DIVISIA SECOND MOMENTS: An Application of Stochastic Index Number Theory[1]

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I. Introduction:

Diewert (1976) defined the class of superlative index numbers. In particular, he showed that the Törnqvist-Theil discrete time approximation to the continuous time Divisia index could provide a second order approximation to any true economic aggregate. Barnett (1978,1980) provided the appropriate formula for the real user cost of monetary assets and applied this theory to aggregation over monetary assets. There is now a large body of empirical research (e.g. Barnett, Offenbacher and Spindt 1984; Chrystal and MacDonald 1994) which suggests that Divisia indices outperform simple sum monetary aggregates.

Exact monetary aggregation requires two restrictions on the utility functions of economic agents. Aggregation over goods requires weak separability of monetary assets in the utility function, and aggregation over consumers requires Gorman's well-known conditions for the existence of a representative agent. If these conditions are satisfied, then economic agents treat economic aggregates as if they were elementary goods. Consequently, the dispersion of the component growth rates of the Divisia index contains information only relevant to allocations within the aggregate. Equivalently, if the aggregation conditions hold, only the growth rates of the Divisia index are relevant for macroeconomics.

Although these conditions are typically maintained hypotheses, few economists regard them as exactly correct. Failure of one or more of the

[1] The authors wish to thank Richard Anderson of the Federal Reserve Bank of St. Louis for kindly providing us with the component quantity and user cost data which we used to compute the Divisia first and second moments.

necessary conditions for exact economic aggregation can create dependency upon dispersion of component growth rates of monetary assets or upon higher order moments. Barnett and Serletis (1990) have argued that this provides a test for the existence of aggregation error: the dispersion-dependency diagnostic test (DDT). Aggregation error introduces an additive remainder term into economic models which is dependent on measures of component dispersion. If dispersion measures are introduced into an economic model and are significant, this is an indication of aggregation error.

Theil (1967) provides direct measures of the dispersion of component growth rates based on stochastic aggregation theory. In stochastic aggregation, the Divisia index can be viewed as a share weighted mean of component growth rates. Thus the relevant dispersion measures are higher-order share weighted (Divisia)moments. Barnett and Serletis (1990) applied the DDT by testing for the significance of the Divisia quantity variance in several economic models. In addition Barnett, Offencacher, and Spindt (1984) provide an initial empirical discussion of the other Divisia second moments[2]. We provide an investigation of the relationship between macroeconomic variables and each of the Divisia first and second moments, based on Granger causality.

II. The Divisia Index

The Tornqvist-Theil discrete time approximation to the continuous time Divisia index is given by the following formula:

$$Q_t = Q_{t-1} \prod_i \left(\frac{m_{it}}{m_{i,t-1}}\right)^{\frac{1}{2}(s_{it} + s_{i,t-1})} \qquad (1)$$

where $s_{it} = \frac{\pi_{it} m_{it}}{M_t}$ and $M_t = \sum_{j=1}^{n} \pi_{jt} m_{jt}$. Observe that s_{it} is the expenditure share on monetary asset m_{it}, where π_{it} is the user cost of asset i at time t. The Divisia index (in discrete time) has a convenient representation as a logarithmic change:

$$\log Q_t - \log Q_{t-1} = \sum_{i=1}^{n} \bar{s}_{it} (\log m_{i,t} - \log m_{i,t-1}) \qquad (2)$$

[2] It should be noted that although significance of Divisia second moments provides a test for the existence of aggregation error, the failure of Divisia second moments to be significant does not prove that aggregation error does not exist. Aggregation error can create dependency on Divisia moments of order greater than two.

where $\bar{s}_{it} = \frac{1}{2}\left(s_{it} + s_{i,t-1}\right)$ is the average expenditure share of asset i in period t[3]. We define the log change operator, D, by $Dz_t = \log z_t - \log z_{t-1}$. We can conveniently write the log change of the Divisia index as:

$$DQ_t = \sum_{i=1}^{n} \bar{s}_{it} Dm_{it} \ . \qquad (2')$$

Aggregation across goods, as noted above, is equivalent to assuming that the bundle of goods over which aggregation takes place is weakly separable in the utility function. Strictly speaking this hypothesis should be tested. However following the general practice we will maintain these assumptions at the M1 level of aggregation. Thus the empirical results will only be for the bundle of assets contained in M1.

In order to make (2') operational Barnett (1978,1980) derived the current period user cost of monetary asset i:

$$\pi_{it} = \frac{R_t - r_{it}}{1 + R_t}, \quad i = 1, \ldots, n \qquad (3)$$

where r_{it} is the current period holding yield on monetary asset i, and R_t is the benchmark rate of return, defined as the maximum expected holding period yield available at time t. The data used in this study is the same as that used in Thornton and Yue (1992) and the benchmark rate is defined as the maximum own rate in the collection of assets in L and the bond rate.

III. Stochastic Index Number Theory

Theil (1967) provided a stochastic interpretation of the Divisia index. The average shares are positive for all assets and sum to one by definition and thus can be interpreted as probabilities. We can treat the quantity growth rates as random drawings from a population with probabilities given by the average shares. The log change of the Divisia index (2') is the expected value of a random drawing of a quantity growth rate. The Divisia index is thus a mean or a first moment of the distribution of quantity growth rates.

Similiarly we can define the Divisia user cost index:

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[3] For the remainder of this paper the Törnqvist-Theil discrete time approximation to the continuous time Divisia index will be referred to as the Divisia index.

$$D\Pi_t = \sum_{i=1}^{n} \bar{s}_{it} D\pi_{it} \tag{4}$$

In stochastic index number theory the Divisia user cost index is the expected value of random drawings of user cost growth rates over the probabilities given by the average shares.

The Divisia share index is defined as:

$$DS_t = \sum_{i=1}^{n} \bar{s}_{it} Ds_{it} \ . \tag{5}$$

The Divisia share index is the expected value of random drawings of share growth rates over the probabilities given by the average shares.

Thus stochastic index number theory gives the convenient interpretation of the three Divisia indexes as means or first moments of probability distributions. This probabilistic interpretation can be extended to define variances in the obvious manner.

The Divisia quantity variance is:

$$K_t = \sum_{i=1}^{n} \bar{s}_{it} (Dm_{it} - DQ_t)^2 \ . \tag{6}$$

The Divisia user cost variance is:

$$J_t = \sum_{i=1}^{n} \bar{s}_{it} (D\pi_{it} - D\Pi_t)^2 \ . \tag{7}$$

The Divisia price-quantity covariance is:

$$\Gamma_t = \sum_{i=1}^{n} \bar{s}_{it} (Dm_{it} - DQ_t)(D\pi_{it} - D\Pi_t) \, . \tag{8}$$

The Divisia share variance is:

$$\psi_t = \sum_{i=1}^{n} \bar{s}_{it} (Ds_{it} - DS_t)^2 \ . \tag{9}$$

Theil (1967) shows that the share variance is related to the other second moments by the following relation:

$$\psi_t = K_t + J_t + 2\Gamma_t \, . \tag{10}$$

It can easily be seen that the Divisia quantity variance will be zero if all component quantity growth rates are equal. Similiarly, the Divisia user cost variance will be zero if all user costs grow at the same rate. As Figures 1a, 1b and 2 show, these two special cases are not satisfied.

Figure 1a
Divisia Second Moments
Quantity Variance

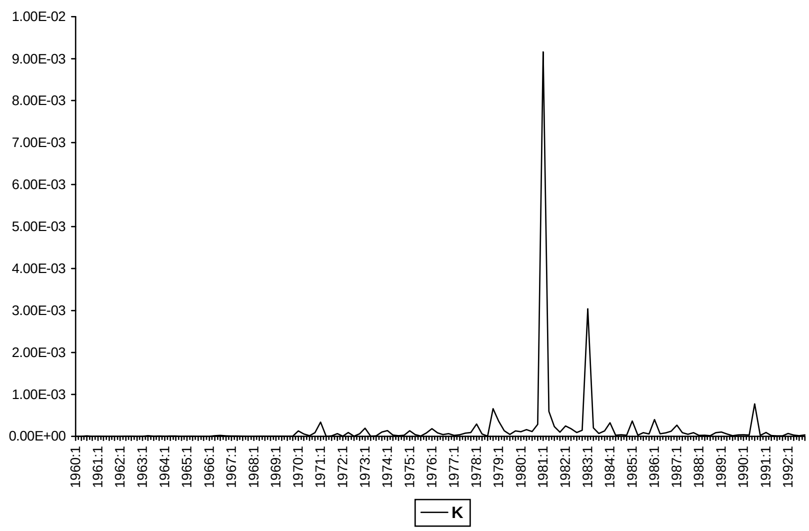


Figure 1b
Divisia Second Moments
Quantity Variance

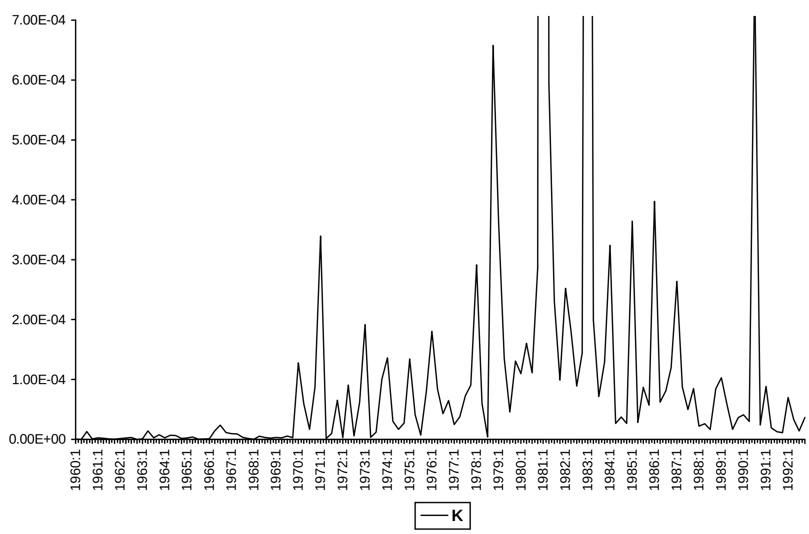
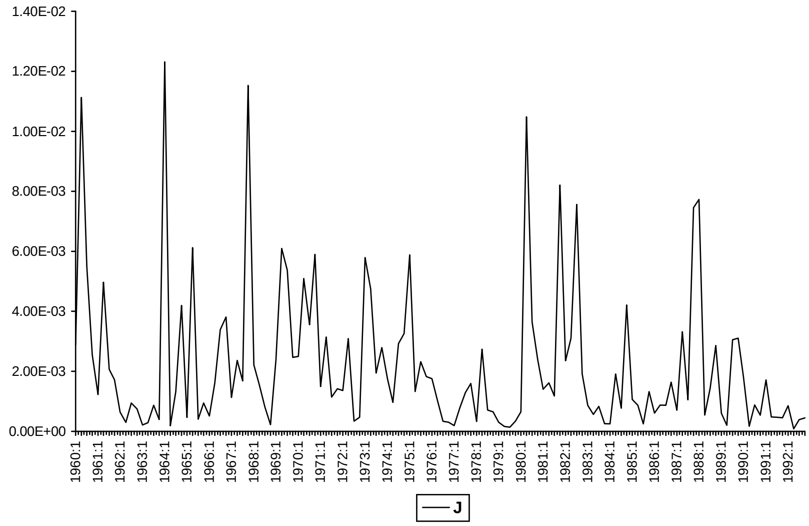


Figure 2
Divisia Second Moments
Price Variance



In addition if the average shares are time-invariant then both the Divisia share mean and variance will be zero[4]. As Figures 3a and 3b demonstrate the Divisia share variance is not trivial.

[4]Theil (1967) has demonstrated that the Divisia quantity and user cost index satisfies the following identity: $DQ_t + D\Pi_t = DM_t + DS_t$. Thus in the special case in which average shares are time invariant, $DS_t = 0$, the Divisia price and quantity indexes will be self dual (satisfy factor reversal).

Figure 3a
Divisia Second Moments
Share Variance

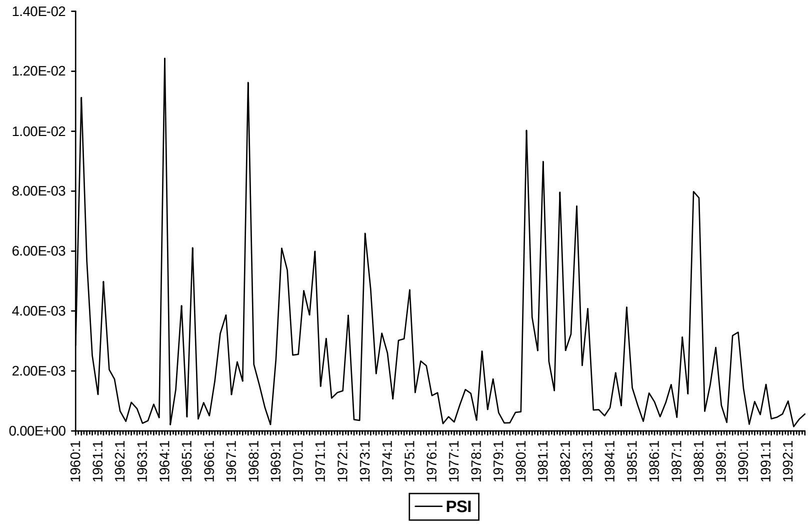
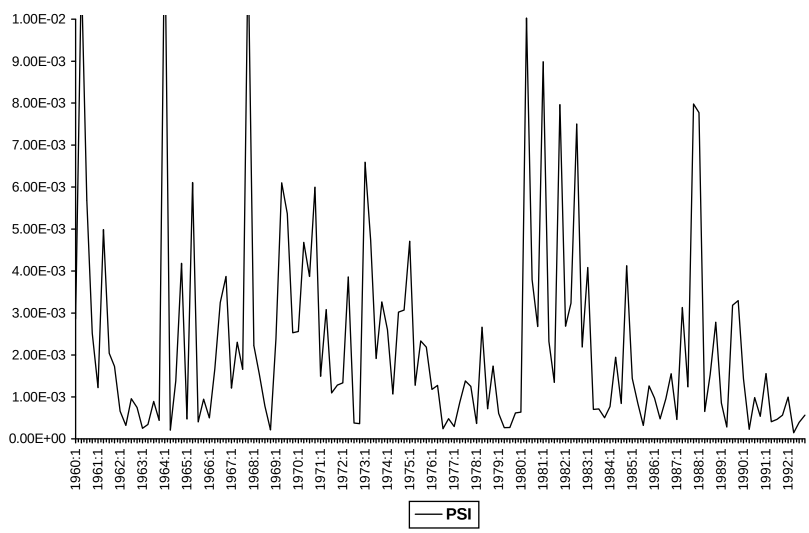


Figure 3b
Divisia Second Moments
Share Variance



If the above special cases are not satisfied then the Divisia second moments may contain information not captured by the first moments, and thus would be relevant to macroeconomic analysis. Barnett and Serlitis (1990) explore this possibility for the Divisia quantity variance, we extend the investigation to all four second moments.

IV. Empirical Analysis:

In this section we implement tests for Granger causality between Divisia first and second moments and four macroeconomic variables: nominal GNP, the unemployment rate, the federal funds rate, and the producer price index. Following Barnett and Serletis (1990) we implement the following trend stationary specification:

$$z_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i z_{t-i} + \sum_{j=1}^{r} \beta_j y_{t-j} + \sum_{k=1}^{r} \gamma_k x_{t-k} + \delta t + \varepsilon_t \qquad (11)$$

where z_t is the macroeconomic variable, the z_{t-i} terms are the lags of macroeconomic variable, the y_{t-j} terms are lags of a Divisia first moment, and the x_{t-k} terms are the lags of a Divisia second moment. Granger methods will be used to test for causality using all three first moments and all four second moments. The error structure is assumed to be white noise. The data is quarterly over the period 1960:1 - 1992:4.

The test procedure is identical for each of the four sections. We will test the hypothesis that Divisia first and second moments Granger cause the relevant variable, z_t and the hypothesis that z_t Granger causes the four Divisia second moments[5]. The results are contained in tables 1-4 of the appendix. Seven test statistics are produced. The statistic ξ_1 tests the null hypothesis that the Divisia first moment does not Granger cause z_t, when the coefficients of the Divisia second moment are maintained to be zero. The statistic ξ_2 tests the null that the Divisia first moment does not Granger cause z_t, when the coefficisents of the Divisia second moment are not maintained to be zero. The statistic ξ_3 tests the null that the Divisia first and second moment jointly do not Granger cause z_t. The statistic ξ_4 tests the null that the Divisia second moment does not Granger cause z_t, when the coefficients of the Divisia first moment are maintained to be zero. The statistic ξ_5 tests the hypothesis that the Divisia second moment does not Granger cause z_t, when the coefficients on the Divisia first moment are not maintained to be zero. The statistic ξ_6 tests the hypothesis that z_t does not

Granger cause the Divisia second moment, when the coefficients on the Divisia first moments are maintained to be zero. The statistic ξ_7 tests the hypothesis that z_t does not Granger cause the Divisia second moment, when the coefficients on the the Divisia first moment are not maintained to be zero. The test procedure is to estimate (11) under the appropriate restrictions by OLS, and then conduct an F test for joint significance. These statistics are presented for all three first moments, although ξ_4 and ξ_6 are redundant and so are presented only once in each table. In addition, ξ_1 is independent of the variance, but is reported in the first column for convenience. All tests are for Granger causality with lag length equal to four.[6]

Nominal GNP

The Granger causality tests provide no evidence that either the user cost variance or the share variance Granger cause nominal GNP. However there is evidence that causality exists between nominal GNP and the quantity variance in both directions. In addition, the tests provide some evidence that nominal GNP Granger causes the user cost variance and the share variance. Therefore it appears that at the M1 level little information is gained in explaining nominal GNP beyond that contained in the quantity variance. See Figure 4.

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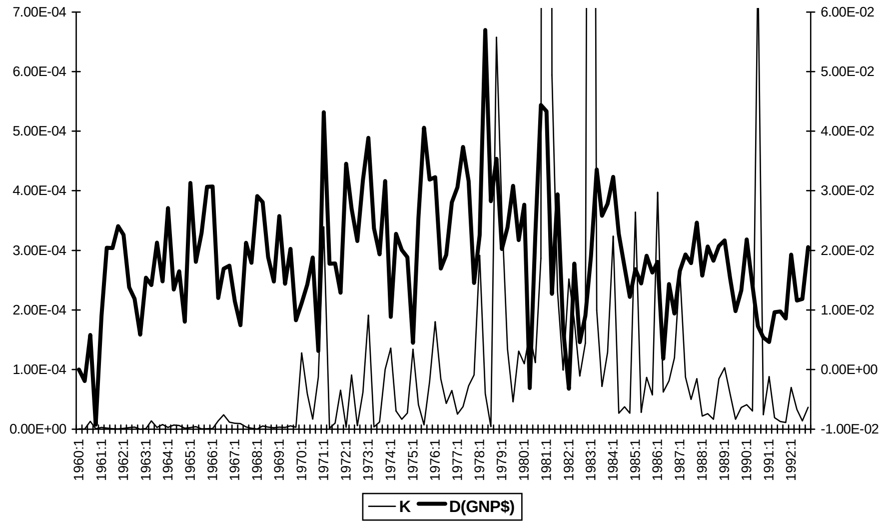
[5] In the test for reverse Granger causality the appropriate model would be

$x_t = \alpha_0 + \sum_{i=1}^{r} \alpha_i z_{t-i} + \sum_{j=1}^{r} \beta_j y_{t-j} + \sum_{k=1}^{r} \gamma_k x_{t-k} + \delta t + \varepsilon_t$ with all variables defined as above.

[6] The authors are aware of the sensitivity of Granger methods to the lag length chosen.

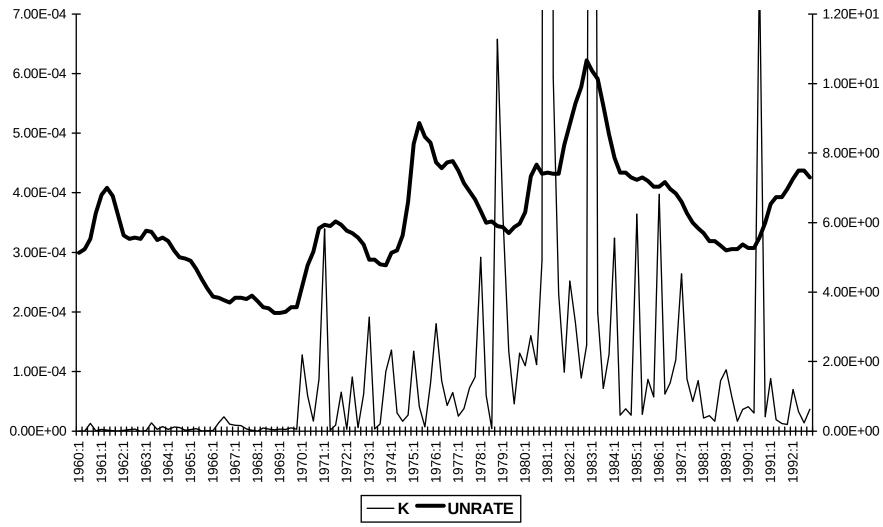
Figure 4
Divisia Second Moments
Quantity Variance vs. Percent Change in Nominal GNP



The Unemployment Rate

The Granger causality tests provide evidence that the Divisia quantity variance Granger cause the unemployment rate. In addition there is evidence that the unemployment rate Granger causes the quantity variance. There appears to be little evidence of interaction in either direction between unemployment and any of the other second moments. See Figure 5.

Figure 5
Divisia Second Moments
Quantity Variance vs. Unemployment Rate



The Federal Funds Rate

Barnett, Offenbacher, and Spindt (1984) have argued that rising interest rates should induce increases in the user cost variance and the share variance, because relative user costs between rate regulated and rate unregulated assets should move away from 1.0. This suggests that the federal funds rate should Granger cause the Divisia user cost variance and the Divisia share variance. Although this hypothesis should undoubtedly be tested at higher levels of aggregation than M1, the results of the Granger causality tests at the M1 level tend to confirm the hypothesis. In addition, there is extremely strong evidence for causality in both directions between the federal funds rate and the quantity variance. See Figures 6 and 7.

Figure 6
Divisia Second Moments
Price Variance vs. Federal Funds Rate

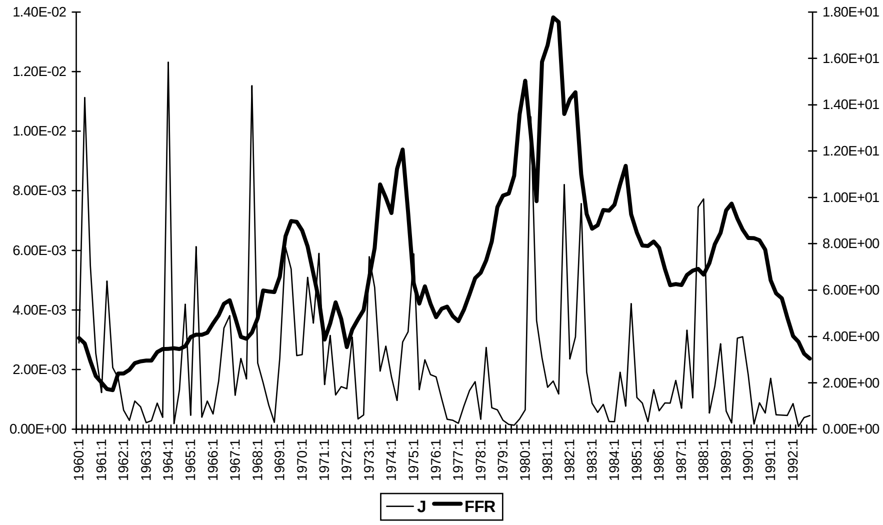
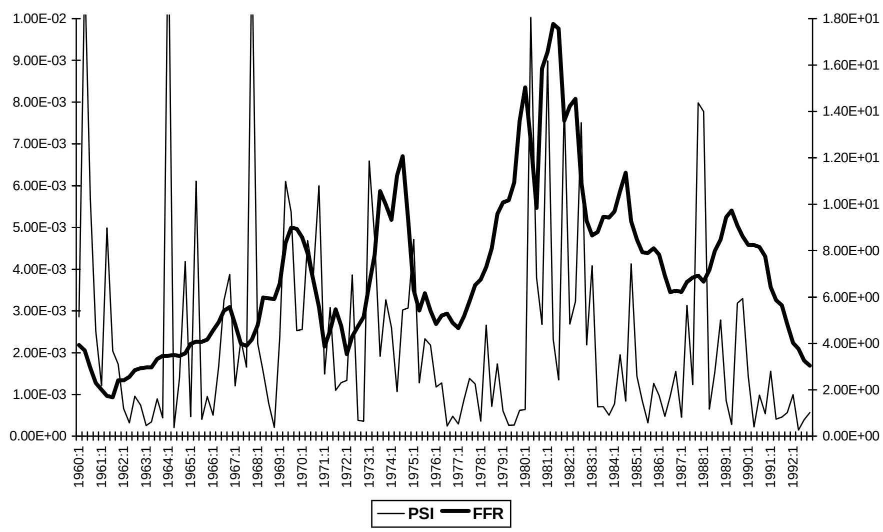


Figure 7
Divisia Second Moments
Share Variance vs. Federal Funds Rate



The Producer Price Index

Theil (1967) has argued that the Divisia quantity-user cost covariance should be negative reflecting substitution effects. The Granger causality tests provide strong evidence of Granger causality between the Producer Price Index and the Divisia covariance in both directions. Although more research should be devoted to this topic, this result suggests that the price level creates aggregation error which can influence the substitutability of monetary assets even at low levels of aggregation. See Figure 8.

V. Conclusions:

In the presence of aggregation error, Divisia second moments can contain relevant information not captured by the Divisia first moments. We have demonstrated that even at the M1 level of aggregation all of the second moments contain information relevant to important macroeconomic variables. This is an important finding, because it demonstrates that aggregation error is present in macroeconomic data, and that Divisia second moments may be useful in correcting for it. These conclusions should be further tested at higher levels of aggregation. In addition, they should be tested using more sophisticated econometric techniques.

As has been noted above, there is abundant evidence that the Divisia monetary aggregates (or any Diewert superlative index) should be used by central banks instead of simple sum monetary aggregates. This paper provides evidence that Divisia second moments should also be used by monetary policy makers, because they contain information relevant to other macroeconomic variables. Although it is well known that differences between Diewert superlative index numbers are typically smaller than the roundoff error of monetary data (see Barnett 1980), only the Divisia index has the generalization to higher moments, which is explained in this paper. We therefore advocate use of the Divisia index and its generalization to higher moments for the design and study of monetary policy.

Figure 8
Divisia Second Moments
Price-Quantity Covariance vs. Percent Change in the Producer Price Index

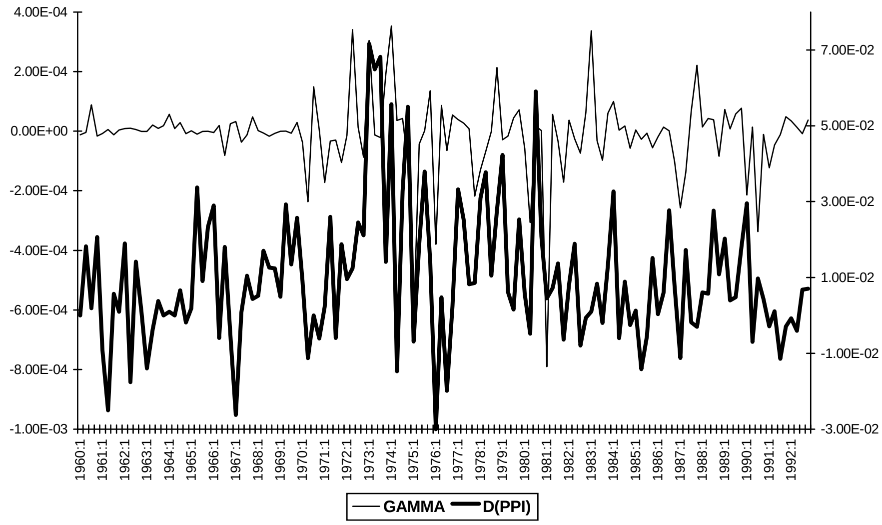


Table 1
Test of Granger causality from Divisia first and second moments to nominal GNP

First Moment	Test Statistic	K	J	ψ	Γ
DQ	ξ_1	1.114751[a] [.352911]	-----------	-----------	----------
	ξ_2	.434728 [.783287]	1.065636 [.376894]	.925219 [.451948]	.533731 [.711197]
	ξ_3	1.473928 [.174371]	.927735 [.496539]	1.098831 [.369289]	1.267652 [.267332]
	ξ_4	2.562226 [.041965]	.788082 [.535142]	1.275676 [.283475]	2.033717 [.094074]
	ξ_5	1.802770 [.133116]	.750160 [.559896]	1.079892 [.369858]	1.405240 [.236713]
	ξ_6	3.921766 [.005020]	2.115490 [.083142]	1.738917 [.145986]	.518233 [.722472]
	ξ_7	3.738424 [.006764]	2.428048 [.051813]	2.142917 [.080003]	.375534 [.825695]
DP	ξ_1	4.222170[a] [.003134]	-----------	-----------	----------
	ξ_2	3.431598 [.010925]	3.634600 [.007956]	3.216860 [.015273]	3.517692 [.009550]
	ξ_3	3.102510 [.003322]	2.246532 [.028888]	2.294201 [.025688]	2.862489 [.006147]
	ξ_5	1.859793 [.122332]	.362182 [.835081]	.445582 [.775415]	1.439854 [.225392]
	ξ_7	5.237780 [.000656]	2.231608 [.069933]	1.276863 [.283236]	.705436 [.589818]
DS	ξ_1	2.153656[a] [.078469]	-----------	-----------	----------
	ξ_2	3.722529 [.006934]	2.000812 [.099107]	2.046554 [.092524]	1.666689 [.162572]
	ξ_3	3.260610 [.002211]	1.407815 [.200641]	1.683744 [.109757]	1.873184 [.070920]
	ξ_5	4.138441 [.003621]	.684972 [.603787]	1.199282 [.315048]	1.552384 [.191884]
	ξ_7	2.218299 [.071362]	1.914627 [.112745]	1.583051 [.183576]	.209159 [.93288]

[a] This statistic is a test of the null that the Divisia first moment Granger causes the macroeconomic variable when the coefficients on the second moment are maintained to be zero. Thus this test does not depend on the variance. The statistic is reported in the column for K merely for convenience

Table 2

Test of causality from Divisia first and second moments to Unemployment

First Moment	Test Statistic	K	J	ψ	Γ
DQ	ξ_1	.574865 [a] [.681401]	-----------	-----------	----------
	ξ_2	.567954 [.686402]	.806196 [.523684]	.741124 [.565871]	.711450 [.585745]
	ξ_3	1.526416 [.155673]	.815301 [.590565]	1.087862 [.376752]	.867223 [.546378]
	ξ_4	2.521811 [.044665]	.829858 [.508748]	1.447300 [.222769]	1.033100 [.393187]
	ξ_5	2.449716 [.050120]	1.054673 [.382377]	1.589374 [.181904]	1.156530 [.333840]
	ξ_6	3.181436 [.016018]	1.795800 [.134227]	1.368359 [.249090]	1.584658 [.182886]
	ξ_7	3.185946 [.016027]	2.071066 [.089171]	1.716951 [.151047]	1.433775 [.227344]
DP	ξ_1	1.979721 [a] [.102032]	-----------	-----------	----------
	ξ_2	2.183396 [.075244]	1.648296 [.166991]	1.547970 [.193107]	2.002766 [.098817]
	ξ_3	2.403185 [.019606]	1.248196 [.277848]	1.511077 [.160948]	2.002766 [.152623]
	ξ_5	2.711772 [.033473]	.547066 [.701515]	1.039765 [.389936]	1.085523 [.367107]
	ξ_7	4.967140 [.000997]	1.920366 [.111784]	1.668421 [.162162]	1.853583 [.123465]
DS	ξ_1	.812388 [a] [.519681]	-----------	-----------	----------
	ξ_2	.402423 [.806566]	.643503 [.632578]	.410237 [.800960]	.927419 [.450692]
	ξ_3	1.436575 [.188827]	.731667 [.663295]	.914302 [.507421]	.978989 [.456112]
	ξ_5	2.032332 [.094525]	.660300 [.620842]	1.015781 [.402343]	1.141687 [.340580]
	ξ_7	1.268741 [.286431]	1.901994 [.114888]	1.539665 [.195429]	1.802678 [.133134]

Table 3
Test of causality from Divisia first and second moments to the Federal Funds Rate

First Moment	Test Statistic	K	J	ψ	Γ
DQ	ξ_1	5.349422[a] [.000540]	-----------	-----------	----------
	ξ_2	5.172114 [.000726]	4.983302 [.000973]	5.295103 [.000600]	5.726630 [.000309]
	ξ_3	7.469806 [.000000]	3.687527 [.000733]	4.541432 [.000081]	3.467984 [.001294]
	ξ_4	8.557265 [.000004]	2.107220 [.084190]	3.306365 [.013174]	1.042330 [.388458]
	ξ_5	8.271588 [.000007]	1.868197 [.120814]	3.313856 [.013129]	1.496511 [.207910]
	ξ_6	19.85595 [.000000]	2.918821 [.024134]	3.709239 [.007006]	2.863208 [.026317]
	ξ_7	15.80154 [.000000]	2.343242 [.058993]	3.754770 [.006594]	1.804376 [.132801]
DP					
	ξ_1	1.970937[a] [.103387]	-----------	-----------	----------
	ξ_2	1.186871 [.320409]	2.006307 [.098293]	1.164973 [.330056]	1.559985 [.189793]
	ξ_3	4.899172 [.000032]	2.092704 [.042034]	2.244914 [.029003]	1.311051 [.245005]
	ξ_5	7.399825 [.000025]	2.138413 [.080550]	2.423767 [.052154]	.673011 [.612027]
	ξ_7	10.22595 [.000000]	2.535772 [.043913]	3.293378 [.013555]	2.427462 [.051859]
DS	ξ_1	4.370681[a] [.002483]	-----------	-----------	----------
	ξ_2	3.671840 [.007506]	3.946982 [.004883]	3.537064 [.009265]	3.888006 [.005354]
	ξ_3	6.502072 [.000001]	3.132354 [.003077]	3.563892 [.001010]	2.516189 [.014780]
	ξ_5	7.648439 [.000017]	1.778662 [.137940]	2.530365 [.044280]	.705353 [.589874]
	ξ_7	15.10814 [.000000]	4.999354 [.000949]	4.193877 [.003321]	1.436905 [.226337]

Table 4
Test of causality from Divisia first and second moments to the Producer Price Index

First Moment	Test Statistic	K	J	ψ	Γ
DQ	ξ_1	.724192 [a] [.576617]	-----------	-----------	----------
	ξ_2	.736421 [.568996]	.765921 [.549562]	.750087 [.559944]	.585526 [.673754]
	ξ_3	.590366 [.784117]	.798866 [.604753]	.586116 [.790248]	1.568758 [.141888]
	ξ_4	.448317 [.773438]	.838464 [.503419]	.425752 [.789793]	2.588357 [.040305]
	ξ_5	.468871 [.758470]	.875870 [.480806]	.460575 [.764513]	2.378724 [.049442]
	ξ_6	4.159786 [.003456]	.314993 [.867477]	.606868 [.658468]	2.849903 [.026867]
	ξ_7	4.160083 [.003501]	.434896 [.783166]	.523595 [.718572]	2.459847 [.049347]
DP					
	ξ_1	1.244364[a] [.295990]	-----------	-----------	----------
	ξ_2	2.598771 [.039853]	1.304844 [.272463]	1.493862 [.208699]	1.299084 [.274651]
	ξ_3	1.535692 [.152556]	1.075986 [.384945]	.963370 [.468240]	1.956841 [.058214]
	ξ_5	1.793547 [.134943]	.911347 [.459931]	.695232 [.596762]	2.601754 [.039670]
	ξ_7	4.359660 [.002564]	.291425 [.883029]	.507722 [.730137]	2.587760 [.040535]
DS	ξ_1	2.143420[a] [.079697]	-----------	-----------	----------
	ξ_2	2.581819 [.040907]	2.674124 [.035478]	2.398780 [.054189]	2.771286 [.030528]
	ξ_3	1.527087 [.155445]	1.780085 [.088068]	1.422360 [.194591]	2.757528 [.008031]
	ξ_5	.916800 [456781]	1.388521 [.242361]	.721532 [.578950]	3.210990 [.015413]
	ξ_7	3.179558 [.016187]	.254373 [.906470]	.609010 [.656962]	1.658930 [.164423]

References:

Barnett, W. A. 1978, “The User Cost of Money,” *Economics Letters* 1: 145-149.

Barnett, W. A. 1980, “Economic Monetary Aggregates: An Application of Index Number and Aggregation Theory, *Journal of Econometrics* 14: 11-48.

Barnett, W. A., E. K. Offenbacher, and P. A. Spindt, 1984. “The New Divisia Monetary Aggregates,” *Journal of Political Economy* 29 (6): 1049-1085.

Barnett, W. A., and A. Serletis. 1990. “A Dispersion-Dependency Diagnostic Test for Aggregation Error: With Applications to Monetary Economics and Income Distribution.” *Journal of Econometrics* 43: 5-34.

Chrystal, K. A., and R. MacDonald. 1984. “Empirical Evidence on the Recent Behavior and Usefulness of Simple Sum and Weighted Measures of the Money Stock.” *Federal Reserve Bank of St. Louis Review* (March/April): 73-109.

Diewert, E. W. 1976. “Exact and Superlative Index Numbers.” *Journal of Econometrics* 4: 115-145.

Gorman, W. M. 1953. “Community Preference Fields. *Econometrica* 21: 63-80.

Theil, H. 1967. *Economics and Information Theory*. Amsterdam: North Holland.

Thornton, D. L., and P. Yue. 1992. “An Extended Series of Divisia Monetary Aggregates.” Federal Reserve Bank of St. Louis Review (November/December): 35-52.