individuals is given by their rank. In order to sensibly represent the rank mobility of subgroups within a given society, we address the problem in terms of partial permutation matrices which include standard (“global”) matrices as a special case. We first provide a characterization of a partial ordering on partial matrices which, in the standard case of global matrices, coincides with the well-known “concordance” ordering. We then provide a characterization of an index of rank mobility based on partial matrices and show that, in the standard case of comparing two global matrices, it is equivalent to Spearman’s index.

JEL Classification: D31, D63.

Keywords: Mobility measurement, Concordance, Partial matrices, Spearman’s Index.

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THE MEASUREMENT OF RANK MOBILITY

MARCELLO D'AGOSTINO AND VALENTINO DARDANONI

ABSTRACT. In this paper we investigate the problem of measuring social mobility when the social status of individuals is given by their rank. In order to sensibly represent the rank mobility of subgroups within a given society, we address the problem in terms of *partial permutation matrices* which include standard (“global”) matrices as a special case. We first provide a characterization of a partial ordering on partial matrices which, in the standard case of global matrices, coincides with the well-known “concordance” ordering. We then provide a characterization of an index of rank mobility based on partial matrices and show that, in the standard case of comparing two global matrices, it is equivalent to Spearman’s ρ index.

KEYWORDS: Mobility measurement, Concordance, Partial matrices, Sperman’s index.

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1. INTRODUCTION

When discussing social mobility issues, a basic distinction is usually made between *intergenerational mobility* (how the distribution of some relevant measure of individual status changes between different generations in a given society) and *intragenerational mobility* (how the distribution of individual status changes among a group of individuals over a given period of their lifetime). As a vehicle of discussion, we shall concentrate on intergenerational mobility, but all our considerations and results could be easily transposed to the intragenerational case. All the information about a *social mobility context* is then contained in a bivariate cumulative distribution function, which describes the distribution of two random variables capturing fathers’ and sons’ *socio-economic status*. It is widely believed that socioeconomic mobility is somewhat an elusive concept, difficult to define, let alone to measure: as remarked by Fields and Ok [FO99a] in a recent survey “. . . the mobility literature does not provide a unified discourse of analysis. . . a considerable rate of confusion confronts a newcomer in the field.”¹ This may be contrasted with the literature on income inequality, where a consensus has emerged on what concepts of inequality mean, on the correct theoretical procedures to measure it, and on how to go from theory to empirical applications.

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¹See also Maasoumi [Maa98] for a survey on mobility measurement.

One of the main challenges in mobility measurement is the precise definition of individual socio-economic status, and its practical implementation using available data. Typically, mobility studies use data such as current or permanent income, consumption, occupational prestige, education etc. For ease of presentation, assume that income data is used for this purpose. In this paper we explore the possibility that the socio-economic status of each individual is given by his *rank*, i.e. by its relative position in the generation to which he belongs. This way of defining individual status seems quite natural and intuitively satisfying. Indeed, the Encyclopedia Britannica defines *social status* as “the relative rank that an individual holds” and some analysts follow this common-sense interpretation by *equating* the concept of social status with that of rank. However, although being an interesting and much used notion, to the best of our knowledge rank mobility has not been thoroughly investigated so far, and this paper attempts to fill this gap.

To be precise, suppose $(x_1, y_1), \dots, (x_n, y_n)$ describe the population distribution of fathers' and sons' incomes in a society of n families. We make the simplifying assumption that there are no ties in the marginal distributions.² Then, for our purposes, the socio-economic status of the i th father is given by the number of fathers with income less than or equal to y_i , and the socio-economic status of the j th son is given by the number of sons with income less than or equal to x_j . Hence, rank mobility comparisons use only the information on the *order* of the x 's and y 's. Mobility comparisons which use *cardinal* information on fathers' and sons' income are axiomatized, among others, by [CBW85, Cow85, FO96, FO99b, GS02, Kin83, MZ86, MO98, RC04] and capture different aspects of social mobility than the present paper. These contributions may be considered more complementary than alternative to our approach. Fields ([Fie02], chapter 6) compares some theoretical properties of various indices of income mobility including some indices of rank mobility, and Buchinsky *et. al.* [BFFK05] compare their empirical properties in an application to French income mobility.

Given our assumption of no ties in the marginal distributions, all the information concerning the rank mobility of a society is contained in a permutation matrix P , with typical element $P(i, j)$ equal to 1 if there is a family in this society whose father has rank i and son has rank j , and 0 otherwise. The problem is how to turn this information into a quantitative measure.³ In order to achieve a faithful and consistent representation of the rank mobility of *subgroups* within a given society, we address the problem in terms of *partial permutation matrices* (defined in Section 2) which include standard (“global”) matrices as a special case, and argue that a representation of the rank mobility of a given subgroup of the population in terms of global matrices would be

²This case typically arises when $(x_1, y_1), \dots, (x_n, y_n)$ is actually a random sample generated from a continuous distribution (so that ties occur with probability zero).

³Note that if $(x_1, y_1), \dots, (x_n, y_n)$ is a random sample and we multiply by $\frac{1}{n}$ the permutation matrix P we obtain the so called *empirical joint rank distribution function* (see Block *et al.* [BCFS90]).

paradoxical. After observing that a standard decomposability property — which is key in the characterization of additively separable indices — cannot be sensibly assumed in the context of measuring rank mobility, we take advantage of our representation in terms of partial matrices to define a weaker form of decomposability which can be safely assumed. As an intermediate step, in section 3 we provide (Theorem 1) a characterization of a partial ordering on partial matrices which in the case of global matrices coincides with the well-known “concordance” ordering. We then provide (in Theorems 2 and 3) a rather natural and simple characterization (up to a monotonic transformation) of an index of rank mobility based on partial matrices and show that, in the standard case of comparing the mobility of two populations, this is equivalent to Spearman’s ρ index. Our characterization seems to provide reasonable grounds for adopting this kind of index (rather than other alternative ones, such as Kendall’s τ or Spearman’s footrule) in the measurement of rank mobility.

2. SUBGROUP MOBILITY AND PARTIAL PERMUTATION MATRICES

Let F denote the set of all families who live in a given society and consider a subset $A \subseteq F$; examples of interesting subsets are the families which live in a given geographical location, or which belong to a given race, or whose fathers have a given education level etc. Sometimes we may be interested in exploring how the status of individuals change from one generation to the next for members of this particular subset. We could call this kind of information *the rank mobility of A with respect to F* . Observe that this is not the same as considering the rank mobility of A w.r.t. A , because individuals’ rank is calculated with respect to the whole of F . A simple example may help to clarify. Consider a society F consisting of six families in which the distributions of fathers’ and sons’ incomes is summarized in Table 1 below:

| | 1 | 2 | 3 | 4 | 5 | 6 |
|----------------|-----|-----|-----|-----|-----|-----|
| <i>Fathers</i> | 100 | 150 | 200 | 250 | 300 | 350 |
| <i>Sons</i> | 150 | 200 | 100 | 250 | 350 | 300 |

TABLE 1. Incomes of fathers and sons in F .

Now, consider the subset A of F consisting only of the third, fourth and fifth families. If we consider only A and calculate the rank of individuals with respect to this specific subset, then there is no rank mobility from one generation to the next. Viceversa, there is clearly a sense in which the families in A exhibit some status mobility, which is made apparent when the status is calculated with respect to the whole of F : the son of the third family has lost two positions with respect to his generation, while the son of the fifth family has gained one position.

This kind of “partial” mobility information, i.e. restricted to a subset of a whole set F of families, will then be described by an $n \times n$ matrix which differs from a

permutation matrix because it can have rows and columns with zeros only. Such matrices are called *partial permutation matrices*.⁴ When necessary for clarity, we shall call ordinary permutation matrices *global*. More formally, the set \mathcal{P}_n of $n \times n$ partial permutation matrices is defined as follows: a matrix P belongs to \mathcal{P}_n if and only if, for all $i = 1, \dots, n$ and $j = 1, \dots, n$, we have: (i) $P(i, j) \in \{0, 1\}$; (ii) $\sum_i P(i, j) \leq 1$; (iii) $\sum_j P(i, j) \leq 1$. Notice that, under this definition, global matrices are nothing but a special case of partial matrices.

Now, suppose A and B are disjoint subsets of a set F of n families. Clearly the partial permutation matrices that describe the rank mobility of A and B with respect to F , call them P and Q , will belong to \mathcal{P}_n and will be disjoint in a related sense that is expressed by the following definition:

Definition 1. $P, Q \in \mathcal{P}_n$ are disjoint if

$$P(i, j) = 1 \text{ and } Q(m, k) = 1 \Rightarrow i \neq m \text{ and } j \neq k.$$

Corollary 1. $P, Q \in \mathcal{P}_n$ are disjoint if and only if $P + Q \in \mathcal{P}_n$, where $+$ is the usual sum of matrices.

Therefore, the rank mobility with respect to F of disjoint subsets of F is represented by disjoint partial matrices. Notice that if we partition F into m mutually exclusive and exhaustive subsets A_1, \dots, A_m , the rank mobility of these subsets with respect to F will be described by mutually disjoint partial matrices P_1, \dots, P_m such that $P = P_1 + \dots + P_m$. Let us say that a partial matrix P is *atomic* if there exists exactly one i and one j such that $P(i, j) = 1$, that is, if we are considering a subset containing exactly one family. We shall use the lower case letter p (possibly with subscripts) to denote atomic matrices. Clearly, any partial permutation matrix in \mathcal{P}_n will be equal to a sum $p_1 + \dots + p_k$ of $k \leq n$ atomic matrices, where $k = n$ only for global matrices.

Observe that *any* $n \times n$ partial matrix P can be regarded as representing the rank mobility of some set of families A with respect to some “society” F of size n that includes it. Indeed, it is always possible to find an F such that the rank mobility determined by the marginal distributions of fathers’ and sons’ income is represented by a global matrix that includes P . So, the above corollary implies that the sum of two disjoint partial matrices can always be regarded as representing the rank mobility of some suitable subgroup A of a possible “society” F . Therefore, in this abstract setting, we can forget about “real” families, groups and societies and concentrate only on the partial matrices that represent their rank mobility. However, to avoid long-winded sentences, we shall often abuse of the more concrete terminology and speak, for instance, of “a society (group) P ” to mean “a society (group) whose rank mobility is represented by the (partial) matrix P ”, or of “the family (i, j) (in a matrix P)” to

⁴See e.g. Horn and Johnson [HJ91] for definitions and some properties of partial permutation matrices.

mean “the family in which the father’s rank is i and the son’s rank is j (in a society whose rank mobility is represented by P)”.

We are interested in axiomatizing the properties of some suitable ordering \preceq_M over \mathcal{P}_n such that $P \preceq_M Q$ can be taken as meaning that the matrix Q exhibits at least the same degree of social mobility as the matrix P .

3. AXIOMATIZING RANK MOBILITY ORDERINGS

In this section we shall start investigating the ordering relation \preceq_M . Given two matrices P and Q in \mathcal{P}_n , when can we say that Q displays at least the same rank mobility as P ? We introduce and discuss some plausible axioms to impose on \preceq_M and then derive characterization theorems following an incremental approach. As a first step, in Section 3.1, we shall only assume that \preceq_M is a *quasi-ordering*, that is a reflexive and transitive binary relation on \mathcal{P}_n , and shall not assume that the ordering is *complete*. Then, we derive, in Theorem 1, a characterization of what we propose as the basic rank mobility ordering from two basic axioms. Next, in Section 3.2, we investigate the possible completions of this basic quasi-ordering, and add further axioms which allow us to obtain sharper characterizations in Theorems 2 and 3.

3.1. The concordance ordering. While it is intuitively clear that it is meaningful to compare two standard (i.e. global) permutation matrices representing the rank mobility of two societies F and F' , it is not quite as clear whether it is equally meaningful to compare partial matrices — representing, say, the rank mobility of some subset $A \in F$ w.r.t. F and the rank mobility of some subset $B \in F'$ w.r.t. F' — when they have different marginal distributions.⁵ We shall therefore start by restricting the comparison to a clear-cut case.

Definition 2.

(1) Two matrices $P, Q \in \mathcal{P}_n$ are similar if

$$\{i | P(i, j) = 1 \text{ for some } j\} = \{i | Q(i, j) = 1 \text{ for some } j\} \text{ and} \\ \{j | P(i, j) = 1 \text{ for some } i\} = \{j | Q(i, j) = 1 \text{ for some } i\}.$$

(2) A matrix $P \in \mathcal{P}_n$ is monotone if, for all i, j, m, k such that $P(i, j) = 1$ and $P(m, k) = 1$, we have $(i - m)(j - k) > 0$.

So, two matrices are similar when they have equal marginal distributions. Note that the definition of similarity induces an equivalence relation on \mathcal{P}_n . Moreover, observe that within each similarity set there is a unique monotone matrix which can be considered as displaying the least amount of mobility:

⁵Loosely speaking, by “marginal distributions” of a partial matrix P we mean the following: the marginal distribution of the fathers is the set of all i such that $P(i, j) = 1$ for some j , and the marginal distribution of the sons is the set of all j such that $P(i, j) = 1$ for some i .

Axiom 1 (Monotonicity). *For any distinct $P, Q \in \mathcal{P}_n$ such that P is monotone and similar to Q we have $P \prec_M Q$.*

Notice also that if a matrix is monotone, then, and only then, there is a strictly increasing function from fathers' rank to sons' rank.⁶

The second axiom requires that the sum of disjoint partial matrices is a monotonic operation:

Axiom 2 (Subgroup Consistency). *For every $P_1, P_2, P_3, P_4 \in \mathcal{P}_n$ such that P_1 is disjoint with P_2 and P_3 is disjoint with P_4*

$$P_1 \preceq_M P_3 \text{ and } P_2 \preceq_M P_4 \Rightarrow P_1 + P_2 \preceq_M P_3 + P_4.$$

Similar axioms are commonly used in the literature on income inequality [Sho88], poverty [FS91] and mobility measurement [FO99b],⁷ where they usually imply a fundamental, and practically useful, *decomposability* property: given an arbitrary partition of a population into k subgroups, the problem of measuring a certain feature in the overall population can be reduced to the k separate problems of measuring that feature in each of the k subgroups.⁸ It must be stressed that the above axiom cannot be interpreted as asserting a similar decomposability of the rank mobility of a society F into the rank mobility of their subgroups. In the terminology used in the introduction, this would amount to asserting that, given a partition of F into A_1, \dots, A_k , the rank mobility of F w.r.t. F , can be decomposed into the rank mobility of A_1 w.r.t. A_1 , A_2 w.r.t. A_2 , etc., where the rank of each individual is evaluated with reference to the subgroup to which it belongs. However, this would clearly be paradoxical in the context of measuring rank mobility.

To see why, recall the simple example given in Section 2 (see Table 1), considering a society made of six families. Let A_1 be the subgroup consisting of the third, fourth and fifth families, and A_2 the subgroup consisting of the first, second and sixth. Now, it is clear that the rank mobility of the whole F w.r.t. F is greater than zero, while the rank mobility of A_1 w.r.t. A_1 and the rank mobility of A_2 w.r.t. A_2 are both, intuitively, equal to zero. Hence, we cannot hope that rank mobility enjoys such a strong decomposability property.⁹ However, our axiom states a *weaker* decomposability

⁶The reader may find it helpful to compare our concept of similar matrices with the well-known *Frechet class* of distributions with fixed marginals, and our monotonicity axiom with the lower bound in the Frechet class (see e.g. Nelsen [Nel99]).

⁷Though such axioms are widely accepted in these contexts, for a critical discussion see Foster and Sen [FS97].

⁸For this interpretation of the decomposability property in the context of social mobility see, for instance, Fields and Ok [FO96]. The term decomposability has different interpretations in other theoretical and applied contexts.

⁹That a mechanical application of standard decomposability properties is clearly nonsensical in this context is remarked, for example, by Cowell [Cow85] page 144, who explicitly states that distance measures based on ranks are not decomposable.

property: given a partition of F into A_1, \dots, A_n , the rank mobility of F *w.r.t.* F , can be decomposed into the rank mobility of A_1 *w.r.t.* F , A_2 *w.r.t.* F , etc., where the rank of each individual is evaluated with reference to the whole society F . Accordingly, the axiom is expressed in terms of *partial* permutation matrices for a society F , which are obtained from the global permutation matrix for F by omitting information concerning some of the families, and are intended to represent the rank mobility of given subgroups with respect to the whole of F . Our subgroup consistency axiom, therefore, cannot be interpreted as allowing us to measure the rank mobility of a population in terms of *independent* measurements of the rank mobility of its subgroups. Indeed, our axiom is better understood as a monotonicity requirement on the sum of disjoint *partial* permutation matrices, and, from this point of view, it expresses a basic logical property that seems quite uncontroversial.

Suppose, now, that we have a matrix P such that, for the indices $k < m$ and $l < n$ we have $P(k, l) = P(m, n) = 1$, and consider another matrix Q such that $P(i, j) = Q(i, j)$ for all $i \neq k, m$ and $j \neq l, n$, and $Q(k, n) = Q(m, l) = 1$. In words, Q differs from P because there has been an inversion of social status between two families, such that before the inversion the higher status father had the higher status son, while after the inversion the lower status father has the higher status son. Intuitively, such an inversion should be mobility-increasing.¹⁰ Under these circumstances we say that Q has been obtained from P by *inverting* (k, l) and (m, n) . We write $P \prec Q$ whenever Q can be obtained from P by means of such an inversion.

Suppose, a given matrix Q can be obtained from P by a *sequence* of inversions. We can intuitively conclude that Q displays more social mobility than P , and write $P \preceq_C Q$. Formally:

Definition 3. $P \preceq_C Q$ if and only if there is a finite sequence of matrices P_0, \dots, P_k , with $k \geq 0$, such that (i) $P_0 = P$, (ii) $P_k = Q$ and (iii) if $k > 0$, $P_{i-1} \prec P_i$ for all $i = 1, \dots, k$.

It can be easily checked that \preceq_C is a partial order¹¹ defined on each set of similar matrices. The reason for the choice of the subscript “ C ” is that, when the similarity class consists of the global matrices in \mathcal{P}_n , \preceq_C is called the *concordance ordering* in the mathematical statistics literature, see e.g. Tchen [Tch80] and Kimeldorf and Sampson [KS87].

Theorem 1. *Within each set of similar matrices, \preceq_C is the smallest¹² quasi-ordering which satisfies Axiom 1 and Axiom 2.*

¹⁰Such swaps are well-known in the mathematical statistics [Tch80] and economics [ET80, Atk83, Dar93] literature and it is often assumed that they are always mobility-increasing.

¹¹That is, besides being reflexive and transitive, it is also *antisymmetric*, namely: $P \preceq_C Q$ and $Q \preceq_C P$ imply that $P = Q$.

¹²In terms of set-inclusion.

A proof of this theorem is given in Appendix A. The concordance ordering \preceq_C is a very well established and much studied ordering of bivariate distributions. Atkinson [Atk83] first applies the concordance ordering to mobility measurement; Dardanoni [Dar93] applies it to a Markov chain model of social mobility, and shows the equivalence of a version of this ordering to some very intuitive concepts of greater social mobility. It is a partial ordering which, in the space of global permutation matrices, is a subrelation of many important complete orders, for example, those induced by the popular nonparametric indices of concordance such as Kendall's τ and Spearman's ρ , see e.g. Schweizer and Wolff [SW81]. The theorem then says that all reflexive and transitive relations \preceq_M which satisfy Axioms 1 and 2 must have a common area of agreement equal to \preceq_C .

On the other hand, \preceq_C allows for comparisons between *similar* matrices only and, while this restriction is immaterial when comparing global matrices, it makes the comparison of partial matrices impossible except for the artificial special case in which the matrices have exactly the same marginal distributions. Moreover, being a *partial* ordering, \preceq_C does not even allow for comparisons of all matrices in a given similarity set. Thus, in order to be able to compare all mobility contexts in \mathcal{P}_n , we must focus on *weak* orderings,¹³ rather than quasi-orderings. Clearly, even assuming that \preceq_M is a weak ordering, Axioms 1 and 2 are not sufficient to uniquely characterize it (since there are several weak orderings, e.g. the above mentioned ρ and τ , that satisfy them.) From this point of view, Theorem 1 only implies that every *weak* ordering \preceq_M that satisfies the axioms must include \preceq_C , and so the properties expressed by the axioms can be considered as minimal requirements on any suitable mobility ordering. So, in the sequel, we shall take our mobility ordering \preceq_M to be a *weak* ordering and seek for extra axiomatic properties that allow us to uniquely characterize it.

3.2. Completing the concordance ordering. In this section we investigate the possible completions of the basic concordance ordering characterized in Theorem 1. We shall therefore assume that our mobility ordering \preceq_M is a weak (i.e. transitive and complete) ordering and consider the class of weak orderings \preceq_M satisfying Axioms 1 and 2. (As implied by Theorem 1, they must all include the concordance ordering.) Our aim, now, is to investigate how these axioms can be expanded to single out a suitable mobility ordering from this class.

There are two distinct intuitive aspects of the notion of “greater mobility” which emerge from its conceptual analysis. One aspect, which is apparent in the standard definitions of some well-known orderings — such as the concordance ordering and the weak ordering based on Kendall's function τ — stems from the idea that there is an increase in mobility when two families interchange their relative position. On the other hand, from a different angle, mobility is related to the distance between father's and

¹³Recall that a weak ordering is a quasi-ordering which is also complete, namely such that, for all $P, Q \in \mathcal{P}_n$, either $P \preceq_M Q$ or $Q \preceq_M P$.

sons's status *within each family*, and overall mobility of a group of families may be construed as the aggregation of the degrees of mobility exhibited by all the families in that group.¹⁴

Now, for a single family in a society F , such that father's rank is i and son's rank is j , we can take $|i - j|$ as measuring the *social distance* between father's and son's social status. This basic intuition is captured by the following:

Axiom 3 (Atomic Monotonicity). *For any two atomic matrices $p, q \in \mathcal{P}_n$ such that $p(i, j) = q(i', j') = 1$,*

$$p \preceq_M q \iff |i - j| \leq |i' - j'|.$$

Notice that, although this axiom forces a unique weak ordering of atomic matrices, it is not sufficient to uniquely characterize \preceq_M in the whole of its domain.

Let's now introduce some notation which will simplify considerably the following discussion. If " P " denotes a matrix in a given space \mathcal{P}_n , then " P^m ", with $m \geq n$, will denote the matrix in \mathcal{P}_m which coincides with P wherever P is defined and contains only 0's everywhere else, i.e. the matrix defined as follows:

$$\begin{aligned} P^m(i, j) &= P(i, j) \text{ for all } i, j \leq n \text{ and} \\ P^m(i, j) &= 0 \text{ for any } i, j \text{ such that } i > n \text{ or } j > n. \end{aligned}$$

On the other hand, if " P " denotes a matrix in \mathcal{P}_m , we shall attach no meaning to the notation " P^k " with $k < m$. Observe that, by definition,

- (1) $(P^k)^m = P^m$ for every $m \geq k$
- (2) $(P + Q)^m = P^m + Q^m$ for every disjoint $P, Q \in \mathcal{P}_k$ with $k \leq m$.

Let us say that a matrix P is *null* if $i = j$ for all (i, j) such that $P(i, j) = 1$. Intuitively, a null-matrix says that the subgroup for which it is defined displays no mobility at all. The following axiom is an adaptation of the well-known Archimedean Property to our setting:

Axiom 4 (Archimedean Property). *For every m and all $P, Q \in \mathcal{P}_m$, the strict inequality $P \prec_M Q$ holds if and only if there is an $n \geq m$ and a non-null $R \in \mathcal{P}_n$, disjoint with P^n , such that*

$$P^n + R \sim_M Q^n.$$

Observe that, within a given space \mathcal{P}_n , a partial permutation matrix P is uniquely determined by the set $S(P) = \{(i, j) | P(i, j) = 1\}$. We call $S(P)$ *the characteristic set of P* .

Now, we can prove the following:

¹⁴Clearly these two concepts of mobility (one which considers the interplay of families and the other which considers families in isolation) are interrelated, since single families cannot change relative positions without affecting other families.

Theorem 2. \preceq_M is a weak ordering satisfying Axioms 1–4 if and only if there is a strictly increasing and strictly convex function $f : \mathbb{N} \rightarrow \mathbb{N}$ such that, for all n and for all $P, Q \in \mathcal{P}_n$,

$$P \preceq_M Q \iff \sum_{(i,j) \in S(P)} f(|i-j|) \leq \sum_{(i,j) \in S(Q)} f(|i-j|).$$

A proof of this theorem is given in Appendix B. Theorem 2 shows that Axioms 1–4 characterize (up to a monotonic transformation) a *class* of additive mobility indices which depends on the choice of an appropriate weighting function f . It is interesting to notice that, within the space of global matrices, two important indices of ordinal association which would seem appropriate to (im)mobility measurement, namely Kendall's τ and Spearman's footrule (see e.g. Kendall and Gibbons [KG90] for definitions and a discussion of their properties) do *not* belong to the class defined in Theorem 2.

Consider for example the global permutation matrices P , P' and P'' in \mathcal{P}_4 with the following characteristic sets:

$$\begin{aligned} S(P) &= \{(1, 1), (2, 4), (3, 3), (4, 2)\} \\ S(P') &= \{(1, 3), (2, 1), (3, 4), (4, 2)\}, \\ S(P'') &= \{(1, 1), (2, 3), (3, 4), (4, 2)\}. \end{aligned}$$

Using any of the mobility indices, say M , in the class characterized by Theorem 2, the mobility of P , P' and P'' will be equal to

$$\begin{aligned} M(P) &= H(f(0) + f(2) + f(0) + f(2)), \\ M(P') &= H(f(2) + f(1) + f(1) + f(2)), \\ M(P'') &= H(f(0) + f(1) + f(1) + f(2)) \end{aligned}$$

for some strictly increasing and strictly convex f and strictly increasing H . Now, if we adopted Spearman's footrule as a mobility measure, which corresponds to letting f be the identity function, P and P'' would display the same amount of mobility, since $0 + 2 + 0 + 2 = 0 + 1 + 1 + 2$. However, P can be derived from P'' by an inversion of the families $(2, 3)$ and $(3, 4)$. Thus $S'' \prec_C S$, and so Spearman's footrule is inconsistent with \preceq_C . This failure of Spearman's footrule to satisfy the basic ordering \preceq_C makes it unsuitable for measuring rank mobility.

On the other hand, it is easy to show that, in the class of global matrices, Kendall's τ does indeed agree with \preceq_C (see e.g. Schweizer and Wolff [SW81]). Nevertheless, it cannot satisfy all our axioms, as can be seen by observing that P and P' have the same value of Kendall's τ , while any of the indices of Theorem 2 would deliver different values, since in P' there are two families with social distance equal to 2 (as in P), but, in addition, there are *also* two families with positive social distance (since f is strictly increasing).

Now, while Theorem 2 characterizes a class of mobility indices which is small enough to exclude some important indices, it clearly contains more than one possible mobility index, so that its practical application is still dependent on the choice of the function f . This choice can, however, be guided by the following argument: consider an inversion of two families (i, j) and $(i + 1, j + 1)$, so that 1 is the distance between the fathers and 1 is the distance between the sons before the inversion. We shall write $P \triangleleft Q$ when Q is obtained from P by means of such an inversion.

It may be reasonable to assume that all the inversions of this type are *minimal* inversions and that they all generate an *equivalent* mobility increase.¹⁵ This assumption is expressed by the following axiom:

Axiom 5 (Minimal Inversion). *For any matrices $P, Q, R, \in \mathcal{P}_n$,*

$$P \triangleleft Q \text{ and } P \triangleleft R \implies Q \sim_M R.$$

Then we can prove the following:

Theorem 3. \preceq_M *is a weak ordering satisfying Axioms 1–5 if and only if for any $P, Q \in \mathcal{P}_n$,*

$$P \preceq_M Q \iff \sum_{(i,j) \in S(P)} (i - j)^2 \leq \sum_{(i',j') \in S(Q)} (i' - j')^2.$$

A proof is given in Appendix C. It can be easily verified that, within the set of global matrices, Theorem 3 characterizes (up to a monotonic transformation) the well-known Spearman index of ordinal association, since the latter (which is better described as an *immobility index*) can be written as

$$\rho(P) = 1 - \frac{6 \sum_{(i,j) \in S(P)} (i - j)^2}{n^3 - n}$$

(see e.g. Kendall and Gibbons [KG90], page 8).

On the other hand, the ordering characterized in Theorem 3 is not restricted to populations' comparisons. For partial permutation matrices, the theorem provides a means for comparing the status mobility of different subgroups when the concept of social status we are interested in refers to the rank of individuals in the *whole* society. As an example, recall again the society F considered in in Section 2 (see Table 1), and assume that the third, fourth and fifth family belong to a first group, while the first, second and sixth belong to a second group. It is then easily calculated that families in the first group exhibit a greater level of rank mobility than those in the second, since applying Theorem 3 we have $4 + 0 + 1 > 1 + 1 + 1$.

¹⁵By contrast, we might assume that the increase of mobility generated by an inversion should depend, somehow, also on the values of the father's rank i and the son's rank j in the inverted families. In this case, of course, not all minimal inversions would be equivalent.

APPENDICES

APPENDIX A. PROOF OF THEOREM 1

Proof. We first show that the \preceq_C ordering (which, we recall, is defined on each set of similar partial matrices) satisfies the axioms. It is obvious that it satisfies Axiom 1. As for Axiom 2, suppose P_1, P_2 and P_3, P_4 are mutually disjoint, and $P_1 \preceq_C P_3$ and $P_2 \preceq_C P_4$. Then there exists a sequence Q_0, \dots, Q_k of partial matrices in \mathcal{P}_n such that (i) $Q_0 = P_1$, (ii) $Q_k = P_3$ and (iii) $Q_i \prec Q_{i+1}$ for $i = 0, \dots, k-1$. Similarly, there exists a sequence $R_0, \dots, R_{k'}$ of partial matrices in \mathcal{P}_n such that (i) $R_0 = P_2$, (ii) $R_{k'} = P_4$ and (iii) $R_i \prec R_{i+1}$ for $i = 0, \dots, k'-1$. Suppose $k' > k$. Then, it is easy to see that, since P_1, P_2 , and P_3, P_4 are mutually disjoint, the sequence

$$Q_0 + R_0, \dots, Q_k + R_k, Q_k + R_{k+1}, \dots, Q_k + R_{k'}$$

is such that (i) $Q_0 + R_0 = P_1 + P_2$, (ii) $Q_k + R_{k'} = P_3 + P_4$, (iii) $Q_i + R_i \prec Q_{i+1} + R_{i+1}$ for $i = 0, \dots, k-1$, and (iv) $Q_k + R_j \prec Q_k + R_{j+1}$ for $j = k, \dots, k'-1$. Hence, by definition of \preceq_C , we have that $P_1 + P_3 \preceq_C P_2 + P_4$. The argument is similar when $k > k'$.

Next, we show that if a quasi-ordering \preceq_M satisfies the axioms, then it must include the concordance ordering. This is sufficient to conclude that \preceq_C is the smallest quasi-ordering satisfying the axioms.

Suppose \preceq_M satisfies the axioms. Let P and Q be two matrices such that $P \preceq_C Q$. By definition, this means that P and Q are similar and there is a sequence of matrices P_0, \dots, P_k , with $k \geq 0$, such that $P_0 = P$, $P_k = Q$ and, if $k > 0$, $P_i \prec P_{i+1}$ for all $i = 1, \dots, k-1$. Now consider the i -th inversion step, and suppose it is such that, for some j, m, l, n with $j < m$ and $l < n$, $P_{i-1}(j, l) = P_{i-1}(m, n) = 1$ and $P_i(j, n) = P_i(m, l) = 1$. Consider the matrix P_{i-1}^* such that only $P_{i-1}^*(j, l) = P_{i-1}^*(m, n) = 1$, while all the other entries are 0 (that is, its characteristic set $S(P_{i-1}^*)$ is equal to $\{(j, l), (m, n)\}$). Let also P_i^* be the similar matrix such that only $P_i^*(j, n) = P_i^*(m, l) = 1$, while all the others are 0 (that is, its characteristic set $S(P_i^*)$ is equal to $\{(j, n), (m, l)\}$).

Clearly

$$P_{i-1} = (P_{i-1} - P_{i-1}^*) + P_{i-1}^* \text{ and } P_i = (P_i - P_i^*) + P_i^*.$$

Moreover, $P_{i-1} - P_{i-1}^* = P_i - P_i^*$ and, since $j < m$ and $l < n$, the matrix P_{i-1}^* is a monotone matrix, so that $P_{i-1}^* \preceq_M P_i^*$ (by Axiom 1). Therefore,

$$P_{i-1} = (P_{i-1} - P_{i-1}^*) + P_{i-1}^* \preceq_M (P_i - P_i^*) + P_i^* = P_i$$

by Axiom 2. Hence, $P_{i-1} \preceq_M P_i$. The same argument holds for all i , therefore $P \preceq_M Q$. This shows that $P \preceq_C Q$ implies $P \preceq_M Q$ for all P, Q , i.e. \preceq_C is included in \preceq_M . Since \preceq_M was an arbitrary quasi-ordering satisfying the axioms, \preceq_C is included in all the quasi-orderings satisfying the axioms. \square

APPENDIX B. PROOF OF THEOREM 2

B.1. Preliminary lemmas. The following lemmas turn out to be useful to make the proof of Theorem 2 more readable.

Lemma 1. *If \preceq_M is a weak ordering satisfying Axiom 4, then*

$$P \preceq_M Q \text{ if and only if } P^m \preceq_M Q^m,$$

for all n , all $P, Q \in \mathcal{P}_n$ and all $m \geq n$.

Proof. Suppose $P \preceq_M Q$ and $P^m \not\preceq_M Q^m$, that is $P^m \succ_M Q^m$, for some $m \geq n$. Then, by Axiom 4, there is an $m' \geq m$ and a non-null $R \in \mathcal{P}_{m'}$ such that R is disjoint with $Q^{m'}$ and

$$Q^{m'} + R \sim_M P^{m'}.$$

Since $m' \geq n$, it follows again from Axiom 4, that $Q \prec_M P$, against the hypothesis. Hence:

$$(1) \quad P \preceq_M Q \implies P^m \preceq_M Q^m.$$

For the converse, suppose that for some $m > n$, $P^m \preceq_M Q^m$ and $P \succ_M Q$. Then, by Axiom 4,

$$(2) \quad P^k \sim_M Q^k + R,$$

for some $k \geq n$ and some non-null $R \in \mathcal{P}_k$. Now, if $k \geq m$, this implies, again by Axiom 4, that $Q^m \prec_M P^m$ against the hypothesis. If $k < m$, it follows from (2), by (1), that $P^m \sim_M (Q^k + R)^m \sim_M Q^m + R^m$. Since R^m is non-null, it follows, again by Axiom 4, that $Q^m \prec_M P^m$, against the hypothesis. \square

Lemma 2. *If \preceq_M is a weak ordering satisfying Axiom 4, then*

$$P \preceq_M P + R,$$

for all n , and all $P, R \in \mathcal{P}_n$ such that $P + R \in \mathcal{P}_n$.

Proof. Suppose $P \succ_M P + R$. Then, by Axiom 4,

$$P^m \sim_M (P + R)^m + S \sim_M P^m + R^m + S$$

for some $m \geq n$ and some non-null $S \in \mathcal{P}_m$. Since $R^m + S$ is non-null, this would imply, again by Axiom 4, that $P \succ_M P$, which is impossible. \square

Let us denote by \perp_n the unique matrix $P \in \mathcal{P}_n$, that we call the *empty matrix*, such that $P(i, j) = 0$ for all i, j , that is the $n \times n$ matrix which is totally undefined. By definition, (i) \perp_n is a null matrix (ii) for every $P \in \mathcal{P}_n$, \perp_n is disjoint with P , and (iii) $\perp_n + P = P$.

Lemma 3. *If \preceq_M is a weak ordering satisfying Axioms 2, 3 and 4, then*

$$\perp_n \sim_M P \prec_M Q$$

for every null matrix $P \in \mathcal{P}_n$ and every non-null matrix Q in \mathcal{P}_n .

Proof. First, recall that (by Axiom 4) $\perp_n \prec_M Q$, for every n and every non-null $Q \in \mathcal{P}_n$, since $\perp_n + Q = Q \sim_M Q$. Hence, we only have to show that $P \sim_M \perp_n$ for every null $P \in \mathcal{P}_n$. If $P = \perp_n$, then it is trivially true that $P \sim_M \perp_n$. Consider, then, the case that $P \neq \perp_n$. Let us first show that $p \sim_M \perp_n$ for every null *atomic* matrix p . Suppose, that $p \succ_M \perp_n$. Then, by Axiom 4, $p^m \sim_M \perp_m + R = R$ for some $m \geq n$ and some non-null $R \in \mathcal{P}_m$. Since R is non-null, $R = r + T$ for some non-null atomic matrix r and some, possibly null, matrix T in \mathcal{P}_m . By Axiom 3, $p^m \prec_M r$ and, by Lemma 2, $p^m \prec_M r + T = R$. This is a contradiction, since we had before concluded that $p^m \sim_M R$. Suppose, then, that $\perp_n \succ_M p$. By Axiom 4, $p^m + R \sim \perp_m$ for some $m \geq n$ and some non-null $R \in \mathcal{P}_m$. Now, since $\perp_m \prec_M Q$ for every non-null $Q \in \mathcal{P}_m$, it follows that $\perp_m \prec_M R$. Then, by Lemma 3, $\perp_m \prec_M p^m + R$ against the previous conclusion that $\perp_m \sim_M p^m + R$. Hence, since \preceq_M is a weak ordering, $\perp_m \sim_M p$.

If P is not an atomic matrix and $P \neq \perp_n$, then $P = p_1 + \dots + p_k$ for some k such that $1 < k \leq n$, with each p_i ($1 \leq i \leq k$) being a null atomic matrix. As we have just established, $p_i \sim_M \perp_n$ for all $i = 1, \dots, k$. Hence, by Axiom 2 (and recalling that $\perp_n + \perp_n = \perp_n$), $P \sim_M \perp_n$. \square

Say that two matrices P and Q are *atomically equivalent* if, for every $k \geq 0$, they contain the same number of non-zero entries (i, j) with $|i - j| = k$. Clearly, if P and Q are atomically equivalent, there are atomic matrices p_1, \dots, p_m and q_1, \dots, q_m , with $m \leq n$, such that $P = p_1 + \dots + p_m$, $Q = q_1 + \dots + q_m$ and, by Axiom 3, $p_i \sim_M q_i$ for $i = 1, \dots, m$. Hence, by Axiom 2, if P and Q are atomically equivalent, then $P \sim_M Q$.

Remark 1. *Given any two matrices $P, Q \in \mathcal{P}_n$, one can always find a sufficiently large m and a matrix R in \mathcal{P}_m such that R is atomically equivalent to Q^m and disjoint with P^m . For this purpose, it is sufficient to take $m = 2n$ and R equal to the matrix such that (i) $R(i, j) = 0$ for all $i, j \leq n$ and (ii) $R(n + i, n + j) = 1$ if and only if $Q(i, j) = 1$. Using this method, if P_1, \dots, P_k are matrices in \mathcal{P}_n , one can always find suitable matrices $P'_1, \dots, P'_k \in \mathcal{P}_{kn}$ such that (i) $P_i^{kn} \sim_M P'_i$ for $i \leq k$ and (ii) all the P'_{kn} are mutually disjoint.*

Lemma 4. *If \preceq_M is a weak ordering satisfying Axioms 2 and 4, then for all n , and all $P, Q, R, S \in \mathcal{P}_n$ such that P is disjoint with R and Q is disjoint with S ,*

$$P \sim_M Q \text{ and } P + R \sim_M Q + S \implies R \sim_M S.$$

Proof. Let us assume that $P \sim_M Q$ and $P + R \sim_M Q + S$. Suppose, ex absurdo, that $R \not\sim_M S$.

Case 1: $R \succ_M S$. Then, it follows from Axiom 4, that $R^m \sim_M S^m + T$ for some $m \geq n$ and some non-null $T \in \mathcal{P}_m$. By Remark 1, there are $m' \geq m$ and $U \in \mathcal{P}_{m'}$ such that $U \sim_M P^{m'}$ and U is disjoint with $S^m + T$. Hence, by Lemma 1 and Axiom 2,

$$P^{m'} + R^{m'} \sim_M U + S^{m'} + T \sim Q^{m'} + S^{m'}.$$

So, by Axiom 4, $U + S^{m'} \prec_M Q^{m'} + S^{m'}$ (since T is non-null). However, by Axiom 2, $U + S^{m'} \sim_M Q^{m'} + S^{m'}$ (since $U \sim_M P^{m'} \sim_M Q^{m'}$ by hypothesis and Lemma 1), which is a contradiction.

Case 2: $R \prec_M S$. This case is similar to Case 1 and is left to the reader. \square

Now, consider the set $\mathcal{P} = \bigcup_{i=1}^{\infty} \mathcal{P}_i$ of all partial matrices. We define the subset Δ_k , $k \geq 0$, of \mathcal{P} as the set of all $P \in \mathcal{P}$ such that for all $(i, j) \in S(P)$, $|i - j| \leq k$. Notice that $\Delta_k \subseteq \Delta_m$ whenever $k \leq m$. The matrices in Δ_0 are the null matrices. We shall also write Δ_k^n for $\Delta_k \cap \mathcal{P}_n$. Moreover, given two matrices $P, Q \in \mathcal{P}_n$, let us say that Q is *contained* in P if $P(i, j) = 1$ for all $i, j \in \{1, \dots, n\}$ such that $Q(i, j) = 1$. Recall that every partial matrix can be uniquely expressed as a sum of atomic matrices.

Lemma 5. *If \preceq_M is a weak ordering satisfying Axioms 2, 3 and 4, then for all n and all $P, Q \in \Delta_1^n$, $P \preceq_M Q$ if and only if the number of non-null atomic matrices contained in P is less than or equal to the number of non-null atomic matrices contained in Q .*

Proof. Suppose first that the number of non-null atomic matrices contained in P is less than or equal to the number of non-null atomic matrices contained in Q . Let p_1, \dots, p_j be the non-null atomic matrices in P and q_1, \dots, q_k the non-null atomic matrices in Q , with $j \leq k \leq n$. Then $P = p_1 + \dots + p_j + R$ for some null $R \in \mathcal{P}_n$ and $Q = q_1 + \dots + q_j + S$ for some possibly non-null $S \in \mathcal{P}_n$. By Axiom 3 all the non-null atomic matrices in Δ_1^n are equivalent to each other and therefore, by Axiom 2, $p_1 + \dots + p_j \sim_M q_1 + \dots + q_j$. Moreover, by Lemma 3, $R \preceq_M S$. Hence, again by Axiom 2, $P \preceq_M Q$. Suppose now that the number of non-null atomic matrices in P is strictly greater than the number of non-null atomic matrices in Q . Let p_1, \dots, p_k the non-null atomic matrices in P and q_1, \dots, q_j the non-null atomic matrices in Q , with $j \leq k \leq n$. So, $P = p_1 + \dots + p_j + R$ for some non-null $R \in \mathcal{P}_n$ and $Q = q_1 + \dots + q_j + S$ for some null $S \in \mathcal{P}_n$. Moreover, by Lemma 3, $S \prec_M R$ and, as argued above, $p_1 + \dots + p_j \sim_M q_1 + \dots + q_j$. So, by Axiom 2, $Q = q_1 + \dots + q_j + S \preceq_M p_1 + \dots + p_j + R = P$. By Lemma 4, $Q \sim_M P$ would imply that $S \sim_M R$ which, given that S is null and R is non-null, is ruled out by Lemma 3. Therefore, we can conclude that $Q \prec_M P$. \square

Lemma 6. *If \preceq_M is a weak ordering satisfying Axioms 2–4, then for every n and every atomic matrix $p \in \mathcal{P}_n$, there is an $m \geq n$ such that $p^m \sim_M Q$ for some $Q \in \Delta_1^m$.*

Proof. Since every atomic matrix $p \in \mathcal{P}_n$ belongs to some Δ_k^n , with $k \in \mathbb{N}$, we prove the lemma by induction on the index k of the smallest class Δ_k^n to which p belongs. In the course of the proof, and for the sake of clarity, we shall reserve the notation \hat{P}, \hat{Q} , etc. to refer to matrices in Δ_1 .

Base: $p \in \Delta_1^n$. Trivial.

Step: $p \in \Delta_{j+1}^n$. Assuming that the lemma holds for all atomic matrices in Δ_j^n we show that it holds also for all atomic matrices in Δ_{j+1}^n .

Suppose p is an atomic matrix which belongs to Δ_{j+1}^n but does not belong to Δ_j^n . Then, p is non-null and, by Axiom 3, all atomic matrices in Δ_j^n are strictly less than p . Let now q be a non-null atomic matrix in Δ_j^n . Then, $q \prec_M p$ and, by Axiom 4, $q^m + R \sim_M p^m$ for some $m \geq n$ and some non-null $R \in \mathcal{P}_m$. Since, q^m is itself non-null, this implies (again by Axiom 4) that $R \prec_M p^m$. Now, we argue that R must be in Δ_j^m . We reason by absurd. Suppose $R \notin \Delta_j^m$, then $R = r + S$ for some atomic $r \in \mathcal{P}_m$ not in Δ_j^m , and some (possibly empty) matrix $S \in \mathcal{P}_m$. However (by Axiom 3) $r \succeq_M p^m$ and (by Lemma 2) $r + S \succeq_M p^m$. Hence, $R \succeq_M p^m$ against the conclusion, reached before, that $R \prec_M p^m$. Thus, R must be in Δ_j^m and so also $q^m + R$ is in Δ_j^m . By inductive hypothesis, there is an $m' \geq m$ such that $q^{m'} + R^{m'} \sim_m \hat{P}$ for some $\hat{P} \in \Delta_1^{m'}$. So, since $p^m \sim_m q^m + R$, by Lemma 1, $p^{m'} \sim_M \hat{P}$. This concludes the proof of the lemma. \square

B.2. Proof of the main theorem.

Proof. We leave it to the reader to prove that the ordering satisfies Axioms 1–4.

To show that any ordering \preceq_M which satisfies the axioms must be of the required form, let P, Q be two matrices in \mathcal{P}_n . First, recall that P and Q can be rewritten as sums of atomic matrices,

$$P = \sum_{(i,j) \in S(P)} p_{(i,j)} \quad \text{and} \quad Q = \sum_{(i,j) \in S(Q)} p_{(i,j)},$$

where $p_{(i,j)}$ is the atomic matrix in \mathcal{P}_n such that $S(p_{(i,j)}) = \{(i,j)\}$.

By Lemma 6, for each atomic $p_{(i,j)} \in \mathcal{P}_n$, there is an $k \geq n$ such that $p_{(i,j)}^k \sim_M \hat{P}_{(i,j)}$ for some $\hat{P}_{(i,j)} \in \Delta_1^k$. Observe that, by Remark 1, one can always find, for each $(i,j) \in S(P) \cup S(Q)$, a suitable matrix $\hat{P}'_{(i,j)}$ in Δ_1^K , for some sufficiently large K , such that (i) $\hat{P}'_{(i,j)}$ is atomically equivalent to $\hat{P}_{(i,j)}^K$ (and therefore, by Lemma 1, also to $p_{(i,j)}^K$), (ii) all the $\hat{P}'_{(i,j)}$ such that $(i,j) \in S(P)$ are mutually disjoint, and (iii) all the $\hat{P}'_{(i,j)}$ such that $(i,j) \in S(Q)$ are mutually disjoint.

Thus, by Axiom 2,

$$\begin{aligned} P^K &= \sum_{(i,j) \in S(P)} p_{(i,j)}^K \sim_M \sum_{(i,j) \in S(P)} \hat{P}'_{(i,j)} \\ Q^K &= \sum_{(i,j) \in S(Q)} p_{(i,j)}^K \sim_M \sum_{(i,j) \in S(Q)} \hat{P}'_{(i,j)}. \end{aligned}$$

Hence, for all $P, Q \in \mathcal{P}_n$

$$(3) \quad P \preceq_M Q \iff P^K \preceq_M Q^K \iff \sum_{(i,j) \in S(P)} \hat{P}'_{(i,j)} \preceq_M \sum_{(i,j) \in S(Q)} \hat{P}'_{(i,j)}$$

Now, let

$$\mathcal{S}_{(i,j)}^k = \{Q \in \Delta_1^k \mid Q \sim_M p_{(i,j)}^k\}$$

and

$$\mathcal{S}_{(i,j)} = \bigcup_{k \in \mathbb{N}} \mathcal{S}_{(i,j)}^k$$

We show that any two matrices in $\mathcal{S}_{(i,j)}$ contain the same number of non-null atomic matrices, and therefore this number depends only on i and j . Let Q_1 and Q_2 be any two matrices in $\mathcal{S}_{(i,j)}$. Then, for some k, k' , $Q_1 \in \Delta_1^k$, $Q_2 \in \Delta_1^{k'}$, $Q_1 \sim_M p_{(i,j)}^k$ and $Q_2 \sim_M p_{(i,j)}^{k'}$. We assume without loss of generality that $k' \geq k$. By Lemma 1, $p_{(i,j)}^{k'} \sim Q_1^{k'}$, and so $Q_1^{k'} \sim_M Q_2$. Since these two matrices are both in $\Delta_1^{k'}$, by Lemma 5, they must contain the same number of non-null atomic matrices. Moreover, the number of non-null atomic matrices contained in $Q_1^{k'}$ is the same as the number of those contained in Q_1 . Thus, all the matrices in the set $\mathcal{S}_{(i,j)}$ contain the same number of non-null atomic matrices which depends only on i and j . Let us denote it by $n_{(i,j)}$ and let f be the function $\mathbb{N} \mapsto \mathbb{N}$ such that, for every i, j , $f(|i - j|) = n_{(i,j)}$. So, since $P'_{(i,j)}$ belongs to $\mathcal{S}_{(i,j)}$, the number of non-null atomic matrices contained in $P'_{(i,j)}$ is equal to $f(|i - j|)$.

Now, the matrices $\sum_{(i,j) \in S(P)} \hat{P}'_{(i,j)}$ and $\sum_{(i,j) \in S(Q)} \hat{P}'_{(i,j)}$ in (3) are in Δ_1^K . So, by Lemma 5, they can be compared by simply counting the number of non-null atomic matrices contained in them. This is equal to the sum of the numbers of non-null atomic matrices contained in each $P'_{(i,j)}$ which is, in turn, equal to $f(|i - j|)$. Therefore:

$$(4) \quad \sum_{(i,j) \in S(P)} \hat{P}'_{(i,j)} \preceq_M \sum_{(i,j) \in S(Q)} \hat{P}'_{(i,j)} \iff \sum_{(i,j) \in S(P)} f(|i - j|) \leq \sum_{(i,j) \in S(Q)} f(|i - j|).$$

Finally, from (3) and (4) it follows that:

$$P \preceq_M Q \iff \sum_{(i,j) \in S(P)} f(|i - j|) \leq \sum_{(i,j) \in S(Q)} f(|i - j|).$$

It is obvious, by Axiom 3, that f must be strictly increasing. To show that f must be strictly convex, consider, for any $k \geq 0$, the matrices P and Q such that:

$$\begin{aligned} S(P) &= \{(1, k + 1), (2, k + 2), (k + 3, k + 3), (k + 4, k + 4), \dots\} \\ S(Q) &= \{(1, k + 2), (2, k + 1), (k + 3, k + 3), (k + 4, k + 4), \dots\}. \end{aligned}$$

Hence, by Axiom 1, we must have that, for all k , $2f(k) < f(k + 1) + f(k - 1)$. \square

APPENDIX C. PROOF OF THEOREM 3

Proof. Given Theorem 2, we can concentrate only on Axiom 5. The reader can easily check that if $f(k) = k^2$ the ordering satisfies Axiom 5.

To show that, in order to satisfy Axiom 5, f must be quadratic, suppose P, Q, R are matrices in \mathcal{P}_n such that:

$$\begin{aligned} S(P) &= \{(0, 0), (1, 1), (0, k), (1, k + 1)\} \\ S(Q) &= \{(0, 1), (1, 0), (0, k), (1, k + 1)\} \\ S(R) &= \{(0, 0), (1, 1), (0, k + 1), (1, k)\}. \end{aligned}$$

Thus, $P \triangleleft Q$ and $P \triangleleft R$, since the inversions that lead from P to Q and from P to R are both minimal. Then, by Axiom 5, $Q \sim_M R$ and therefore:

$$\sum_{(i,j) \in S(Q)} f(|i - j|) = 2f(1) + 2f(k) = \sum_{(i,j) \in S(R)} f(|i - j|) = 2f(0) + f(k + 1) + f(k - 1).$$

Observe that, by definition of f , $f(0) = 0$, and $f(1) = 1$ (see above, Appendix B.2). Therefore, to satisfy Axiom 5, since k is arbitrary and f is fixed for all n , we must have that, for all k ,

$$f(k + 1) - f(k) = f(k) - f(k - 1) + 2.$$

This difference equation has a unique solution, i.e. $f(k) = k^2$. □

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