A novel result on the revenue equivalence theorem

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Abstract

This paper gives two examples to break through the revelation principle. Furthermore, the revenue equivalence theorem does not hold.

Key words: Quantum game theory; Algorithmic Bayesian mechanism; Revelation principle; Revenue equivalence theorem.

1 Introduction

Recently, several papers on quantum mechanism have generalized the traditional framework of mechanism design theory, not only in the quantum world, but also in the real world [1,2]. Moreover, Ref. [3] claims that the well-known revelation principle may not hold by using an algorithmic Bayesian mechanism.

In the rest of this paper, I will consider two formats of auctions (i.e, first-price sealed-bid procurement auction and first-price sealed-bid auction) respectively, and give two examples to break through the revelation principle. Furthermore, I will show that the revenue equivalence theorem does not hold.

2 First-price sealed-bid procurement auction

According to Example 2.30 and Example 2.37 in Ref. [4], consider a first-price sealed-bid procurement auction of a single indivisible resource with one buyer (call the agent 0) and two sellers (call them agent 1 and 2). Both sellers' privately observed valuations θ_i ($i = 1, 2$) are drawn independently from the uniform distribution on $[0, 1]$. This fact is common knowledge among the agents. Each seller submits a sealed bid, $b_i \geq 0$ ($i = 1, 2$). The sealed bids are examined and the seller

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with the lower bid is declared the winner. If there is a tie, seller 1 is declared the winner. The winning seller receives an amount equal to his bid from the buyer. The losing seller does not receive anything.

Consider the social choice function SCF-PROC3 (page 58, [4]) which is described as follows: $f(\theta) = (y_0(\theta), y_1(\theta), y_2(\theta), t_0(\theta), t_1(\theta), t_2(\theta))$,

$$
\begin{aligned}
y_1(\theta) &= 1, \quad \text{if } \theta_1 \leq \theta_2; \quad = 0 \text{ else;}\\
y_2(\theta) &= 1, \quad \text{if } \theta_1 > \theta_2; \quad = 0 \text{ else;}\\
y_0(\theta) &= 0, \quad \text{for all } \theta;\\
t_1(\theta) &= \frac{1+\theta_1}{2} y_1(\theta);\\
t_2(\theta) &= \frac{1+\theta_2}{2} y_2(\theta);\\
t_0(\theta) &= -(t_1(\theta) + t_2(\theta)).
\end{aligned}
$$

According to Example 2.37 (page 66, [4]), the strategies $b_i(\theta_i) = (1+\theta_i)/2$ (for $i = 1, 2$) constitute a Bayesian Nash equilibrium of this auction that indirectly yields the outcomes specified by $f(\theta)$. Thus, according to Proposition 9 and 10 [5], f is incentive compatible and Bayesian monotonic. Since the two sellers are symmetric, according to the definition of multi-Bayesian monotonicity [3], f is multi-Bayesian monotonic.

In order to obtain more profits, it is beneficial for two sellers to both increase their bids. A possible deception is $\alpha_i(\theta_i) = \sqrt{\theta_i}$ $(i = 1, 2)$. If both sellers choose this deception, $\alpha(\theta) \equiv (\alpha_1(\theta_1), \alpha_2(\theta_2)) = (\sqrt{\theta_1}, \sqrt{\theta_2})$. Let $h(\theta) \equiv [f \circ \alpha](\theta)$, then $h(\theta) = f(\alpha(\theta)) = (z_0(\theta), z_1(\theta), z_2(\theta), r_0(\theta), r_1(\theta), r_2(\theta))$:

$$
\begin{aligned}
z_1(\theta) &= 1, \quad \text{if } \theta_1 \leq \theta_2; \quad = 0 \text{ else;}\\
z_2(\theta) &= 1, \quad \text{if } \theta_1 > \theta_2; \quad = 0 \text{ else;}\\
z_0(\theta) &= 0, \quad \text{for all } \theta;\\
r_1(\theta) &= \frac{1+\sqrt{\theta_1}}{2} z_1(\theta);\\
r_2(\theta) &= \frac{1+\sqrt{\theta_2}}{2} z_2(\theta);\\
r_0(\theta) &= -(r_1(\theta) + r_2(\theta)).
\end{aligned}
$$

For each seller i $(i = 1, 2)$, he faces two strategies (denoted as C and D): strategy C means seller i uses the deception $\alpha_i(\theta_i) = \sqrt{\theta_i}$, and strategy D means seller i does not deceive. Consider the expected payoff of seller 2, we denote by $\$_{CC}$ the expected payoff when each agent i choose the deception α_i, and denote by $\$_{CD}$ the expected payoff when agent 1 chooses the deception α_1 and agent 2 does not deceive. $\$_{DD}$

and $\$_{DC}$ are defined similarly.

$$\$_{DD} = E[\frac{1+\theta_2}{2} - \theta_2]y_2(\theta_1, \theta_2) = (1-\theta_2)^2/2,$$
$$\$_{CC} = E[\frac{1+\sqrt{\theta_2}}{2} - \theta_2]y_2(\sqrt{\theta_1}, \sqrt{\theta_2}) = [\frac{1+\sqrt{\theta_2}}{2} - \theta_2](1-\theta_2),$$
$$\$_{DC} = E[\frac{1+\sqrt{\theta_2}}{2} - \theta_2]y_2(\theta_1, \sqrt{\theta_2}) = [\frac{1+\sqrt{\theta_2}}{2} - \theta_2](1-\sqrt{\theta_2}).$$

Since $\theta_1, \theta_2 \in [0,1]$, then $\$_{CC} > \$_{DD}$, $\$_{CC} > \$_{DC}$. Thus, condition $\lambda^{B\pi/2}$ is satisfied for seller 2. Similarly, condition $\lambda^{B\pi/2}$ is also satisfied for seller 1. By using the algorithmic Bayesian mechanism [3], h can be implemented in Bayesian Nash equilibrium. Note that h is not incentive compatible (since f is incentive compatible), so the revelation principle does not hold for this auction.

3 First-price sealed-bid auction

According to Example 23.B.5 in Ref. [6], consider an auction setting with one seller (i.e., agent 0) and two buyers (i.e., agent 1 and 2). Both buyers' privately observed valuations θ_i are drawn independently from the uniform distribution on $[0,1]$ and this fact is common knowledge among the agents. Each buyer submits a sealed bid, $b_i \geq 0$ ($i = 1,2$). The sealed bids are examined and the buyer with the higher bid is declared the winner. If there is a tie, buyer 1 is declared the winner. The winning seller pays an amount equal to his bid to the seller. The losing buyer does not pay anything.

Consider the social choice function $f'(\theta) = (y'_0(\theta), y'_1(\theta), y'_2(\theta), t'_0(\theta), t'_1(\theta), t'_2(\theta))$, in which

$$\begin{aligned}
y'_1(\theta) &= 1, \quad \text{if } \theta_1 \geq \theta_2; \quad = 0 \text{ else};\\
y'_2(\theta) &= 1, \quad \text{if } \theta_1 < \theta_2; \quad = 0 \text{ else};\\
y'_0(\theta) &= 0, \quad \text{for all } \theta;\\
t'_1(\theta) &= -\frac{1}{2}\theta_1 y'_1(\theta);\\
t'_2(\theta) &= -\frac{1}{2}\theta_2 y'_2(\theta);\\
t'_0(\theta) &= -(t'_1(\theta) + t'_2(\theta)).
\end{aligned}$$

As specified in Example 23.B.5, the strategies $b_i(\theta_i) = \theta_i/2$ (for $i = 1,2$) constitute a Bayesian Nash equilibrium of this auction that indirectly yields the outcomes specified by $f'(\theta)$. Thus, according to Proposition 9 and 10 [5], f' is incentive compatible and Bayesian monotonic. Since the two buyer are symmetric, then according to the definition of multi-Bayesian monotonicity [3], f' is multi-Bayesian monotonic.

In order to obtain more profits, it is beneficial for two buyers to both decrease their bids. A possible deception is $\alpha_i(\theta_i) = \theta_i^2$ $(i = 1, 2)$. If both buyers choose this deception, $\alpha'(\theta) \equiv (\alpha'_1(\theta_1), \alpha'_2(\theta_2)) = (\theta_1^2, \theta_2^2)$. Let $h'(\theta) \equiv [f' \circ \alpha'](\theta)$, then $h'(\theta) = f'(\alpha'(\theta)) = (z'_0(\theta), z'_1(\theta), z'_2(\theta), r'_0(\theta), r'_1(\theta), r'_2(\theta))$:

$$
\begin{aligned}
&z'_1(\theta) = 1, && \text{if } \theta_1 \geq \theta_2; && = 0 \text{ else;} \\
&z'_2(\theta) = 1, && \text{if } \theta_1 < \theta_2; && = 0 \text{ else;} \\
&z'_0(\theta) = 0, && \text{for all } \theta; \\
&r'_1(\theta) = -\frac{1}{2}\theta_1^2 z'_1(\theta); \\
&r'_2(\theta) = -\frac{1}{2}\theta_2^2 z'_2(\theta); \\
&r'_0(\theta) = -(r'_1(\theta) + r'_2(\theta)).
\end{aligned}
$$

For each buyer i $(i = 1, 2)$, he faces two strategies (denoted as C and D): strategy C means buyer i uses the deception $\alpha'_i(\theta_i) = \theta_i^2$, and strategy D means buyer i does not deceive. Consider the expected payoff of buyer 2,

$$
\begin{aligned}
\$_{DD} &= E[\theta_2 - \frac{\theta_2}{2}]y'_2(\theta_1, \theta_2) = [\theta_2 - \frac{\theta_2}{2}]\theta_2, \\
\$_{CC} &= E[\theta_2 - \frac{\theta_2^2}{2}]y'_2(\theta_1^2, \theta_2^2) = [\theta_2 - \frac{\theta_2^2}{2}]\theta_2, \\
\$_{DC} &= E[\theta_2 - \frac{\theta_2^2}{2}]y'_2(\theta_1, \theta_2^2) = [\theta_2 - \frac{\theta_2^2}{2}]\theta_2^2.
\end{aligned}
$$

Obviously, $\$_{CC} > \$_{DD}$, $\$_{CC} > \$_{DC}$. Thus, condition $\lambda^{B\pi/2}$ is satisfied for buyer 2. Similarly, condition $\lambda^{B\pi/2}$ is also satisfied for buyer 1. By using the algorithmic Bayesian mechanism [3], h' can be implemented in Bayesian Nash equilibrium. Note that h' is not incentive compatible (since f' is incentive compatible), so the revelation principle does not hold for this auction.

4 Breaking through the revenue equivalence theorem

Here I cite Proposition 23.D.3 from Ref. [6].

Proposition 23.D.3: (The Revenue Equivalence Theorem) Consider an auction setting with I risk-neutral buyers, in which buyer i's valuation is drawn from an interval $[\underline{\theta}_i, \overline{\theta}_i]$ with $\underline{\theta}_i \neq \overline{\theta}_i$ and a strictly positive density $\phi_i(\cdot) > 0$, and in which buyers' types are statistically independent. Suppose that a given pair of Bayesian Nash equilibria of two different auction procedures are such that for every buyer i: (i) For each possible realization of $(\theta_1, \cdots, \theta_I)$, buyer i has an identical probability of getting the good in the two auctions; and (ii) Buyer i has the same expected utility level in the two auctions when his valuation for the object is at its lowest possible level. Then these equilibria of the two auctions generate the same expected revenue for the seller.

Obviously, for $i = 0, 1, 2$, $z'_i(\theta) = y'_i(\theta)$ for all $\theta \in [0, 1] \times [0, 1]$, and $r'_1(0, 0) = t'_1(0, 0) = 0$, $r'_2(0, 0) = t'_2(0, 0) = 0$. Since $r'_i(\theta) > t'_i(\theta)$ for any $\theta \in (0, 1) \times (0, 1)$ $(i = 1, 2)$, the deception α' does decrease the expected revenue for the seller. So the revenue equivalence theorem does not hold.

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