Convexity of Bertrand oligopoly TU-games with differentiated products

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Abstract

In this article we consider Bertrand oligopoly TU-games with differentiated products. We assume that the demand system is Shubik's (1980) and that firms operate at a constant and identical marginal and average cost. First, we show that the α and β-characteristic functions (Aumann 1959) lead to the same class of Bertrand oligopoly TU-games and we prove that the convexity property holds for this class of games. Then, following Chander and Tulkens (1997) we consider the γ-characteristic function where firms react to a deviating coalition by choosing individual best reply strategies. For this class of games, we show that the Equal Division Solution belongs to the core and we provide a sufficient condition under which such games are convex.

Keywords: Bertrand oligopoly TU-games; Core; Convexity; Equal Division Solution
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1 Introduction

Usually, oligopoly situations are modeled by means of non-cooperative games. Every profit-maximizing firm pursues Nash strategies and the resulting outcome is not Pareto optimal. Yet, it is known that firms are better off by forming cartels and that Pareto efficiency is achieved when all the firms merge together. A problem faced by the members of a cartel is the stability of the agreement and non-cooperative game theory predicts that member firms have always an incentive to deviate from the agreed-upon decision.

However, in some oligopoly situations firms don't always behave non-cooperatively and if sufficient communication is feasible it may be possible for firms to sign agreements. A question is then whether it is possible for firms to agree all together and coordinate their

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decision to achieve Pareto efficiency. For that, a cooperative approach for oligopoly situations can be considered by converting the normal form oligopoly game into an oligopoly TU-game in which firms can form cartels acting as a single player. Since every individual profit function depends on each firm's decision, the profit of a cartel depends on outsiders' behavior. Hence, the determination of the profit that a cartel can obtain requires to specify how the other firms react. Aumann (1959) proposes two approaches: according to the first, every cartel computes the total profit which it can guarantee itself regardless of what outsiders do; the second approach consists in computing the minimal profit for which outsiders can prevent the firms in the cartel from getting more. These two assumptions lead to consider the α and β-characteristic functions respectively. However, these two approaches can be questioned since outsiders probably cause substantial damages upon themselves by minimizing the profit of the cartel. This is why Chander and Tulkens (1997) propose to consider an alternative blocking rule where external firms choose their strategy individually as a best reply to the coalitional action. This leads to consider the "partial agreement characteristic function" or, for short, the γ-characteristic function.

Until now, many works have dealt with Cournot oligopoly TU-games. With or without transferable technologies,[1] Zhao (1999a,b) shows that the α and β-characteristic functions lead to the same class of Cournot oligopoly TU-games. When technologies are transferable, Zhao (1999a) provides a necessary and sufficient condition to establish the convexity property in case the inverse demand function and cost functions are linear. Although these games may fail to be convex in general, Norde et al. (2002) show they are nevertheless totally balanced. When technologies are not transferable, Zhao (1999b) proves that the core of such games is non-empty if every individual profit function is continuous and concave.[2] Furthermore, Norde et al. (2002) show that these games are convex in case the inverse demand function and cost functions are linear, and Driessen and Meinhardt (2005) provide economically meaningful sufficient conditions to guarantee the convexity property in a more general case.
For the class of Cournot oligopoly TU-games in γ-characteristic function form, Lardon (2009) shows that the differentiability of the inverse demand function ensures that these games are well-defined and provides two core existence results. The first result establishes that such games are balanced, and therefore have a non-empty core, if every individual profit function is concave. When cost functions are linear, the second result provides a single-valued allocation rule in the core, called NP(Nash Pro rata)-value, which is characterized by four axioms: efficiency, null firm, monotonicity and non-cooperative fairness.[3] In continuation of this work, Lardon (2010) shows that if the inverse demand function is continuous but not necessarily differentiable, it is always possible to define a Cournot

[1]We refer to Norde et al. (2002) for a detailed discussion of this distinction.

[2]Zhao shows that the core is non-empty for general TU-games in β-characteristic function form in which every strategy set is compact and convex, every utility function is continuous and concave, and satisfying the strong separability condition that requires that the utility function of a coalition and each of its members' utility functions have the same minimizers. Zhao proves that Cournot oligopoly TU-games satisfy this latter condition.

[3]We refer to Lardon (2009) for a precise description of these axioms.

oligopoly interval game[4] in γ-characteristic function form. This class of Cournot oligopoly interval games includes the class of Cournot oligopoly TU-games in γ-characteristic function form. Lardon considers two extensions of the core, the interval core and the standard core, and provides a necessary and sufficient condition for the non-emptiness of each of these core solution concepts.

Unfortunately, few works have dealt with Bertrand oligopoly TU-games. Kaneko (1978) considers a Bertrand oligopoly situation where a finite number of firms sell a homogeneous product to a continuum of consumers. Kaneko assumes that a subset of firms and consumers can cooperate by trading the good among themselves. The main result establishes that the core is empty when there are more than two firms. Deneckere and Davidson (1985) consider a Bertrand oligopoly situation with differentiated products in which the demand system is Shubik's (1980) and firms operate at a constant and identical marginal and average cost. They study the equilibrium distribution of prices and profits among cartels and show that a merger of two cartels implies that all the firms charge higher prices, and so benefits all the industry. They also prove that these games have a superadditive property in the sense that a merger of two disjoint cartels results in a joint after-merger profit for them which is greater than the sum of their pre-merger profits. For the same model, Huang and Sjöström (2003) provide a necessary and sufficient condition for the non-emptiness of the recursive core[5] which requires that the substitutability parameter must be greater than or equal to some number that depends on the size of the industry. They conclude that the recursive core is empty when there are more than ten firms.

As a counterpart to this lack of interest in the study of Bertrand oligopoly TU-games, we consider the same model as Deneckere and Davidson (1985) and substantially extend their results. In order to define Bertrand oligopoly TU-games, we consider successively the α, β and γ-characteristic function as introduced above. As for Cournot oligopoly TU-games, we show that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games and we prove that the convexity property holds for this class of games. Then, following Chander and Tulkens (1997) we consider the γ-characteristic function where firms react to a deviating coalition by choosing individual best reply strategies. For this class of games, we show that the Equal Division Solution belongs to the core and we provide a sufficient condition under which such games are convex. This finding generalizes the superadditivity result of Deneckere and Davidson (1985) and contrasts sharply with the negative core existence results of Kaneko (1978) and Huang and Sjöström (2003). Note that these properties are also satisfied for Cournot oligopoly TU-games in γ-characteristic function form.

In non-cooperative game theory, an important distinction between a normal form Cournot oligopoly game and a normal form Bertrand oligopoly game is that the former has strategic substitutabilities and the latter has strategic complementarities. Thus, although Cournot and Bertrand oligopoly games are basically different in their non-cooperative

[4] Interval games are introduced by Branzei et al. (2003).

[5] The worth of a coalition is defined in a recursive procedure applying the core solution concept to a "reduced game" in order to predict outsiders' behavior.

forms, it appears that their cooperative forms have the same core geometrical structure.[6]

The remainder of the article is structured as follows. In section 2 we introduce the model and some notations. Section 3 establishes that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games and shows that the convexity property holds for this class of games. Section 4 proves that the Equal Division Solution belongs to the core of Bertrand oligopoly TU-games in γ-characteristic function form and provides a sufficient condition under which such games are convex. Section 5 gives some concluding remarks.

2 The model

Consider a **Bertrand oligopoly situation** $(N, (D_i, C_i)_{i\in N})$ where $N = \{1, 2, \ldots, n\}$ is the finite **set of firms**, $D_i : \mathbb{R}^n_+ \longrightarrow \mathbb{R}$, $i \in N$, is firm i's **demand function** and $C_i : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, $i \in N$, is firm i's **cost function**. Throughout this article, we assume that:

(a) the demand system is Shubik's (1980), i.e.

$$\forall i \in N,\ D_i(p_1, \ldots, p_n) = V - p_i - r\left(p_i - \frac{1}{n}\sum_{j\in N} p_j\right),$$

where p_j is the price charged by firm j, $V \in \mathbb{R}_+$ is the intercept of demand and $r \in \mathbb{R}_{++}$ is the substitutability parameter.[7] The quantity demanded of firm i's brand depends on its own price p_i and on the difference between p_i and the average price in the industry $\sum_{j\in N} p_j/n$. This quantity is decreasing with respect to p_i and increasing with respect to p_j, $j \neq i$;

(b) firms operate at a constant and identical marginal and average cost, i.e.

$$\forall i \in N,\ C_i(x) = cx,$$

where $c \in \mathbb{R}_+$ is firm i's marginal and average cost, and $x \in \mathbb{R}_+$ is the quantity demanded of firm i's brand.

Given assumptions (a) and (b), a Bertrand oligopoly situation is summarized by the 4-tuple (N, V, r, c).

The **normal form Bertrand oligopoly game** $(N, (X_i, \pi_i)_{i\in N})$ associated with the Bertrand oligopoly situation (N, V, r, c) is defined as follows:

1. the set of firms is $N = \{1, \ldots, n\}$;

[6] The convexity property ensures that the core is the convex combination of the marginal vectors (Shapley 1971).

[7] When r approaches zero, products become unrelated, and when r approaches infinity, products become perfect substitutes.

2. for every $i \in N$, the **individual strategy set** is $X_i = \mathbb{R}_+$ where $p_i \in X_i$ represents the price charged by firm i;

3. the **set of strategy profiles** is $X_N = \prod_{i \in N} X_i$ where $p = (p_i)_{i \in N}$ is a representative element of X_N; for every $i \in N$, the **individual profit function** $\pi_i : X_N \longrightarrow \mathbb{R}$ is defined as

$$\pi_i(p) = D_i(p)(p_i - c).$$

As mentioned above, we want to analyze the stability of cartels and the incentive for merger in Bertrand oligopoly situations. To this end, we have to convert the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i \in N})$ into a Bertrand oligopoly TU-game. This is the purpose of the following two sections. In the remainder of this section, we recall the definition of a TU-game, the definition of the core and the notion of a convex TU-game. We denote by 2^N the power set of N and call a subset $S \in 2^N$, $S \neq \emptyset$, a **coalition**. The **size** $s = |S|$ of coalition S is the number of players in S. A **TU-game** (N, v) is a **set function** $v : 2^N \longrightarrow \mathbb{R}$ with the convention that $v(\emptyset) = 0$, which assigns a number $v(S) \in \mathbb{R}$ to every coalition $S \in 2^N$. The number $v(S)$ is the worth of coalition S. For a fixed set of players N, we denote by G^N the **set of TU-games** where v is a representative element of G^N.
In a TU-game $v \in G^N$, every player $i \in N$ may receive a **payoff** $\sigma_i \in \mathbb{R}$. A vector $\sigma = (\sigma_1, \ldots, \sigma_n)$ is a **payoff vector**. We say that a payoff vector $\sigma \in \mathbb{R}^n$ is **acceptable** if for every coalition $S \in 2^N$, $\sum_{i \in S} \sigma_i \geq v(S)$, i.e. the payoff vector provides a total payoff to the members of coalition S that is at least as great as its worth. We say that a payoff vector $\sigma \in \mathbb{R}^n$ is **efficient** if $\sum_{i \in N} \sigma_i = v(N)$, i.e. the payoff vector provides a total payoff to all the players that is equal to the worth of the grand coalition N. The **core** $C(v)$ of a TU-game $v \in G^N$ is the set of payoff vectors that are both acceptable and efficient, i.e.

$$C(v) = \left\{ \sigma \in \mathbb{R}^n : \forall S \in 2^N,\ \sum_{i \in S} \sigma_i \geq v(S) \text{ and } \sum_{i \in N} \sigma_i = v(N) \right\}.$$

Given a payoff vector in the core, the grand coalition can form and distribute its worth as payoffs to its members in such a way that no coalition can contest this sharing by breaking off from the grand coalition.
The notion of a convex TU-game is introduced by Shapley (1971) and provides a natural way to formalize the idea that it is worthwhile for a player to join larger coalitions. A TU-game $v \in G^N$ is **convex** if one of the following two equivalent conditions is satisfied:

$$\forall S, T \in 2^N,\ v(S) + v(T) \leq v(S \cup T) + v(S \cap T),$$

or equivalently,

$$\forall i, j \in N,\ \forall S \subseteq N \backslash \{i, j\},\ v(S \cup \{i\}) - v(S) \leq v(S \cup \{i, j\}) - v(S \cup \{j\}) \qquad (1)$$

The convexity property means that the marginal contribution of a player to some coalition increases if the coalition which he joins becomes larger. Thus, the convexity property can be regarded as a strong incentive for large scale cooperation.

3 Bertrand oligopoly TU-games with pessimistic expectations.

In this section, we convert a normal form Bertrand oligopoly game into a Bertrand oligopoly TU-game in such a way that every coalition has pessimistic expectations on its feasible profits. Traditionally, there are two main ways of converting a normal form game into a cooperative game called game in α and β-characteristic function form respectively (Aumann 1959). In the first case, the worth of a coalition is obtained by computing the profit which its members can guarantee themselves regardless of what outsiders do. In the second case, the worth of a coalition can be derived by computing the minimal profit such that outsiders can prevent its members from getting more. First, we show that the α and β-characteristic functions are well-defined and lead to the same class of Bertrand oligopoly TU-games. This equality between the α and β-characteristic functions is a useful property, as it relieves us of the burden of choosing between the α and β-characteristic functions when describing collusive profits. Then, we prove that the convexity property holds for this class of games, i.e. when every coalition has pessimistic expectations on its feasible profits there exists a strong incentive to form the grand coalition.

In order to define the α and β-characteristic functions, we denote by $X_S = \prod_{i \in S} X_i$ the **coalition strategy set** of coalition $S \in 2^N$ and $X_{N \backslash S} = \prod_{i \in N \backslash S} X_i$ the **set of outsiders' strategy profiles** for which $p_S = (p_i)_{i \in S}$ and $p_{N \backslash S} = (p_i)_{i \in N \backslash S}$ are the representative elements respectively. For every coalition $S \in 2^N$, the **coalition profit function** $\pi_S : X_S \times X_{N \backslash S} \longrightarrow \mathbb{R}$ is defined as

$$\pi_S(p_S, p_{N \backslash S}) = \sum_{i \in S} \pi_i(p).$$

Given the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i \in N})$, the α and β-characteristic functions are defined for every coalition $S \in 2^N$ as

$$v_\alpha(S) = \max_{p_S \in X_S} \min_{p_{N \backslash S} \in X_{N \backslash S}} \pi_S(p_S, p_{N \backslash S}) \qquad (2)$$

and

$$v_\beta(S) = \min_{p_{N \backslash S} \in X_{N \backslash S}} \max_{p_S \in X_S} \pi_S(p_S, p_{N \backslash S}) \qquad (3)$$

respectively. For a fixed set of firms N, we denote by $G_o^N \subseteq G^N$ the **set of Bertrand oligopoly TU-games**.

The following proposition states that Bertrand oligopoly TU-games in β-characteristic function form are well-defined.

Proposition 3.1 *Let $(N, (X_i, \pi_i)_{i \in N})$ be a normal form Bertrand oligopoly game. Then, for every coalition $S \in 2^N$, it holds that*

$$v_\beta(S) = \pi_S(\overline{p}_S, \overline{p}_{N \backslash S}),$$

where $(\overline{p}_S, \overline{p}_{N \backslash S}) \in X_S \times X_{N \backslash S}$ is given by

$$\forall i \in S,\ \overline{p}_i = \max\left\{c, \frac{V}{2\big(1 + r(n-s)/n\big)} + \frac{c}{2}\right\} \tag{4}$$

and

$$\sum_{j \in N \backslash S} \overline{p}_j = \max\left\{0, \frac{n}{r}\left(c\left(1 + r\frac{(n-s)}{n}\right) - V\right)\right\} \tag{5}$$

Proof: Take a coalition $S \in 2^N$. Define $b_S : X_{N \backslash S} \longrightarrow X_S$ the **best reply function of coalition** S as

$$\forall p_{N \backslash S} \in X_{N \backslash S},\ \forall p_S \in X_S,\ \pi_S(b_S(p_{N \backslash S}), p_{N \backslash S}) \geq \pi_S(p_S, p_{N \backslash S}) \tag{6}$$

It follows from (6) that

$$v_\beta(S) = \min_{p_{N \backslash S} \in X_{N \backslash S}} \pi_S(b_S(p_{N \backslash S}), p_{N \backslash S}).$$

In order to compute the worth $v_\beta(S)$ of coalition S, we have to successively solve the maximization and the minimization problems defined in (3). First, for every $p_{N \backslash S} \in X_{N \backslash S}$ consider the profit maximization program of coalition S, i.e.

$$\forall p_{N \backslash S} \in X_{N \backslash S},\ \max_{p_S \in X_S} \pi_S(p_S, p_{N \backslash S}).$$

The first-order conditions for a maximum are

$$\forall p_{N \backslash S} \in X_{N \backslash S},\ \forall i \in S,\ \frac{\partial \pi_S}{\partial p_i}(p_S, p_{N \backslash S}) = 0,$$

and imply that the unique maximizer $b_S(p_{N \backslash S})$ is given by

$$\forall p_{N \backslash S} \in X_{N \backslash S},\ \forall i \in S,\ b_i(p_{N \backslash S}) = \frac{V + (r/n)\sum_{j \in N \backslash S} p_j}{2\big(1 + r(n-s)/n\big)} + \frac{c}{2} \tag{7}$$

Then, given $b_S(p_{N \backslash S}) \in X_S$ consider the profit minimization program of the complementary coalition $N \backslash S$, i.e.

$$\min_{p_{N \backslash S} \in X_{N \backslash S}} \pi_S(b_S(p_{N \backslash S}), p_{N \backslash S}).$$

The first-order conditions for a minimum are

$$\forall j \in N\backslash S,\ \frac{\partial \pi_S}{\partial p_j}(b_S(p_{N\backslash S}), p_{N\backslash S}) = 0,$$

which are equivalent, for all $j \in N\backslash S$, to the following equality

$$\sum_{j \in N\backslash S} p_j = \frac{n}{r}\Bigg(c\bigg(1 + r\frac{(n-s)}{n}\bigg) - V\Bigg).$$

Since for every $i \in N$, $X_i = \mathbb{R}_+$, it follows that any minimizer $\overline{p}_{N\backslash S} \in X_{N\backslash S}$ satisfies

$$\sum_{j \in N\backslash S} \overline{p}_j = \max\Bigg\{0, \frac{n}{r}\Bigg(c\bigg(1 + r\frac{(n-s)}{n}\bigg) - V\Bigg)\Bigg\},$$

which proves (5). By substituting (5) into (7), we deduce that

$$\forall i \in S,\ \overline{p}_i = b_i(\overline{p}_{N\backslash S}) = \max\Bigg\{c, \frac{V}{2\big(1 + r(n-s)/n\big)} + \frac{c}{2}\Bigg\},$$

which proves (4) and completes the proof. ■

Proposition 3.1 calls for some comments which will be useful for the sequel.

Remark 3.2

For every coalition $S \in 2^N$, it holds that:

1. If $V \leq c\big(1 + r(n-s)/n\big)$, then by (4) each member $i \in S$ charges prices equal to their marginal cost, $\overline{p}_i = c$, and by (5) the outsiders charge a non-negative average price, $\sum_{j \in N\backslash S} \overline{p}_j/(n-s) \geq 0$. In this case, coalition S obtains a zero profit, $v_\beta(S) = 0$.

2. If $V > c\big(1 + r(n-s)/n\big)$, then by (4) each member $i \in S$ charges prices strictly greater than their marginal cost, $\overline{p}_i > c$, and by (5) the outsiders charge a zero average price, $\sum_{j \in N\backslash S} \overline{p}_j/(n-s) = 0$. In this case, coalition S obtains a positive profit, $v_\beta(S) > 0$.

3. The computation of the worth $v_\beta(S)$ is consistent with the fact that the quantity demanded of firm i's brand, $i \in S$, is positive since for every $i \in S$, $D_i(\overline{p}) \geq 0$.

By solving successively the minimization and the maximization problems defined in (2), we can show that Bertrand oligopoly TU-games in α-characteristic function form are well-defined. The proof is similar to the one for Proposition 3.1, and so it is not detailed. A useful property is that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games as enunciated in the following proposition.

Proposition 3.3 *Let $(N,(X_i,\pi_i)_{i\in N})$ be a normal form Bertrand oligopoly game. Then, for every coalition $S \in 2^N$, it holds that*

$$v_\alpha(S) = v_\beta(S).$$

Proof: First, for every coalition $S \in 2^N$, it holds that

$$\begin{aligned} v_\alpha(S) &= \max_{p_S \in X_S} \min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S(p_S, p_{N\setminus S}) \\ &\leq \min_{p_{N\setminus S} \in X_{N\setminus S}} \max_{p_S \in X_S} \pi_S(p_S, p_{N\setminus S}) \\ &= v_\beta(S). \end{aligned}$$

Then, it remains to show that for every coalition $S \in 2^N$, $v_\beta(S) \leq v_\alpha(S)$. We distinguish two cases:
- assume that $V \leq c\big(1 + r(n-s)/n\big)$. It follows from point 1 of Remark 3.2 that for every $i \in S$, $\overline{p}_i = c$. Hence, for every $p_{N\setminus S} \in X_{N\setminus S}$ it holds that $\pi_S(\overline{p}_S, p_{N\setminus S}) = 0$.
- assume that $V > c\big(1 + r(n-s)/n\big)$. It follows from point 2 of Remark 3.2 that for every $i \in S$, $\overline{p}_i > c$, and $\overline{p}_{N\setminus S} = 0_{N\setminus S}$. Since for every $i \in S$, D_i is increasing on $X_{N\setminus S}$ we deduce that for every $p_{N\setminus S} \in X_{N\setminus S}$, $\pi_S(\overline{p}_S, p_{N\setminus S}) \geq \pi_S(\overline{p}_S, 0_{N\setminus S})$.

In both cases, it holds that

$$\overline{p}_{N\setminus S} \in \arg \min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S(\overline{p}_S, p_{N\setminus S}) \tag{8}$$

For every coalition $S \in 2^N$, it follows from (8) that

$$\begin{aligned} v_\beta(S) &= \pi_S(\overline{p}_S, \overline{p}_{N\setminus S}) \\ &= \min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S(\overline{p}_S, p_{N\setminus S}) \\ &\leq \max_{p_S \in X_S} \min_{p_{N\setminus S} \in X_{N\setminus S}} \pi_S(p_S, p_{N\setminus S}) \\ &= v_\alpha(S), \end{aligned}$$

which completes the proof. ■

Proposition 3.3 implies that outsiders' strategy profile $\overline{p}_{N\setminus S}$ that best punishes coalition S as a first mover (α-characteristic function) also best punishes S as a second mover (β-characteristic function). Zhao (1999b) obtains a similar result for general TU-games in which every strategy set is compact, every utility function is continuous, and satisfying the strong separability condition that requires that the utility function of a coalition S and each of its members' utility functions have the same minimizers. We can use Zhao's result (1999b) in order to prove Proposition 3.3. First, we compactify the strategy sets by assuming that for every $i \in N$, $X_i = [0, \mathbf{p}]$ where $\mathbf{p}$ is sufficiently large so that the maximization/minimization problems defined in (2) and (3) have interior solutions. Then, it is clear that every individual profit function π_i is continuous. Finally, since the demand system is symmetric and firms operate at a constant and identical marginal and average cost, we can verify that Bertrand oligopoly TU-games satisfy the strong separability condition. In this article, in order to be shorter and perfectly rigorous we prefer to give a constructive proof of Proposition 3.3 without using Zhao's result (1999b). We deduce from Proposition 3.3 the following corollary.

Corollary 3.4 *Let $v_\alpha \in G_o^N$ and $v_\beta \in G_o^N$ be Bertrand oligopoly TU-games associated with the Bertrand oligopoly situation (N, V, r, c). Then*

$$C(v_\alpha) = C(v_\beta).$$

Now, we want to prove that Bertrand oligopoly TU-games in α or β-characteristic function form are convex. Proposition 3.1 implies that Bertrand oligopoly TU-games in β-characteristic function form are symmetric.[8] It follows from (4) that the members of a coalition $S \in 2^N$ charge identical prices, i.e. for every $i \in S$, there exists $\overline{p}^s \in \mathbb{R}_+$ such that $\overline{p}_i = \overline{p}^s$. It follows from (5) that outsiders charge an average price $\overline{p}_{[n-s]}/(n-s)$ where $\overline{p}_{[n-s]} = \sum_{j \in N \setminus S} \overline{p}_j$. Hence, the worth $v_\beta(S)$ depends only on the size s of coalition S, i.e. there exists a function $f_\beta : \mathbb{N} \longrightarrow \mathbb{R}$ such that for every coalition $S \in 2^N$, it holds that

$$v_\beta(S) = f_\beta(s) = s(\overline{p}^s - c)\left(V - \overline{p}^s\left(1 + r\frac{(n-s)}{n}\right) + \frac{r}{n}\overline{p}_{[n-s]}\right).$$

For a symmetric TU-game $v \in G^N$ for which for every coalition $S \in 2^N$, $v(S) = f(s)$, condition (1) becomes:

$$\forall S \in 2^N : s \leq n-2,\ f(s+1) - f(s) \leq f(s+2) - f(s+1) \qquad (9)$$

The following theorem states that the convexity property holds for the class of Bertrand oligopoly TU-games in β-characteristic function form.

Theorem 3.5 *Every Bertrand oligopoly TU-game $v_\beta \in G_o^N$ is convex.*

Proof: We want to prove (9). Take a coalition $S \in 2^N$ of size s such that $s \leq n-2$. First, we distinguish two cases:
- assume that $V \leq c\big(1 + r(n-s-1)/n\big)$. It follows from point 1 of Remark 3.2 that $\overline{p}^{s+1} = c$.
- assume that $V > c\big(1 + r(n-s-1)/n\big)$. This implies that $V > c\big(1 + r(n-s-2)/n\big)$, and it follows from point 2 of Remark 3.2 that $\overline{p}_{[n-s-1]} = \overline{p}_{[n-s-2]} = 0$.
In both cases, it holds that

$$(\overline{p}^{s+1} - c)\overline{p}_{[n-s-2]} = (\overline{p}^{s+1} - c)\overline{p}_{[n-s-1]} \qquad (10)$$

[8] A TU-game $v \in G^N$ is symmetric if there exists a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ such that for every coalition $S \in 2^N$, $v(S) = f(s)$.

Since $\overline{p}^{s+2}$ is the unique maximizer for any coalition of size $s+2$ and from (10), it holds that

$$\begin{aligned} f_\beta(s+2) &= (s+2)(\overline{p}^{s+2}-c)\left(V-\overline{p}^{s+2}\left(1+r\frac{(n-s-2)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-2]}\right) \\ &\geq (s+2)(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s-2)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-2]}\right) \\ &= (s+2)(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s-2)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \\ &= f_\beta(s+1)+(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-3)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \end{aligned} \tag{11}$$

Moreover, since $\overline{p}^s$ is the unique maximizer for any coalition of size s and $\overline{p}_{[n-s]} \geq \overline{p}_{[n-s-1]}$, we deduce that

$$\begin{aligned} f_\beta(s) &= s(\overline{p}^{s}-c)\left(V-\overline{p}^{s}\left(1+r\frac{(n-s)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s]}\right) \\ &\geq s(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s]}\right) \\ &\geq s(\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-s)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \end{aligned} \tag{12}$$

It follows from the expression of $f_\beta(s+1)$ and (12) that

$$f_\beta(s+1)-f_\beta(s) \leq (\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-1)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \tag{13}$$

We conclude from (11) and (13) that

$$\begin{aligned} f_\beta(s+1)-f_\beta(s) &\leq (\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-1)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \\ &\leq (\overline{p}^{s+1}-c)\left(V-\overline{p}^{s+1}\left(1+r\frac{(n-2s-3)}{n}\right)+\frac{r}{n}\overline{p}_{[n-s-1]}\right) \\ &\leq f_\beta(s+2)-f_\beta(s+1), \end{aligned}$$

which completes the proof. ■

The convexity property does not hold in Bertrand oligopoly TU-games in β-characteristic function form when the firms operate at different marginal costs as illustrated in the following example.

Example 3.6 Consider the Bertrand oligopoly situation $(N, V, r, (c_i)_{i\in N})$ where $N = \{1, 2, 3\}$, $V = 2$, $r = 5$, $c_1 = 1$, $c_2 = 3$ and $c_3 = 5$. For every coalition $S \in 2^N$, the worth $v_\beta(S)$ is given in the following table:

S	$\{1\}$	$\{2\}$	$\{3\}$	$\{1,2\}$	$\{1,3\}$	$\{2,3\}$	$\{1,2,3\}$
$v_\beta(S)$	0	0	0	3	12	3	12

Note that $v_\beta(\{1,2\})+v_\beta(\{1,3\}) = 15 > 12 = v_\beta(\{1,2,3\})+v_\beta(\{1\})$, and so $v_\beta \in G_o^N$ is not convex. □

When the firms have different marginal costs, Driessen et al. (2010) provide a sufficient condition under which the convexity property holds for Bertrand oligopoly TU-games in β-characteristic function form.

4 Bertrand oligopoly TU-games with rational expectations

In this section, we question the resorting to the α and β-characteristic functions in order to define Bertrand oligopoly TU-games. Indeed, the reaction of outsiders to minimize the profit of a deviating coalition probably implies substantial damages upon themselves. As in Chander and Tulkens (1997), we consider the alternative blocking rule for which outsiders choose their strategy individually as a best reply facing the deviating coalition. Such an equilibrium is called a partial agreement equilibrium and leads to consider the "partial agreement characteristic function" or, for short, the γ-characteristic function. First, we show that the γ-characteristic function is well-defined and that the associated core is included in the core associated with the β-characteristic function. Then, for this class of games we prove that the core is non-empty by showing that the Equal Division Solution belongs to the core and we provide a sufficient condition under which such games are convex.

Given the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i\in N})$ and a coalition $S \in 2^N$, a strategy profile $(p^*_S, \tilde{p}_{N\backslash S}) \in X_S \times X_{N\backslash S}$ is a **partial agreement equilibrium** under S if

$$\forall p_S \in X_S,\ \pi_S(p^*_S, \tilde{p}_{N\backslash S}) \geq \pi_S(p_S, \tilde{p}_{N\backslash S}) \quad (14)$$

and

$$\forall j \in N\backslash S,\ \forall p_j \in X_j,\ \pi_j(p^*_S, \tilde{p}_{N\backslash S}) \geq \pi_j(p^*_S, \tilde{p}_{N\backslash(S\cup\{j\})}, p_j) \quad (15)$$

The γ-characteristic function is defined for every coalition $S \in 2^N$ as

$$v_\gamma(S) = \pi_S(p^*_S, \tilde{p}_{N\setminus S}),$$

where $(p^*_S, \tilde{p}_{N\setminus S}) \in X_S \times X_{N\setminus S}$ is a partial agreement equilibrium under S.

Throughout this section, in addition to assumptions (a) and (b) we assume that:

(c) the intercept of demand $V \in \mathbb{R}_+$ is strictly greater than the marginal cost $c \in \mathbb{R}_+$.

For every partial agreement equilibrium under S, assumption (c) ensures that the quantity demanded of every firm's brand is non-negative as discussed later.

Deneckere and Davidson (1985) study normal form Bertrand oligopoly games with general coalition structures. A **coalition structure** $\mathcal{P}$ is a partition of the set of firms N, i.e. $\mathcal{P} = \{S_1, \dots, S_k\}$, $k \in \{1, \dots, n\}$. An element of a coalition structure, $S \in \mathcal{P}$, is called an **admissible coalition** in $\mathcal{P}$. We denote by $\Pi(N)$ the **set of coalition structures** on N. We introduce a binary relation $\leq^F$ on $\Pi(N)$ defined as follows: we say that a coalition structure $\mathcal{P}' \in \Pi(N)$ is finer than a coalition structure $\mathcal{P} \in \Pi(N)$ (or $\mathcal{P}$ is coarser than $\mathcal{P}'$) which we write $\mathcal{P} \leq^F \mathcal{P}'$ if for every admissible coalition S in $\mathcal{P}'$ there exists an admissible coalition T in $\mathcal{P}$ such that $T \supseteq S$. Note that $(\Pi(N), \leq^F)$ is a complete lattice.
Given $\mathcal{P} \in \Pi(N)$, the normal form Bertrand oligopoly game $(\mathcal{P}, (X_S, \pi_S)_{S\in\mathcal{P}})$ associated with the Bertrand oligopoly situation (N, V, r, c) is defined as follows:

1. the **set of players** (or admissible coalitions) is $\mathcal{P}$;
2. for every $S \in \mathcal{P}$, the **coalition strategy set** is $X_S = \prod_{i\in S} X_i$;
3. for every $S \in \mathcal{P}$, the **coalition profit function** is $\pi_S = \sum_{i\in S} \pi_i$.

The following proposition is a compilation of different results in Deneckere and Davidson (1985).

Proposition 4.1 (Deneckere and Davidson 1985)

- *Let $\mathcal{P} \in \Pi(N)$ be a coalition structure and let $(\mathcal{P}, (X_S, \pi_S)_{S\in\mathcal{P}})$ be the associated normal form Bertrand oligopoly game. Then,*

1. *there exists a unique Nash equilibrium $p^* \in X_N$ such that*

$$\forall S \in \mathcal{P},\ \exists p^{*S} \in \mathbb{R}_+ : \forall i \in S,\ p^*_i = p^{*S}.$$

2. *it holds that*

$$\forall S, T \in \mathcal{P} : s \leq t,\ p^{*S} \leq p^{*T},$$

with strict inequality if $s < t$.

- *Let $\mathcal{P}, \mathcal{P}' \in \Pi(N)$ be two coalition structures such that $\mathcal{P} \leq^F \mathcal{P}'$. Let $p^* \in X_N$ and $p^{**} \in X_N$ be the unique Nash equilibria of the normal form Bertrand oligopoly games $(\mathcal{P}, (X_S, \pi_S)_{S \in \mathcal{P}})$ and $(\mathcal{P}', (X_S, \pi_S)_{S \in \mathcal{P}'})$ respectively. Then,*

3. *it holds that for every $i \in N$, $p_i^* \geq p_i^{**}$.*

Point 1 of Proposition 4.1 establishes the existence of a unique Nash equilibrium for every normal form Bertrand oligopoly game $(\mathcal{P}, (X_S, \pi_S)_{S \in \mathcal{P}})$ and stipulates that the members of an admissible coalition $S \in \mathcal{P}$ charge identical prices. Point 2 of Proposition 4.1 characterizes the distribution of prices within a coalition structure and states that if the size t of an admissible coalition $T \in \mathcal{P}$ is greater than or equal to the size s of an admissible coalition $S \in \mathcal{P}$, then the firms in T charge higher prices than the firms in S. Point 3 of Proposition 4.1 analyses the variations in equilibrium prices according to the coarseness of the coalition structure and specifies that all the firms charge higher prices when the coalition structure becomes coarser.

The following proposition states that Bertrand oligopoly TU-games in γ-characteristic function form are well-defined.

Proposition 4.2 *Let $(N, (X_i, \pi_i)_{i \in N})$ be a normal form Bertrand oligopoly game. Then, for every coalition $S \in 2^N$, there exists a unique partial agreement equilibrium under S.*

Proof: Take a coalition $S \in 2^N$ and consider the coalition structure $\mathcal{P}^S = \{S\} \cup \{\{j\} : j \in N \backslash S\}$. It follows from (14) and (15) that a strategy profile $(p_S^*, \tilde{p}_{N \backslash S}) \in X_S \times X_{N \backslash S}$ is a partial agreement equilibrium for the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i \in N})$ if and only if it is a Nash equilibrium for the normal form Bertrand oligopoly game $(\mathcal{P}^S, (X_T, \pi_T)_{T \in \mathcal{P}^S})$. By point 1 of Proposition 4.1 we conclude that there exists a unique partial agreement equilibrium under S. ■

By solving the maximization problems derived from (14) and (15), the unique partial agreement equilibrium under S, $(p_S^*, \tilde{p}_{N \backslash S}) \in X_S \times X_{N \backslash S}$, is given by

$$\forall i \in S,\ p_i^* = \frac{(V-c)\big(2n(1+r)-r\big)n}{2\big(2n+r(n+s-1)\big)\big(n+r(n-s)\big)-r^2 s(n-s)} + c \tag{16}$$

and

$$\forall j \in N \backslash S,\ \tilde{p}_j = \frac{(V-c)\big(2n(1+r)-rs\big)n}{2\big(2n+r(n+s-1)\big)\big(n+r(n-s)\big)-r^2 s(n-s)} + c \tag{17}$$

When $c = 0$, Deneckere and Davidson (1985) provide equivalent expressions of (16) and (17). We deduce from (16) and (17) that Bertrand oligopoly TU-games in γ-characteristic function form are symmetric. It follows from (16) that the members of a coalition $S \in 2^N$ charge identical prices, i.e. for every $i \in S$, there exists $p^{*s} \in \mathbb{R}_+$ such that $p_i^* = p^{*s}$. It follows from (17) that outsiders charge identical prices, i.e. for every $j \in N \backslash S$, there exists $\tilde{p}^s \in \mathbb{R}_+$ such that $\tilde{p}_j = \tilde{p}^s$. Hence, the worth $v_\gamma(S)$ depends only on the size s of coalition S, i.e. there exists a function $f_\gamma : \mathbb{N} \longrightarrow \mathbb{R}$ such that for every coalition $S \in 2^N$, it holds that

$$v_\gamma(S) = f_\gamma(s) = s(p^{*s} - c)\bigg(V - p^{*s} + r\frac{(n-s)}{n}(\tilde{p}^s - p^{*s})\bigg).$$

With these notations in mind, Proposition 4.1 calls for some comments which will be useful for the sequel.

Remark 4.3

1. For every coalition $S \in 2^N$, we deduce from point 2 of Proposition 4.1 that $p^{*s} \geq \tilde{p}^s$, i.e. the members of coalition S charge higher prices than the outsiders.

2. For every coalition $S, T \in 2^N$ such that $S \subseteq T$, it follows from point 3 of Proposition 4.1 that $p^{*s} \leq p^{*t}$ and $\tilde{p}^s \leq \tilde{p}^t$.

3. For every coalition $S \in 2^N$, let $(p^*_S, \tilde{p}_{N\backslash S}) \in X_S \times X_{N\backslash S}$ be the unique partial agreement equilibrium under S. If $p^{*s} > c$ and $\tilde{p}^s > c$ then for every $i \in N$, $D_i(p^*_S, \tilde{p}_{N\backslash S}) \geq 0$. In order to prove this result, for the sake of contradiction, assume that there exists $i \in N$ such that $D_i(p^*_S, \tilde{p}_{N\backslash S}) < 0$, and $p^{*s} > c$ and $\tilde{p}^s > c$. We distinguish two cases:
   - if $i \in S$ then we deduce from point 1 of Proposition 4.1 that for every $j \in S$, $D_j(p^*_S, \tilde{p}_{N\backslash S}) = D_i(p^*_S, \tilde{p}_{N\backslash S}) < 0$. Hence, it follows from $p^{*s} > c$ that coalition S obtains a negative profit.
   - if $i \in N\backslash S$ then it follows from $\tilde{p}^s > c$ that outsider i obtains a negative profit.

   In both cases, since coalition S or every outsider can guarantee a non-negative profit by charging $p^{*s} = c$ or $\tilde{p}^s = c$ respectively, we conclude that $(p^*_S, \tilde{p}_{N\backslash S}) \in X_S \times X_{N\backslash S}$ is not a partial agreement equilibrium under S, a contradiction.
   By (16) and (17), note that $p^{*s} > c$ and $\tilde{p}^s > c$ is satisfied if and only if $V > c$, which corresponds to assumption (c). Thus, assumption (c) ensures that the quantity demanded of every firm's brand is non-negative.

The following proposition states that the core associated with the γ-characteristic function is included in the core associated with the β-characteristic function.[9]

Proposition 4.4 *Let $v_\gamma \in G^N_o$ and $v_\beta \in G^N_o$ be Bertrand oligopoly TU-games associated with the Bertrand oligopoly situation (N, V, r, c). Then*

$$C(v_\gamma) \subseteq C(v_\beta).$$

[9] Chander and Tulkens (1997) obtain this result for TU-games derived from an economy with multilateral environmental externalities. Although their result can be easily generalized to general TU-games, for the sake of completeness, we prefer to give the proof of this result for Bertrand oligopoly TU-games.

Proof: By the definition of the core, we have to show that for every coalition $S \in 2^N \backslash \{N\}$, $v_\gamma(S) \geq v_\beta(S)$ and $v_\gamma(N) = v_\beta(N)$. Clearly,

$$\begin{aligned} v_\gamma(N) &= \max_{p \in X_N} \pi_N(p) \\ &= v_\beta(N). \end{aligned}$$

Moreover, for every coalition $S \in 2^N \backslash \{N\}$, it holds that

$$\begin{aligned} v_\gamma(S) &= \pi_S(p^*_S, \tilde{p}_{N\backslash S}) \\ &= \max_{p_S \in X_S} \pi_S(p_S, \tilde{p}_{N\backslash S}) \\ &\geq \min_{p_{N\backslash S} \in X_{N\backslash S}} \max_{p_S \in X_S} \pi_S(p_S, p_{N\backslash S}) \\ &= v_\beta(S), \end{aligned}$$

which completes the proof. ■

Proposition 4.4 is illustrated in the following example.

Example 4.5 Consider the Bertrand oligopoly situation (N, V, r, c) where $N = \{1, 2, 3\}$, $V = 5$, $r = 2$ and $c = 1$. For every coalition $S \in 2^N$, the worths $v_\beta(S)$ and $v_\gamma(S)$ are given in the following table:

s	1	2	3
$f_\beta(s)$	0.76	3.33	12
$f_\gamma(s)$	3.36	7.05	12

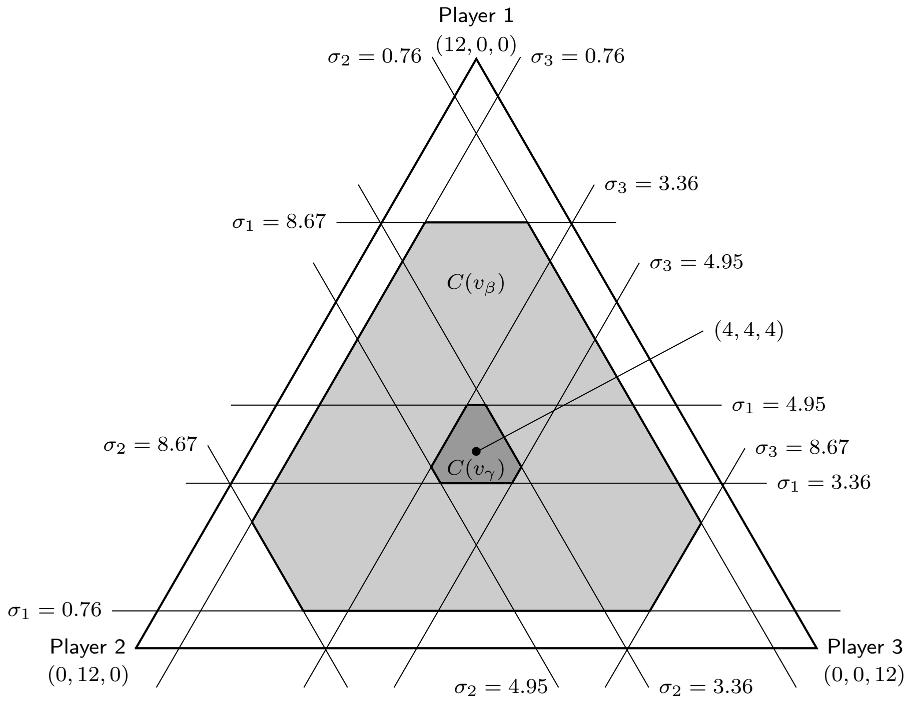
It follows that the cores $C(v_\beta)$ and $C(v_\gamma)$ are given by

$$C(v_\beta) = \left\{ \sigma \in \mathbb{R}^3 : \sum_{i \in N} \sigma_i = 12 \text{ and } \forall i \in \{1, 2, 3\},\ 0.76 \leq \sigma_i \leq 8.67 \right\},$$

and

$$C(v_\gamma) = \left\{ \sigma \in \mathbb{R}^3 : \sum_{i \in N} \sigma_i = 12 \text{ and } \forall i \in \{1, 2, 3\},\ 3.36 \leq \sigma_i \leq 4.95 \right\}.$$

The 2-simplex below represents these two core configurations:



Thus, from the Bertrand oligopoly TU-game $v_\beta \in G_o^N$ to the Bertrand oligopoly TU-game $v_\gamma \in G_o^N$, we see that the core is substantially reduced. Two features must be noticed. The first is that the payoff vector $(4, 4, 4) \in \mathbb{R}^3$ that distributes the worth of the grand coalition equally among the players is the center of gravity of both cores. The second is that the Bertrand oligopoly TU-game $v_\gamma \in G_o^N$ is convex. In the remainder of this section, we show that these properties hold for some class of Bertrand oligopoly TU-games. □

Now, we show that a single-valued allocation rule,[10] the Equal Division Solution, belongs to the core of Bertrand oligopoly TU-games in γ-characteristic function form. For every TU-game $v \in G^N$ the **Equal Division Solution** denoted by $ED(v)$ is defined as

$$\forall i \in N,\ ED_i(v) = \frac{v(N)}{n}.$$

The Equal Division Solution distributes the worth of the grand coalition equally among the players. In order to prove that Bertrand oligopoly TU-games in γ-characteristic function form have a non-empty core, we show that the Equal Division Solution belongs to the core as enunciated in the following theorem.

Theorem 4.6 *Let $v_\gamma \in G_o^N$ be a Bertrand oligopoly TU-game. Then $ED(v_\gamma) \in C(v_\gamma)$.*

[10] A single-valued allocation rule on G^N is a mapping $\rho : G^N \longrightarrow \mathbb{R}^n$ that associates to every TU-game $v \in G^N$ a payoff vector $\rho(v) \in \mathbb{R}^n$.

Proof: In order to prove that $ED(v_\gamma) \in C(v_\gamma)$, we have to show that for every coalition $S \in 2^N$, $v_\gamma(N)/n \geq v_\gamma(S)/s$. First, it follows from (16) that $p^{*n} = (V+c)/2$. Then, take a coalition $S \in 2^N$. We deduce from points 1 and 2 of Remark 4.3 that

$$\begin{aligned}\frac{v_\gamma(N)}{n} - \frac{v_\gamma(S)}{s} &= (p^{*n}-c)(V-p^{*n}) - (p^{*s}-c)\left(V - p^{*s} + r\frac{(n-s)}{n}(\tilde{p}^s - p^{*s})\right)\\ &\geq (p^{*n}-c)(V-p^{*n}) - (p^{*s}-c)(V-p^{*s})\\ &= (p^{*n}-p^{*s})(V+c-p^{*n}-p^{*s})\\ &\geq (p^{*n}-p^{*s})(V+c-2p^{*n})\\ &= (p^{*n}-p^{*s})(V+c-V-c)\\ &= 0,\end{aligned}$$

which completes the proof. ■

Then, we provide a sufficient condition under which Bertrand oligopoly TU-games in γ-characteristic function form are convex. For any $v_\gamma \in G_o^N$, this condition is defined as:

$$\forall s \leq n-2,\ \frac{(s+2)(n-s-2)\tilde{p}^{s+2} + s(n-s)\tilde{p}^s + 2p^{*s+1}}{2(s+1)(n-s-1)} \geq \tilde{p}^{s+1} \qquad (18)$$

By noting that $(s+2)(n-s-2)+s(n-s)+2 = 2(s+1)(n-s-1)$, condition (18) means that the convex combination of $\tilde{p}^{s+2}$, $\tilde{p}^s$ and p^{*s+1} must be greater than or equal to $\tilde{p}^{s+1}$. It follows from point 1 of Remark 4.3 that $p^{*s+1} \geq \tilde{p}^{s+1}$. It follows from point 2 of Remark 4.3 that $\tilde{p}^{s+2} \geq \tilde{p}^{s+1} \geq \tilde{p}^s$. Hence, condition (18) holds if the difference between $\tilde{p}^{s+1}$ and $\tilde{p}^s$ is sufficiently small. For instance, condition (18) is satisfied in Example 4.5.

Theorem 4.7 *Let $v_\gamma \in G_o^N$ be a Bertrand oligopoly TU-game such that condition* (18) *is satisfied. Then v_γ is convex.*

Proof: We want to prove (9), i.e. $f_\gamma(s+2) + f_\gamma(s) \geq 2f_\gamma(s+1)$. Take a coalition $S \in 2^N$ of size s such that $s \leq n-2$. Since p^{*s+2} is the unique maximizer for any coalition of size $s+2$, it holds that

$$\begin{aligned}f_\gamma(s+2) &= (s+2)(p^{*s+2}-c)\left(V - p^{*s+2} + r\frac{(n-s-2)}{n}(\tilde{p}^{s+2} - p^{*s+2})\right)\\ &\geq (s+2)(p^{*s+1}-c)\left(V - p^{*s+1} + r\frac{(n-s-2)}{n}(\tilde{p}^{s+2} - p^{*s+1})\right).\end{aligned}$$

Similarly, since p^{*s} is the unique maximizer for any coalition of size s, it holds that

$$\begin{aligned}f_\gamma(s) &= s(p^{*s}-c)\left(V - p^{*s} + r\frac{(n-s)}{n}(\tilde{p}^{s} - p^{*s})\right)\\ &\geq s(p^{*s+1}-c)\left(V - p^{*s+1} + r\frac{(n-s)}{n}(\tilde{p}^{s} - p^{*s+1})\right).\end{aligned}$$

For notational convenience, for all $s \leq n-2$, we denote by $A(s) = (s+2)(n-s-2)\tilde{p}^{s+2} + s(n-s)\tilde{p}^{s} + 2p^{*s+1}$, and so condition (18) becomes

$$\forall s \leq n-2,\ A(s) \geq 2(s+1)(n-s-1)\tilde{p}^{s+1} \tag{19}$$

By the above two inequalities and (19) it holds that

$$\begin{aligned} f_\gamma(s+2) + f_\gamma(s) &\geq (p^{*s+1} - c)\Big(2(s+1)(V - p^{*s+1}) + \frac{r}{n}\big(A(s) - 2(s+1)(n-s-1)p^{*s+1}\big)\Big) \\ &\geq (p^{*s+1} - c)\Big(2(s+1)(V - p^{*s+1}) + \frac{r}{n}\big(2(s+1)(n-s-1)(\tilde{p}^{s+1} - p^{*s+1})\big)\Big) \\ &= 2(s+1)(p^{*s+1} - c)\Big(V - p^{*s+1} + r\frac{(n-s-1)}{n}(\tilde{p}^{s+1} - p^{*s+1})\Big) \\ &= 2f_\gamma(s+1), \end{aligned}$$

which completes the proof. ■

Note that condition (18) is not necessary for the convexity property as illustrated in the following example.

Example 4.8 Consider the Bertrand oligopoly situation (N, V, r, c) where $N = \{1, 2, 3, 4\}$, $V = 5$, $r = 6$ and $c = 0$. For every coalition $S \in 2^N$, the worth $v_\gamma(S)$ is given in the following table:

s	1	2	3	4
$f_\gamma(s)$	3.25	6.96	12.58	25

Although $v_\gamma \in G_o^N$ is convex, condition (18) does not hold for $s = 2$ since $A(2) - 2(2+1)(n-2-1)\tilde{p}^{2+1} = -0.03$ where $A(s)$ is defined as in the proof of Theorem 4.7. □

5 Concluding remarks

In this article, we have focused on Bertrand oligopoly situations where the demand system is Shubik's (1980) and firms operate at a constant and identical marginal and average cost. In order to define Bertrand oligopoly TU-games we have considered successively the α, β and γ-characteristic functions and we have showed that these functions are well-defined. The first two are suggested by Aumann (1959) and the last one is proposed by Chander and Tulkens (1997). First, we have showed that the α and β-characteristic functions lead to the same class of Bertrand oligopoly TU-games. Moreover, we have proved that the convexity property holds for this class of games. Then, for the class of Bertrand oligopoly TU-games in γ-characteristic function form, we have showed that the Equal Division Solution belongs to the core and we have provided a sufficient condition under which such games are convex, which substantially extends the superadditivity result of Deneckere and Davidson (1985) and contrasts sharply with the negative core existence results of Kaneko (1978) and Huang and Sjöström (2003). Thus, although Cournot and Bertrand oligopoly games are basically different in their non-cooperative forms, it appears that their cooperative forms have the same core geometrical structure. Hence, it follows from the convexity property that there exists a strong incentive for large scale cooperation in such games.

We have directly assumed that products are differentiated. Two other cases can be considered: when products are unrelated ($r = 0$) and when products are perfect substitutes ($r = +\infty$).

In the first case, the quantity demanded of firm i's brand only depends on its own price. Hence, the profit of a coalition does not depend on outsiders' behavior, and so the α, β and γ-characteristic functions are equal. Moreover, for every coalition $S \in 2^N$, the coalition profit function π_S is separable, i.e.

$$\forall x_S \in X_S,\ \pi_S(x_S) = \sum_{i \in S} \pi_i(x_i).$$

Thus, for every coalition $S \in 2^N$ the unique Nash equilibrium $p^* \in X_N$ of the normal form Bertrand oligopoly game $(N, (X_i, \pi_i)_{i \in N})$ is also the unique partial agreement equilibrium under S. Hence, Bertrand oligopoly TU-games are additive,[11] and so $(v(\{i\}))_{i \in N} \in \mathbb{R}^n$ is the unique core element.

In the second case, firms sell a homogeneous product. It follows that firm i's quantity demanded is positive if and only if it charges the smallest price. Since firms operate at a constant and identical marginal and average cost, for every coalition $S \in 2^N \backslash \{N\}$, outsiders charge prices equal to their marginal cost,[12] and so coalition S obtains a zero profit. By charging the monopoly price, the grand coalition obtains a non-negative profit, and we conclude that the core is equal to the set of imputations.

Other alternative blocking rules can be considered. For instance, firms in $N \backslash S$ can choose coalitional rather than individual best reply strategies. In this case, the worth of coalition S is given by the unique Nash equilibrium of the normal form Bertrand duopoly game $(\{S, N \backslash S\}, (X_T, \pi_T)_{T \in \{S, N\backslash S\}})$. However, the following example shows that the non-emptiness of the core crucially depends on the substitutability parameter.

Example 5.1 Consider the two Bertrand oligopoly situations (N, V, r_1, c) and (N, V, r_2, c) where $N = \{1, 2, 3, 4\}$, $V = 1$, $r_1 = 1$, $r_2 = 3$ and $c = 0$. The two Bertrand oligopoly TU-games associated with the Bertrand oligopoly situations (N, V, r_1, c) and (N, V, r_2, c) are symmetric, and so the worths of every coalition $S \in 2^N$ are summarized by the functions $f^{r_1} : \mathbb{N} \longrightarrow \mathbb{R}$ and $f^{r_2} : \mathbb{N} \longrightarrow \mathbb{R}$ respectively. The worths of every coalition $S \in 2^N$ are given in the following table:

s	1	2	3	4
$f^{r_1}(s)$	0.252	0.480	0.719	1
$f^{r_2}(s)$	0.242	0.408	0.622	1

[11] A TU-game $v \in G^N$ is additive if for every coalition $S \in 2^N$, $v(S) = \sum_{i \in S} v(\{i\})$.

[12] This outsiders' behavior is consistent with the α, β and γ-characteristic functions.

For the Bertrand oligopoly TU-game associated with the Bertrand oligopoly situation (N, V, r_1, c), we have $4f^{r_1}(1) = 1.008 > 1 = f^{r_1}(4)$, and so the core is empty. For the one associated with the Bertrand oligopoly situation (N, V, r_2, c), the payoff vector $(0.25, 0.25, 0.25, 0.25) \in \mathbb{R}^4$ is a core element, and so the core is non-empty. □

According to Example 5.1, it is easier to obtain a non-empty core when the substitutability parameter increases. A similar argument is used by Huang and Sjöström (2003) in order to guarantee the non-emptiness of the recursive core. They prove that the recursive core is non-empty if and only if the substitutability parameter is greater than or equal to some number that depends on the size of the industry. When firms in $N \backslash S$ choose coalitional best reply strategies, we expect that a similar condition would ensure the non-emptiness of the core. This is left for future work.

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