

# The Differential Approach to Demand Analysis and the Rotterdam Model\*

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## Abstract

This paper presents the differential approach to applied demand analysis. The demand systems of this approach are general, having coefficients which are not necessarily constant. We consider the Rotterdam parameterization of differential demand systems and derive the absolute and relative price versions of the Rotterdam model, due to Theil (1965) and Barten (1966). We address estimation issues and point out that, unlike most parametric and semi-nonparametric demand systems, the Rotterdam model is econometrically regular.

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# 1 Introduction

There is an old tradition in applied demand analysis which specifies the demand equations directly with no reference to any utility function. Under this approach, the demand for a good  $i$ ,  $x_i$ , is specified as a function of nominal income,  $y$ , and prices,  $p_1, \dots, p_n$ , where  $n$  is the number of goods.

Consider, for example, the log-log demand system,

$$\log x_i = \alpha_i + \eta_{iy} \log y + \sum_{j=1}^n \eta_{ij} \log p_j, \quad i = 1, \dots, n, \quad (1)$$

where  $\alpha_i$ ,  $\eta_{iy}$ , and  $\eta_{ij}$  are constant coefficients. The coefficient  $\eta_{iy}$  is the income elasticity of demand for good  $i$ ,  $\eta_{iy} = d \log x_i / d \log y$ , and the coefficient  $\eta_{ij}$  is the uncompensated (Cournot) cross-price elasticity of good  $i$ ,  $\eta_{ij} = d \log x_i / d \log p_j$ , including both the income and substitution effects of the changes in prices.

Another example of a demand system without reference to the utility function is Working's (1943) model,

$$w_i = \alpha_i + \beta_i \log y, \quad i = 1, \dots, n, \quad (2)$$

expressing the budget share of good  $i$ ,  $w_i = p_i x_i / y$ , as a linear function of logged income,  $\log y$ . As equation (2) does not involve prices, it is applicable to cross sectional data that offer limited variation in relative prices but substantial variation in income levels. To apply equation (2) to time series data that offer substantial variation in relative prices but less variation in income, the model has to be extended by adding a substitution term, as in equation (1).

Unlike this traditional single equation approach to demand analysis, neoclassical consumer theory assumes a representative economic agent with preferences over consumption goods, captured by a utility function. The representative consumer maximizes utility subject to a budget constraint and the solution to this problem is a unique demand system. This system-wide approach to empirical demand analysis allows for the imposition and testing of cross-equation restrictions (such as, for example, symmetry), unlike the traditional single equation approach which ignores such restrictions. The modern, system-wide approach to demand analysis has its origins in the work of Theil (1965) and the Rotterdam model, although that model avoids the necessity of using a particular functional form for the utility function.

This paper discusses the differential approach to demand analysis and the Rotterdam model. It is organized as follows. Section 2 reviews neoclassical consumption theory and utility based demand analysis. Section 3 presents the differential approach to applied demand analysis and presents differential demand systems in relative and absolute prices. In section 4, we consider the Rotterdam parameterization of differential demand systems and derive the relative and absolute price versions of the Rotterdam model, due to Theil (1965) and Barten

(1966). In section 5, we address estimation issues, and in Section 6 we emphasize the need for economic theory to inform econometric research, and point out that, unlike most parametric and semi-nonparametric demand systems, the Rotterdam model is econometrically regular. The final section concludes the paper.

## 2 Neoclassical Consumer Theory

Consider  $n$  consumption goods that can be selected by a consuming household. The household's problem is

$$\max_{\mathbf{x}} u(\mathbf{x}) \quad (3)$$

subject to

$$\mathbf{p}'\mathbf{x} = y, \quad (4)$$

where  $\mathbf{x}$  is the  $n \times 1$  vector of goods;  $\mathbf{p}$  is the corresponding vector of prices;  $y$  is the household's total nominal income; and  $u(\mathbf{x})$  is the utility function.

The first order conditions for a maximum can be found by forming an auxiliary function, known as the Lagrangian,

$$\mathcal{L} = u(\mathbf{x}) + \lambda(y - \mathbf{p}'\mathbf{x}),$$

where  $\lambda$  is the Lagrange multiplier. By differentiating  $\mathcal{L}$  with respect to  $x_i$ , and using the budget constraint, we obtain the  $(n + 1)$  first order conditions

$$\frac{\partial u(\mathbf{x})}{\partial x_i} = \lambda p_i, \quad i = 1, \dots, n; \quad (5)$$

$$\mathbf{p}'\mathbf{x} = y, \quad (6)$$

where  $\partial u(\mathbf{x})/\partial x_i$  is the marginal utility of good  $i$ .

The first order conditions can be solved for the  $n$  optimal (i.e., equilibrium) values of  $x_i$

$$x_i = x_i(\mathbf{p}, y), \quad i = 1, \dots, n, \quad (7)$$

and the optimal value of  $\lambda$ ,

$$\lambda = \lambda(\mathbf{p}, y). \quad (8)$$

System (7) is the demand system, giving the quantity demanded as a function of the prices of all goods and money income.

Total differentiation of the first order conditions for utility maximization, (5) and (6), gives

$$\begin{bmatrix} \mathbf{U} & \mathbf{p} \\ \mathbf{p}' & 0 \end{bmatrix} \begin{bmatrix} d\mathbf{x} \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \lambda \mathbf{I} \\ 1 & -\mathbf{x}' \end{bmatrix} \begin{bmatrix} dy \\ d\mathbf{p} \end{bmatrix}, \quad (9)$$

where  $\mathbf{U}$  is the  $n \times n$  Hessian matrix of the utility function,

$$\mathbf{U} = \begin{bmatrix} \frac{\partial^2 u(\mathbf{x})}{\partial x_1^2} & \cdots & \frac{\partial^2 u(\mathbf{x})}{\partial x_1 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u(\mathbf{x})}{\partial x_n \partial x_1} & \cdots & \frac{\partial^2 u(\mathbf{x})}{\partial x_n^2} \end{bmatrix};$$

the Hessian matrix is a symmetric and negative definite matrix. Also, total differentiation of the demand system (7) and (8), yields

$$\begin{bmatrix} d\mathbf{x} \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{x}_y & \mathbf{X}_p \\ -\lambda_y & -\lambda'_p \end{bmatrix} \begin{bmatrix} dy \\ d\mathbf{p} \end{bmatrix}, \quad (10)$$

where

$$\lambda_p = \begin{bmatrix} \frac{\partial \lambda}{\partial p_1} \\ \vdots \\ \frac{\partial \lambda}{\partial p_n} \end{bmatrix}, \quad \mathbf{x}_y = \begin{bmatrix} \frac{\partial x_1}{\partial y} \\ \vdots \\ \frac{\partial x_n}{\partial y} \end{bmatrix}, \quad \mathbf{X}_p = \begin{bmatrix} \frac{\partial x_1}{\partial p_1} & \cdots & \frac{\partial x_1}{\partial p_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial x_n}{\partial p_1} & \cdots & \frac{\partial x_n}{\partial p_n} \end{bmatrix},$$

and  $\lambda_y = \partial \lambda / \partial y$ .

Substitution of (10) into (9) leads to

$$\begin{bmatrix} \mathbf{U} & \mathbf{p} \\ \mathbf{p}' & 0 \end{bmatrix} \begin{bmatrix} \mathbf{x}_y & \mathbf{X}_p \\ -\lambda_y & -\lambda'_p \end{bmatrix} = \begin{bmatrix} \mathbf{0} & \lambda \mathbf{I} \\ 1 & -\mathbf{x}' \end{bmatrix}. \quad (11)$$

This equation is known as ‘Barten’s fundamental matrix equation’ — see Barten (1964). The solution to equation (11) can be written in the form

$$\begin{aligned} \begin{bmatrix} \mathbf{x}_y & \mathbf{X}_p \\ -\lambda_y & -\lambda'_p \end{bmatrix} &= \begin{bmatrix} \mathbf{U} & \mathbf{p} \\ \mathbf{p}' & 0 \end{bmatrix}^{-1} \begin{bmatrix} \mathbf{0} & \lambda \mathbf{I} \\ 1 & -\mathbf{x}' \end{bmatrix} \\ &= \frac{1}{\mathbf{p}' \mathbf{U}^{-1} \mathbf{p}} \begin{bmatrix} (\mathbf{p}' \mathbf{U}^{-1} \mathbf{p}) \mathbf{U}^{-1} - \mathbf{U}^{-1} \mathbf{p} (\mathbf{U}^{-1} \mathbf{p})' & \mathbf{U}^{-1} \mathbf{p} \\ (\mathbf{U}^{-1} \mathbf{p})' & -1 \end{bmatrix} \begin{bmatrix} \mathbf{0} & \lambda \mathbf{I} \\ 1 & -\mathbf{x}' \end{bmatrix}, \end{aligned}$$

which implies [see Barten (1964), Philips (1974), or Selvanathan and Selvanathan (2005) for more details]

$$\mathbf{x}_y = \lambda_y \mathbf{U}^{-1} \mathbf{p}; \quad (12)$$

$$\mathbf{X}_p = \lambda \mathbf{U}^{-1} - (\lambda/\lambda_y) \mathbf{x}_y \mathbf{x}'_y - \mathbf{x}_y \mathbf{x}', \quad (13)$$

where  $\mathbf{U}^{-1}$  is the inverse of the Hessian matrix of the utility function and is symmetric negative definite.

Equations (12) and (13) give the income and price derivatives of the demand functions. Equation (13) is known as the ‘Slutsky equation.’ It shows that the total effect of a change in  $p_j$  on  $x_i$  is made up of two terms — the ‘income effect’ of the price change,  $-\mathbf{x}_y \mathbf{x}'$ , and the ‘total substitution effect,’  $\lambda \mathbf{U}^{-1} - (\lambda/\lambda_y) \mathbf{x}_y \mathbf{x}'_y$ , which gives the response of  $x_i$  to a change in  $p_j$  with real income and all the other prices held constant. The total substitution effect consists of the ‘specific substitution effect,’  $\lambda \mathbf{U}^{-1}$ , and the ‘general substitution effect,’  $-(\lambda/\lambda_y) \mathbf{x}_y \mathbf{x}'_y$ , in the terminology of Houthakker (1960).

The Slutsky equation (13) can be written as

$$\mathbf{X}_p = \mathbf{K} - \mathbf{x}_y \mathbf{x}', \quad (14)$$

where  $\mathbf{K} = \lambda \mathbf{U}^{-1} - (\lambda/\lambda_y) \mathbf{x}_y \mathbf{x}'_y$  is the ‘substitution matrix’ (also known as the ‘Slutsky matrix’) of income-compensated (equivalently, utility-held-constant) price changes and  $\mathbf{x}_y \mathbf{x}'$  is the ‘matrix of income effects.’ Writing equation (14) in scalar form we get

$$\frac{\partial x_i}{\partial p_j} = k_{ij} - \frac{\partial x_i}{\partial y} x_j, \quad i, j = 1, \dots, n,$$

where  $\partial x_i / \partial p_j$  is the total effect of a price change on demand,  $k_{ij}$  (i.e., the  $i, j$  element of  $\mathbf{K}$ ) is the substitution effect of a compensated price change on demand, and  $-(\partial x_i / \partial y) x_j$  is the income effect, resulting from a change in price (not in income).

Substitution of (13) into (10) yields

$$\begin{bmatrix} d\mathbf{x} \\ -d\lambda \end{bmatrix} = \begin{bmatrix} \mathbf{x}_y & \mathbf{K} - \mathbf{x}_y \mathbf{x}' \\ -\lambda_y & -\lambda'_p \end{bmatrix} \begin{bmatrix} dy \\ d\mathbf{p} \end{bmatrix}$$

which implies, after solving for  $d\mathbf{x}$ ,

$$d\mathbf{x} = \mathbf{x}_y dy + (\mathbf{K} - \mathbf{x}_y \mathbf{x}') d\mathbf{p}. \quad (15)$$

### 3 The Differential Approach to Demand Analysis

The differential approach to demand analysis was introduced by Theil (1965) and Barten (1966) and explored by Theil (1967, 1975, 1976, 1980). To briefly review this modeling approach, we write equation (15) in scalar form as

$$dx_i = \frac{\partial x_i}{\partial y} dy + \sum_{j=1}^n \frac{\partial x_i}{\partial p_j} dp_j, \quad i = 1, \dots, n. \quad (16)$$

Multiplying both sides by  $p_i/y$  and using the identity  $dz = z d \log z$ , equation (16) can be written in logarithmic differentials as

$$w_i d \log x_i = \theta_i d \log y + \sum_{j=1}^n \frac{p_i p_j}{y} \frac{\partial x_i}{\partial p_j} d \log p_j, \quad i = 1, \dots, n, \quad (17)$$

where  $w_i = p_i x_i / y$  is the budget share of the  $i$ th use of income and  $\theta_i = w_i \eta_{iy}$  is the marginal budget share of the  $i$ th use of money income ( $p_i \partial x_i / \partial y$ ). The budget shares are always positive and sum to unity,  $\sum_{i=1}^n w_i = 1$ . The marginal budget shares are not always positive (for example,  $\theta_i < 0$  if good  $i$  is an inferior good) but like the budget shares sum to unity,  $\sum_{i=1}^n \theta_i = 1$ .

Writing equation (13) in scalar form as (for  $i, j = 1, \dots, n$ )

$$\partial x_i / \partial p_j = \lambda u_{ij} - \frac{\lambda}{\lambda_y} \frac{\partial x_i}{\partial y} \frac{\partial x_j}{\partial y} - \frac{\partial x_i}{\partial y} x_j,$$

where  $u_{ij}$  is the  $(i, j)$ <sup>th</sup> element of  $\mathbf{U}^{-1}$ , substituting in (17) to eliminate  $\partial x_i / \partial p_j$ , and rearranging yields (for  $i = 1, \dots, n$ )

$$w_i d \log x_i = \theta_i \left( d \log y - \sum_{j=1}^n w_j d \log p_j \right) + \sum_{j=1}^n \left( \frac{\lambda p_i p_j u_{ij}}{y} - \frac{\lambda / y}{\partial \lambda / \partial y} \theta_i \theta_j \right) d \log p_j. \quad (18)$$

In equation (18),  $\sum_{j=1}^n w_j d \log p_j$  is the budget share weighted average of the  $n$  logged price changes and defines the Divisia (1925) price index, that is

$$d \log P = \sum_{j=1}^n w_j d \log p_j. \quad (19)$$

Moreover, the first term in parenthesis on the right of (18), which can now be written as  $(d \log y - d \log P)$ , gives the Divisia quantity (volume) index. To see this, take the differential of the budget constraint (4) to obtain

$$\sum_{j=1}^n p_j dx_j + \sum_{j=1}^n x_j dp_j = dy.$$

Dividing both sides of the above by  $y$  and writing it in logarithmic differential (using the identity,  $dz/z = d \log z$ ) yields

$$\sum_{j=1}^n w_j d \log x_j + \sum_{j=1}^n w_j d \log p_j = d \log y, \quad (20)$$

where the first term on the left of equation (20) is the Divisia quantity index, denoted here as  $d \log Q$ . That is,

$$d \log Q = \sum_{j=1}^n w_j d \log x_j. \quad (21)$$

Hence, equation (20) decomposes the change in income into a volume and price index. Moreover, since  $d \log Q = d \log y - d \log P$ , the Divisia price index,  $d \log P$ , transforms the change in money income into the change in real income.

To further simplify equation (18), we set  $\lambda p_i p_j u_{ij} / y = v_{ij}$  and  $(\lambda / y) / \partial \lambda / \partial y = \phi$  and write it as

$$w_i d \log x_i = \theta_i d \log Q + \sum_{j=1}^n v_{ij} d \log p_j - \phi \theta_i \sum_{j=1}^n \theta_j d \log p_j, \quad i = 1, \dots, n. \quad (22)$$

For later use, we can also define the  $n \times n$  matrix

$$[v_{ij}] = \frac{\lambda}{y} \mathbf{P}' \mathbf{U}^{-1} \mathbf{P},$$

where  $\mathbf{P}$  is an  $n \times n$  symmetric positive definite matrix with diagonal elements  $p_1, \dots, p_n$  and off-diagonal elements of zero. Hence,  $[v_{ij}]$  is a symmetric negative definite  $n \times n$  matrix. Also, writing equation (12) in scalar form as

$$\frac{\partial x_i}{\partial y} = \frac{\partial \lambda}{\partial y} \sum_{j=1}^n p_j u_{ij},$$

multiplying both sides of the above by  $p_i$ , and rearranging, yields

$$\sum_{j=1}^n v_{ij} = \phi \theta_i, \quad i = 1, \dots, n. \quad (23)$$

### 3.1 A Differential Demand System in Relative Prices

In equation (22),  $\sum_{j=1}^n \theta_j d \log p_j$  is the Frisch (1932) price index, denoted here as  $d \log P^f$ . That is,

$$d \log P^f = \sum_{j=1}^n \theta_j d \log p_j. \quad (24)$$

As can be seen, the Frisch price index (24) uses marginal shares as weights instead of budget shares used by the Divisia price index (19). Using (23) and (24), equation (22) can be written as

$$w_i d \log x_i = \theta_i d \log Q + \sum_{j=1}^n v_{ij} (d \log p_j - d \log P^f), \quad i = 1, \dots, n. \quad (25)$$

Equation (25) is a differential demand system in relative prices. In particular, the Frisch price index,  $d \log P^f$ , transforms absolute prices into relative prices, by deflating each price change in the second term on the right of equation (25); we refer to  $(d \log p_j - d \log P^f) = d \log (p_j/P^f)$  as the Frisch-deflated price of good  $j$ .

In equation (25),  $\theta_i$  gives the effect of real income,  $d \log Q (= d \log y - d \log P)$ , on the demand for good  $i$ . In fact, since the Divisia price index,  $d \log P$ , is a budget share weighted price index,  $\theta_i$  in equation (25) measures the income effect of the  $n$  price changes on the demand for good  $i$ . Also,  $v_{ij}$  is the coefficient of the  $j$ th relative price,  $d \log (p_j/P^f)$ .

### 3.2 A Differential Demand System in Absolute Prices

To express the demand system in terms of absolute prices, we express the substitution terms in equation (25),  $\sum_{j=1}^n v_{ij} (d \log p_j - d \log P^f)$ , in absolute (or undeflated) prices as follows<sup>1</sup>

$$\sum_{j=1}^n v_{ij} (d \log p_j - d \log P^f) = \sum_{j=1}^n \pi_{ij} d \log p_j,$$

where  $\pi_{ij} = v_{ij} - \phi \theta_i \theta_j$ . Then equation (25) can be written as

$$w_i d \log x_i = \theta_i d \log Q + \sum_{j=1}^n \pi_{ij} d \log p_j, \quad i = 1, \dots, n. \quad (26)$$

In equation (26),  $\pi_{ij} (= v_{ij} - \phi \theta_i \theta_j)$  is the Slutsky (1915) coefficient; it gives the total substitution effect on the demand for good  $i$  of a change in the price of good  $j$ .

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<sup>1</sup>In doing so, we use the definition of the Frisch price index,  $d \log P^f = \sum_{j=1}^n \theta_j d \log p_j$ , to write the substitution terms in equation (25) as

$$\begin{aligned} \sum_{j=1}^n v_{ij} (d \log p_j - d \log P^f) &= \sum_{j=1}^n v_{ij} d \log p_j - \sum_{j=1}^n v_{ij} d \log P^f = \sum_{j=1}^n v_{ij} d \log p_j - \sum_{j=1}^n v_{ij} \sum_{j=1}^n \theta_j d \log p_j \\ &= \sum_{j=1}^n \left( v_{ij} - \sum_{j=1}^n v_{ij} \theta_j \right) d \log p_j = \sum_{j=1}^n (v_{ij} - \phi \theta_i \theta_j) d \log p_j = \sum_{j=1}^n \pi_{ij} d \log p_j, \end{aligned}$$

where  $\pi_{ij} = v_{ij} - \phi \theta_i \theta_j$ .



The income elasticities,  $\eta_{iy}$ , and the compensated price elasticities of good  $i$  with respect to price  $j$ ,  $\eta_{ij}^*$ , can be easily calculated as follows

$$\eta_{iy} = \frac{d \log x_i}{d \log Q} = \frac{\theta_i}{w_i}, \quad i = 1, \dots, n; \quad (27)$$

$$\eta_{ij}^* = \frac{d \log x_i}{d \log p_j} = \frac{\pi_{ij}}{w_i}, \quad i, j = 1, \dots, n. \quad (28)$$

## 4 The Rotterdam Parameterization

Demand systems (25) and (26) have been formulated in infinitesimal changes. Economic data, however, are available in finite time intervals such as, for example, monthly, quarterly, or yearly. By converting the infinitesimal changes in (25) and (26) to finite-change form, and assuming that the parameters are constant over the period of observation, we get the Rotterdam model, due to Theil (1965) and Barten (1966). It is to be noted that the parameterization (the assumption regarding the constancy of the parameters) is an assumption as important as the choice of a model. For example, the parameterization that  $\theta_i$  is constant implies linear Engel curves, which defines a particular model.

### 4.1 The Relative Price Version of the Rotterdam Model

When formulated in terms of finite changes, equation (25) is written as

$$w_{it}^* D x_{it} = \theta_i D Q_t + \sum_{j=1}^n v_{ij} \left( D p_{jt} - D P_t^f \right), \quad i = 1, \dots, n, \quad (29)$$

where the subscript  $t$  indexes time,  $D$  is the log-change operator,  $D z_t = \Delta (\log z_t) = \log z_t - \log z_{t-1} = \log (z_t / z_{t-1})$ , and  $w_{it}^*$  is the  $i$ th good's (arithmetic) average value share over two successive time periods,  $t-1$  and  $t$ , that is,

$$w_{it}^* = \frac{1}{2} (w_{it} + w_{i,t-1}).$$

In equation (29),  $D Q_t$  is a finite-change version of the Divisia quantity index, known as the Törnqvist-Theil Divisia quantity index, defined as

$$D Q_t = \sum_{j=1}^n w_{jt}^* D x_{jt}, \quad (30)$$

and  $DP_t^f$  is a finite-change version of the Frisch price index, defined as

$$DP_t^f = \sum_{j=1}^n \theta_j Dp_{jt}. \quad (31)$$

For later use notice that writing (20) in finite-change form yields

$$\sum_{j=1}^n w_{jt}^* Dx_{jt} + \sum_{j=1}^n w_{jt}^* Dp_{jt} = Dy_t, \quad (32)$$

where the first term on the left defines the Törnqvist-Theil Divisia quantity index (30) and the second term defines the Törnqvist-Theil Divisia price index,

$$DP_t = \sum_{j=1}^n w_{jt}^* Dp_{jt}. \quad (33)$$

Hence, equation (32), like equation 20, decomposes the change in income into a volume and price index.

Under the assumption that the coefficients  $\theta_i$  and  $v_{ij}$  are constant, equation (29) is the relative price version of the Rotterdam model — see Theil (1975, 1976). It uses real income and price variables, since in equation (29), the income variable is deflated by the Divisia price index, defined in (33), and the price variables are deflated by the Frisch price index, defined in (31).

As noted earlier, the matrix  $[v_{ij}]$  is a symmetric and negative definite  $n \times n$  matrix, and restrictions (23) hold, implying that  $\phi$  is also constant. However, equation (29) is not identified, unless the  $v_{ij}$ s are restricted, as noted by Theil (1971, pp. 579-80). The reason is the ordinality of utility under perfect certainty. Hence there exists an infinite number of utility functions, all monotonic transformations of each other, which are in the same equivalence class producing the same preference preorderings. A normalization is necessary to select one from the infinite number of cardinal utility functions in the equivalence class. One possible identifying restriction is preference independence. In that case, the consumer's utility function (3) is additive in the  $n$  goods, as follows

$$u(\mathbf{x}) = \sum_{i=1}^n u_i(x_i), \quad (34)$$

implying that the marginal utility of good  $i$  is independent of the consumption of good  $j$ ,  $j \neq i$ . Under preference independence, the Hessian matrix  $\mathbf{U}$  is an  $n \times n$  diagonal matrix, as  $u_{ij} = 0$  for  $i \neq j$ . This also implies that  $v_{ij} = 0$  for  $i \neq j$  and equation (23) reduces to  $v_{ii} = \phi\theta_i$ , so that the demand system (29) takes the form

$$w_{it}^* Dx_{it} = \theta_i DQ_t + \phi\theta_i \left( Dp_{jt} - DP_t^f \right), \quad i = 1, \dots, n. \quad (35)$$

That is, under preference independence, only the own Frisch-deflated price appears in each demand equation, ruling out the possibility of either a specific substitute or a specific complement; according to Houthakker (1960), goods  $i$  and  $j$  are specific substitutes if  $v_{ij} > 0$  and they are specific complements if  $v_{ij} < 0$ . Moreover, under preference independence, for the  $[v_{ij}]$  matrix to be a negative definite  $n \times n$  diagonal matrix with elements  $\phi\theta_1, \dots, \phi\theta_n$ , each marginal share,  $\theta_i$  must be positive, thereby ruling out inferior goods.

As can be seen, preference independence identifies the relative price version of the Rotterdam model and significantly reduces the number of parameters to be estimated. For example, the number of parameters in demand systems (25) and (26) is in the order of  $n^2$ , where  $n$  is the number of goods, whereas in the demand system (35) it is in the order of  $2n$ . It is, however, an extremely restrictive assumption and might be a reasonable maintained hypothesis only if the commodities are broad commodity groups, such as, for example, ‘food,’ ‘clothing,’ ‘recreation,’ and so on.

A weaker version of preference independence is block independence (also known as block additivity). Under block independence, the additive specification (34) is applied to groups of goods and the utility function is written as

$$u(\mathbf{x}) = \sum_{r=1}^R u_r(\mathbf{x}^r),$$

where  $R < n$  is the number of groups and  $n$  the total number of goods. Under block independent preferences, the demand equations for an aggregate group of goods (called group or composite demand equations) can be derived as well as the demand equations for goods within a group (called conditional demand equations). See Theil (1975, 1976) or Selvanathan and Selvanathan (2005) for more details.

The discussion above follows imposition of an identifying restriction that cardinalizes the utility function. It is very important, following the use of the relative price version of the Rotterdam model, to reach only those conclusions that are invariant to monotonic transformations of the utility function and are thereby ordinal. For example, the concepts of specific complements and specific substitutes are cardinal, since they are conditional upon the cardinalizing normalization and are not invariant to monotonic transformations of the utility function. During the estimation procedure, the concepts can be used. But there cannot be a conclusion of specific complements or specific substitutes at the completion of the analysis. Similarly the concepts of block independence and block additivity are cardinal. The ordinal version that can be a valid conclusion is called blockwise strong separability, which is defined by the class of all utility functions that are monotonic transformations of a block additive or block preference independent cardinal utility function.

## 4.2 The Absolute Price Version of the Rotterdam Model

Writing equation (26) in terms of finite changes yields

$$w_{it}^* Dx_{it} = \theta_i DQ_t + \sum_{j=1}^n \pi_{ij} Dp_{jt}, \quad i = 1, \dots, n, \quad (36)$$

where (as before)  $\pi_{ij} = v_{ij} - \phi\theta_i\theta_j$  is the Slutsky (1915) coefficient and  $DQ_t$  is defined as in (30). When the coefficients  $\theta_i$  and  $\pi_{ij}$  are treated as constants, (36) is known as the absolute price version of the Rotterdam model.

There are two sets of restrictions on the parameters of (36). The first set of ‘weak’ restrictions on consumer demand follows from the budget constraint (adding-up) and the homogeneity of the demand equations:

- Adding-up requires

$$\sum_{i=1}^n \theta_i = 1 \quad \text{and} \quad \sum_{j=1}^n \pi_{ji} = 0, \quad \text{for all } i = 1, \dots, n. \quad (37)$$

- Demand homogeneity follows from  $\sum_{j=1}^n v_{ij} = \phi\theta_i$  and  $\pi_{ij} = v_{ij} - \phi\theta_i\theta_j$ , and requires

$$\sum_{j=1}^n \pi_{ij} = 0, \quad \text{for all } i = 1, \dots, n. \quad (38)$$

Under the standard assumptions of economic theory, if the household solves problem (3)-(4), then the  $\theta_i$  and  $\pi_{ij}$  coefficients in (36) must also satisfy the second set of ‘strong’ restrictions:

- Slutsky symmetry requires

$$\pi_{ij} = \pi_{ji}, \quad i, j = 1, \dots, n. \quad (39)$$

- Concavity requires that the Slutsky matrix,  $[\pi_{ij}]$ , is negative semi-definite  $n \times n$  matrix with rank  $n - 1$ .

It is to be noted, however, that the above restrictions are not independent. Typically, adding-up, homogeneity, and symmetry are imposed in estimation, and the negative semi-definiteness of the  $[\pi_{ij}]$  matrix is empirically confirmed — see, for example, Fayyad (1986). The income elasticities,  $\eta_{iy}$ , and the compensated price elasticities of good  $i$  with respect to price  $j$ ,  $\eta_{ij}^*$ , are calculated using equations (27) and (28). In this case, however, since the

parameters are assumed to be constant under the Rotterdam parameterization, the average budget shares over the sample period are used.

The Rotterdam model in absolute prices, equation (36), is linear in the parameters, unlike the Rotterdam model in relative prices, equation (29), which is nonlinear in the parameters. This makes estimation of (36) and hypotheses testing straightforward. However, as the number of goods,  $n$ , increases, the number of the  $\pi_{ij}$  parameters in (36) increases rapidly. In such cases, the relative price version of the Rotterdam model, equation (29) with suitable restrictions on the  $v_{ij}$  parameters, might be more appealing. No cardinalizing normalization of parameters is needed with the absolute price version, since all parameters of that version of the Rotterdam model are invariant to monotonic transformations of the utility function. Hence all of the model's inferences are ordinal, unlike the relative price version, with which it is important to use only the model's noncardinal conclusions.

## 5 Estimation

The relative and absolute price versions of the Rotterdam model can be estimated in a number of ways. In what follows we discuss a procedure for estimating the absolute price version of the Rotterdam model, keeping in mind that the relative price version of the model can be estimated in a similar manner. For more details regarding different estimation procedures, see the recent survey article by Barnett and Serletis (2008).

In order to estimate the absolute price version of the Rotterdam model, equation (36), a stochastic version is specified as follows

$$w_{it}^* Dx_{it} = \theta_i DQ_t + \sum_{j=1}^n \pi_{ij} Dp_{jt} + \epsilon_{it}, \quad i = 1, \dots, n, \quad t = 1, \dots, T, \quad (40)$$

where  $T$  is the number of observations. The disturbance term,  $\epsilon_{it}$ , is assumed to capture the random effects of all variables other than those of  $DQ_t$  and  $Dp_{jt}$ ,  $j = 1, \dots, n$ .

Summing up both sides of (40) over  $i = 1, \dots, n$  we get

$$\sum_{i=1}^n w_{it}^* Dx_{it} = \sum_{i=1}^n \theta_i DQ_t + \sum_{i=1}^n \sum_{j=1}^n \pi_{ij} Dp_{jt} + \sum_{i=1}^n \epsilon_{it}.$$

Since  $DQ_t = \sum_{i=1}^n w_{it}^* Dx_{it}$ , the adding up restrictions (37) imply that

$$\sum_{i=1}^n \epsilon_{it} = 0, \quad t = 1, \dots, T,$$

meaning that the disturbances are not linearly independent and that the error covariance matrix is singular. This suggests that one of the equations can be deleted. Barten (1969)

has shown that any equation can be deleted; the parameter estimates of the deleted equation can be recovered from the restrictions imposed. If we delete the last equation from (40), we can then write it as

$$w_{it}^* Dx_{it} = \theta_i DQ_t + \sum_{j=1}^n \pi_{ij} Dp_{jt} + \epsilon_{it}, \quad i = 1, \dots, n-1, \quad t = 1, \dots, T. \quad (41)$$

It is usually assumed that  $\boldsymbol{\epsilon} = (\epsilon_1, \dots, \epsilon_{n-1})' \sim N(\mathbf{0}, \boldsymbol{\Omega} \otimes \mathbf{I}_T)$  where  $\mathbf{0}$  is the null vector,  $\otimes$  is the Kronecker product,  $\boldsymbol{\Omega}$  is the  $(n-1) \times (n-1)$  symmetric positive definite error variance-covariance matrix, and  $\mathbf{I}_T$  is a  $T \times T$  identity matrix. This assumption permits correlation among the disturbances at time  $t$  but rules out the possibility of autocorrelated disturbances.

For notational convenience, equation (41) is written as

$$\mathbf{s}_t = \mathbf{g}(\mathbf{v}_t, \boldsymbol{\vartheta}) + \boldsymbol{\epsilon}_t, \quad (42)$$

where  $\mathbf{s}_t = (w_{1t}^* Dx_{1t}, \dots, w_{n-1,t}^* Dx_{n-1,t})'$  is the vector of the left-hand-side variables of (41),  $\mathbf{v}_t = (DQ_t, Dp_{1t}, \dots, Dp_{nt})'$  is the vector of the right-hand-side variables of (41),  $\mathbf{g}(\mathbf{v}, \boldsymbol{\vartheta}) = (\mathbf{g}_1(\mathbf{v}, \boldsymbol{\vartheta}), \dots, \mathbf{g}_{n-1}(\mathbf{v}, \boldsymbol{\vartheta}))'$ ,  $\boldsymbol{\vartheta}$  is the vector of parameters,  $\theta_i$  and  $\pi_{ij}$ , to be estimated, and  $\mathbf{g}_i(\mathbf{v}, \boldsymbol{\vartheta})$  is given by the right-hand side of the  $i$ th equation in (41).

Given the observed data on  $\mathbf{s}$  and  $\mathbf{v}$ , the log-likelihood function on  $\boldsymbol{\vartheta}$  and  $\boldsymbol{\Omega}$  is given by

$$\begin{aligned} \log L(\boldsymbol{\vartheta}, \boldsymbol{\Omega} | \mathbf{s}, \mathbf{v}) &= -\frac{(n-1)T}{2} \log(2\pi) |\boldsymbol{\Omega}| \\ &\quad - \frac{1}{2} \sum_{t=1}^T [(\mathbf{s}_t - \mathbf{g}(\mathbf{v}_t, \boldsymbol{\vartheta}))' \boldsymbol{\Omega}^{-1} (\mathbf{s}_t - \mathbf{g}(\mathbf{v}_t, \boldsymbol{\vartheta}))]. \end{aligned}$$

This function is maximized with respect to the elements of the parameter vector,  $\boldsymbol{\vartheta}$ , and the elements of the variance-covariance matrix,  $\boldsymbol{\Omega}$ .

## 5.1 An Example

As an example, let us consider the case of four goods,  $n = 4$ . In equation (41), let  $s_{it} = w_{it}^* Dx_{it}$ ,  $x = DQ_t$ , and  $v_{jt} = Dp_{jt}$ . Equation (41) can then be written as (ignoring time subscripts)

$$\begin{aligned} s_1 &= \theta_1 x + \pi_{11} v_1 + \pi_{12} v_2 + \pi_{13} v_3 + \pi_{14} v_4 + \epsilon_1; \\ s_2 &= \theta_2 x + \pi_{21} v_1 + \pi_{22} v_2 + \pi_{23} v_3 + \pi_{24} v_4 + \epsilon_2; \\ s_3 &= \theta_3 x + \pi_{31} v_1 + \pi_{32} v_2 + \pi_{33} v_3 + \pi_{34} v_4 + \epsilon_3; \\ s_4 &= \theta_4 x + \pi_{41} v_1 + \pi_{42} v_2 + \pi_{43} v_3 + \pi_{44} v_4 + \epsilon_4. \end{aligned}$$

This system has 20 parameters. In view of the fact that the disturbances are not linearly independent and that one of the equations can be deleted, delete the 4th equation, to get

$$\begin{aligned} s_1 &= \theta_1 x + \pi_{11} v_1 + \pi_{12} v_2 + \pi_{13} v_3 + \pi_{14} v_4 + \epsilon_1; \\ s_2 &= \theta_2 x + \pi_{21} v_1 + \pi_{22} v_2 + \pi_{23} v_3 + \pi_{24} v_4 + \epsilon_2; \\ s_3 &= \theta_3 x + \pi_{31} v_1 + \pi_{32} v_2 + \pi_{33} v_3 + \pi_{34} v_4 + \epsilon_3. \end{aligned} \tag{43}$$

The homogeneity property (38) implies the following restrictions

$$\begin{aligned} \pi_{11} + \pi_{12} + \pi_{13} + \pi_{14} &= 0; \\ \pi_{21} + \pi_{22} + \pi_{23} + \pi_{24} &= 0; \\ \pi_{31} + \pi_{32} + \pi_{33} + \pi_{34} &= 0. \end{aligned} \tag{44}$$

Moreover, symmetry (39) implies

$$\pi_{12} = \pi_{21}; \quad \pi_{13} = \pi_{31}; \quad \pi_{23} = \pi_{32}. \tag{45}$$

Combining the homogeneity and symmetry restrictions, (44) and (45), yields  $\pi_{i4} = -\sum_{j=1}^3 \pi_{ij}$  ( $i = 1, 2, 3$ ), or written out in full,

$$\begin{aligned} \pi_{14} &= -(\pi_{11} + \pi_{12} + \pi_{13}); \\ \pi_{24} &= -(\pi_{12} + \pi_{22} + \pi_{23}); \\ \pi_{34} &= -(\pi_{13} + \pi_{23} + \pi_{33}). \end{aligned}$$

Hence, the demand system (43) can now be written as

$$\begin{aligned} s_1 &= \theta_1 x + \pi_{11} v_1 + \pi_{12} v_2 + \pi_{13} v_3 - (\pi_{11} + \pi_{12} + \pi_{13}) v_4 + \epsilon_1; \\ s_2 &= \theta_2 x + \pi_{12} v_1 + \pi_{22} v_2 + \pi_{23} v_3 - (\pi_{12} + \pi_{22} + \pi_{23}) v_4 + \epsilon_2; \\ s_3 &= \theta_3 x + \pi_{13} v_1 + \pi_{23} v_2 + \pi_{33} v_3 - (\pi_{13} + \pi_{23} + \pi_{33}) v_4 + \epsilon_3, \end{aligned}$$

which has 9 free parameters (that is, parameters estimated directly),  $\theta_1, \theta_2, \theta_3, \pi_{11}, \pi_{12}, \pi_{13}, \pi_{22}, \pi_{23}$ , and  $\pi_{33}$ .

As can be seen, by deleting the 4th equation,  $\theta_4$  and  $\pi_{4i}$  ( $i = 1, \dots, 4$ ) are no longer parameters of the estimated system. Hence, none of the constraints  $\sum_{i=1}^4 \theta_i = 1$ ,  $\sum_{j=1}^4 \pi_{4j} = 0$ , and  $\pi_{i4} = \pi_{4i}$  are imposed for estimation purposes. These constraints are used to recover the parameters of the deleted equation,  $\theta_4, \pi_{41}, \pi_{42}, \pi_{43}$ , and  $\pi_{44}$ , as follows

$$\begin{aligned} \theta_4 &= 1 - \theta_1 - \theta_2 - \theta_3; \\ \pi_{41} (= \pi_{14}) &= -\pi_{11} - \pi_{12} - \pi_{13}; \\ \pi_{42} (= \pi_{24}) &= -\pi_{12} - \pi_{22} - \pi_{23}; \\ \pi_{43} (= \pi_{34}) &= -\pi_{13} - \pi_{23} - \pi_{33}; \\ \pi_{44} &= \pi_{11} + 2\pi_{12} + 2\pi_{13} + \pi_{22} + 2\pi_{23} + \pi_{33}. \end{aligned}$$

## 6 Regularity

### 6.1 Theoretical Regularity

As already noted, adding-up, linear homogeneity, and symmetry are imposed in estimation, and the negative semidefiniteness of the  $[\pi_{ij}]$  matrix is left unimposed, but is empirically confirmed. For example, with four goods ( $n = 4$ ), negative semidefiniteness of the  $[\pi_{ij}]$  matrix requires that:

- all four  $\pi_{ii}$  are negative at each observation
- each of the six possible  $2 \times 2$  matrices

$$\begin{bmatrix} \pi_{ii} & \pi_{ij} \\ \pi_{ij} & \pi_{jj} \end{bmatrix}$$

for  $i, j = 1, 2, 3, 4$  but  $i \neq j$ , has a positive determinant at every observation

- each of the four possible  $3 \times 3$  matrices

$$\begin{bmatrix} \pi_{ii} & \pi_{ij} & \pi_{ik} \\ \pi_{ij} & \pi_{jj} & \pi_{jk} \\ \pi_{ik} & \pi_{jk} & \pi_{kk} \end{bmatrix}$$

for  $i, j, k = 1, 2, 3, 4$  but  $i \neq j$ ,  $i \neq k$ ,  $j \neq k$ , has a negative determinant at every observation, and

- the  $4 \times 4$  matrix consisting of all the  $\pi_{ij}$ ,  $i, j = 1, 2, 3, 4$ ,

$$\begin{bmatrix} \pi_{11} & \pi_{12} & \pi_{13} & \pi_{14} \\ \pi_{12} & \pi_{22} & \pi_{23} & \pi_{24} \\ \pi_{13} & \pi_{23} & \pi_{33} & \pi_{34} \\ \pi_{14} & \pi_{24} & \pi_{34} & \pi_{44} \end{bmatrix}$$

has a determinant whose value is zero (or near zero).

If theoretical regularity (that is, negative semidefiniteness of the  $[\pi_{ij}]$  matrix) is not attained by luck, the model should be estimated by imposing regularity, as suggested by Barnett (2002) and Barnett and Pasupathy (2003), thereby treating the curvature property as maintained hypothesis. This can be accomplished using methods discussed in Barnett and Serletis (2008); see also Barnett and Seck (2008) for a comparison of the Rotterdam model with the Almost Ideal Demand System (AIDS).

It is to be noted that the first tests of the Rotterdam model by Barten (1967, 1969) and Byron (1970) seemed to suggest rejection of the theoretical restrictions. Deaton (1972),



however, showed that these rejections were due to the inappropriate use of asymptotic test criteria and after appropriate finite sample correction the conflict between theory and empirical evidence was removed, except for the homogeneity restriction.

Finally, we should note that although the Rotterdam model avoids the necessity of using a particular functional form for the utility function, the specified demand equations may imply the adoption of particular restrictions on preferences typical for a certain class of utility functions. For example, it has been argued by Phlips (1974), based on earlier research by McFadden (1964), that the Rotterdam model is globally exactly consistent with utility maximization only if the utility function is linear logarithmic. As with the translog, the Rotterdam model is globally exact only in the Cobb Douglas special case, but both are local approximations of the same order to any demand system. Moreover Barnett (1979a, 1981) has shown that the Rotterdam model has a uniquely rigorous connection with demand after aggregation over consumers, based upon taking probability limits of Slutsky equations as the number of consumers increases. No other model has been shown to have such an attractive connection with theory after aggregation over consumers under weak assumptions.

## 6.2 Econometric Regularity

In most industrialized economies time series of prices and income are nonstationary and as recently argued by Lewbel and Ng (2005), the vast majority of the existing utility based empirical demand system studies, with either household- or aggregate-level data, has failed to cope with the issue of nonstationary variables, mainly because standard methods for dealing with nonstationarity in linear models cannot be used with nonstationary data and nonlinear estimation in large demand systems. For these reasons, the problem of nonstationarity has either been ignored (treating the data as if they were stationary) or dealt with using cointegration methods that apply to linear models, as in Ogaki (1992) and Attfield (1997). See Barnett and Serletis (2008) for more details regarding this issue.

The Rotterdam model, however, is not subject to the substantive criticisms relating to nonstationary variables, because it uses logarithmic first differences of the variables, which are typically stationary. In this regard, the Rotterdam model compares favorably against the currently popular parametric demand systems based on locally flexible functional forms such as the generalized Leontief [see Diewert (1974)], the translogs [see Christensen *et al.* (1975)], the almost ideal demand system [see Deaton and Muellbauer (1980)], the minflex Laurent [see Barnett (1983)], the quadratic AIDS [see Banks *et al.* (1997)], and the normalized quadratic [see Diewert and Wales (1988)]. It also compares favorably with the two semi-nonparametric flexible functional forms: the Fourier, introduced by Gallant (1981), and the Asymptotically Ideal Model (AIM), introduced by Barnett and Jonas (1983). In addition, systematic tests of the properties of the error structure of the Rotterdam model have consistently reflected more favorably on the maintained hypotheses about the model's error structure than about any other consumer demand model's error structure. See, for example, Barnett (1979b,

appendix).

## 7 Conclusion

The Rotterdam model was the turning point in empirical demand analysis, offering many features not available in modeling efforts that had been used up to that time, such as the double-log demand system and Working's (1943) model, both briefly discussed in the introduction. In particular, the Rotterdam model is entirely based on consumer demand theory, has the ability to model the whole substitution matrix, has parameters that can easily be related to underlying theoretical restrictions, is linear in parameters and therefore easy to econometrically estimate, and is econometrically regular. However, after the publication of Diewert's (1971) important paper, most of the demand modeling literature has taken the approach of specifying the aggregator function with the utility function of the representative consumer, despite the fact that theorists have shown that the representative consumer does not exist under reasonable assumptions.

## References

- [1] Attfield, C., 1997, Estimating a cointegrated demand system. *European Economic Review* 41, 61-73.
- [2] Banks, J., R. Blundell, and A. Lewbel, 1997, Quadratic Engel curves and consumer demand. *Review of Economics and Statistics* 79, 527-539.
- [3] Barnett, W.A., 1979a, Theoretical foundations for the Rotterdam model. *Review of Economic Studies* 46, 109-130. Reprinted in Barnett, W.A. and J. Binner, (Eds.), *Functional structure and approximation in econometrics*, Elsevier, Amsterdam, 2004.
- [4] Barnett, W.A., 1979b, The Joint Allocation of Leisure and Goods Expenditure. *Econometrica* 45, 1117-1136. Reprinted in Barnett, W.A. and J. Binner, (Eds.), *Functional structure and approximation in econometrics*, Elsevier, Amsterdam, 2004.
- [5] Barnett, W.A., 1981, *Consumer theory and labor supply*. North Holland, Amsterdam.
- [6] Barnett, W.A., 1983, New indices of money supply and the flexible Laurent demand system. *Journal of Business and Economic Statistics* 1, 7-23. Reprinted in Barnett, W.A. and J. Binner, (Eds.), *Functional structure and approximation in econometrics*, Elsevier, Amsterdam, 2004.
- [7] Barnett, W.A., 2002, Tastes and technology: Curvature is not sufficient for regularity. *Journal of Econometrics* 108, 199-202. Reprinted in Barnett, W.A. and J. Binner, (Eds.), *Functional structure and approximation in econometrics*, Elsevier, Amsterdam, 2004.
- [8] Barnett, W.A. and A. Jonas, 1983, The Müntz-Szatz demand system: An application of a globally well behaved series expansion. *Economics Letters* 11, 337-342. Reprinted in Barnett, W.A. and J. Binner, (Eds.), *Functional structure and approximation in econometrics*, Elsevier, Amsterdam, 2004.
- [9] Barnett, W.A. and M. Pasupathy, 2003, Regularity of the generalized quadratic production model: A counterexample. *Econometric Reviews* 22, 135-154.
- [10] Barnett, W.A. and O. Seck, 2008, Rotterdam model versus Almost Ideal Demand System: Will the best specification please stand up? *Journal of Applied Econometrics* 23, 699-728.
- [11] Barnett, W.A. and A. Serletis, 2008, Consumer preferences and demand systems. *Journal of Econometrics* 147, 210-224.

- [12] Barten, A.P., 1964, Consumer demand functions under conditions of almost additive preferences. *Econometrica* 32, 1-38.
- [13] Barten, A.P., 1966, Theorie en empirie van een volledig stelsel van vraagvergelijkingen (Theory and empirics of a complete system of demand equations). Doctoral dissertation (Netherlands School of Economics, Rotterdam).
- [14] Barten, A.P., 1967, Evidence on the Slutsky conditions for demand equations. *Review of Economics and Statistics* 49, 77-84.
- [15] Barten, A.P., 1969, Maximum likelihood estimation of a complete system of demand equations. *European Economic Review* 1, 7-73.
- [16] Byron, R.P., 1970, A simple method for estimating demand systems under separable utility assumptions. *Review of Economic Studies* 37, 261-274.
- [17] Christensen, L.R., D.W. Jorgenson, and L.J. Lau, 1975, Transcendental logarithmic utility functions. *American Economic Review* 65, 367-383.
- [18] Deaton, A.S., 1972, The estimation and testing of systems of demand equations. *European Economic Review* 3, 390-411.
- [19] Deaton, A.S. and J.N. Muellbauer, 1980, An almost ideal demand system. *American Economic Review* 70, 312-326.
- [20] Diewert, W.E., 1971, An application of the Shephard duality theorem: a generalized Leontief production function. *Journal of Political Economy* 79, 481-507.
- [21] Diewert, W.E., 1974, Applications of duality theory, in: M. Intriligator and D. Kendrick, (Eds.), *Frontiers in quantitative economics*, Vol. 2. North-Holland, Amsterdam.
- [22] Diewert, W.E. and T.J. Wales, 1988, Normalized quadratic systems of consumer demand functions. *Journal of Business and Economic Statistics* 6, 303-312.
- [23] Divisia, F., 1925, L' indice monetaire et la theorie de la monnaie. *Revue d' Economie Politique* 39, 980-1008.
- [24] Engle, R.F. and C.W.J. Granger, 1987, Cointegration and error correction: representation, estimation and testing. *Econometrica* 55, 251-276.
- [25] Fayyad, S.K., 1986, A microeconomic system-wide approach to the estimation of the demand for money. *Federal Reserve Bank of St. Louis Review* August/September, 22-33.
- [26] Frisch, R., 1932, *New methods of measuring marginal utility*. J.C.B. Mohr, Tiibingen.

- [27] Gallant, A.R., 1981, On the bias of flexible functional forms and an essentially unbiased form: The Fourier functional form. *Journal of Econometrics* 15, 211-245.
- [28] Houthakker, H.S., 1960, Additive preferences. *Econometrica* 28, 244-257.
- [29] Lewbel, A. and S. Ng, 2005, Demand systems with nonstationary prices. *Review of Economics and Statistics* 87, 479-494.
- [30] McFadden, D.L., 1964, Existence conditions for Theil-type preferences. University of California, Berkeley, mimeograph.
- [31] Ogaki, M., 1992, Engel's law and cointegration. *Journal of Political Economy* 100, 1027-1046.
- [32] Philips, L., 1974, *Applied consumption analysis*. North Holland, Amsterdam.
- [33] Selvanathan, S. and E.A. Selvanathan, 2005, *The Demand for alcohol, tobacco and marijuana: International evidence*. Ashgate, England.
- [34] Slutsky, E., 1915, Sulla teoria del bilancio del consumator. *Giornale degli Economisti* 51, 1-26; Translation, On the theory of the budget of the consumer, in: G.J. Stigler and K.E. Boulding (Eds.) *Readings in price theory*. Richard D. Irwin, Chicago (1952).
- [35] Theil, H., 1965, The information approach to demand analysis. *Econometrica* 33, 67-87.
- [36] Theil, H., 1967, *Economics and Information theory*. North Holland, Amsterdam.
- [37] Theil, H., 1971, *Principles of econometrics*. John Wiley & Sons Inc., New York.
- [38] Theil, H., 1975, *Theory and measurement of consumer demand*, Vol. 1. North Holland, Amsterdam.
- [39] Theil, H., 1976, *Theory and measurement of consumer demand*, Vol. 2. North Holland, Amsterdam.
- [40] Theil, H., 1980, *The system-wide approach to microeconomics*. The University of Chicago Press, Chicago.
- [41] Working, H., 1943, Statistical laws of family expenditure. *Journal of the American Statistical Association* 38, 43-56.