Preface

Stochastic differential equations (SDE) of different kinds are recognized to play an important role in a wide range of problems encountered in mathematics, physics, engineering, mathematical finance, etc. The most important role has been played by Itô diffusions, i.e., by solutions of SDEs driven by Gaussian random measures, derived from a standard Brownian motion process. Recent developments show that in many practical applications leading to acceptable stochastic models more general classes of infinitely divisible (ID) random measures are more appropriate. Of particular interest is a class of ID random measures defined by increments of α–stable Lévy motion processes. From the point of view of classical stochastic analysis these problems can be regarded as special cases of SDEs involving stochastic integrals driven by semimartingales, so we can relay on the theoretical results (existence, uniqueness, regularity and stability of solutions) concerning of SDEs with semimartingale measures.

However, our main field of interest here is *Numerical Probability*, a new domain in the probability theory. We focus our attention on numerical and statistical methods providing approximate solutions of Itô type SDEs driven by some classes of ID random measures. This leads to various problems concerning convergence of such methods, and some of them are considered here.

Most of several problems presented here are treated in the style of *Computational Mathematics* or *Computational Physics*, two other comparatively new fields of science which have been developing rapidly for some time. This can be explained by the fact that attempts to get solutions in analytical closed form lead quite often to severe difficulties, while very useful and powerful approximate numerical and computer simulation techniques are available. Computer experiments providing information on solutions of complicated stochastic models should stimulate theoretical investigations of their mathematical features. In this work we present several examples of this kind. Some of them do not belong to the range of problems described by the finite dimensional Itô type SDEs.

This monograph exploits some methods and facts from such fields as theory of SDEs of different kinds; stochastic modeling in computational physics, engineering, mathematical finance, etc.; discrete and other approximate methods in stochastic analysis; statistical estimation methods, computer simulations techniques based on generation of pseudorandom variates and Monte–Carlo type approximations and averaging; and, last but not least, nowadays very powerful computer graphics tools.

After preliminary remarks included in Chapter 1, in Chapter 2 we review various properties of the most important infinitely divisible laws and recall some methods of construction of basic stochastic processes such as Brownian motion, α–stable Lévy motion or Poisson process, as well.

It turns out that with the use of suitable statistical estimation techniques, computer simulation procedures, and numerical discretization methods described in Chapters 3 – 6 it is possible to construct approximations of stochastic integrals and SDEs with infinitely divisible random measures as integrators. Their updated mathematical theory is systematically presented in Chapters 4 and 6. The most valuable statistical estimation methods are briefly recalled in Chapter 5. As a consequence we obtain an effective general method allowing us to construct approximate solutions to a wide class of SDEs involving such integrals. Some new results on convergence of discrete methods approximating stochastic integrals and SDEs with α–stable integrators are contained in Chapter 6. Applications of computer graphics provide useful quantitative and visual information on various features of SDEs with jumps and show what distinguishes them from their commonly used Gaussian counterparts. It is possible to demonstrate time evolution of densities with heavy tails of various stochastic processes, visualize the effect of jumps of trajectories, etc. We try to demonstrate that especially α–stable variates can be very useful in stochastic modeling of various problems arising in science and engineering.

Chapters 6 and 7 contain various examples of application of our approach to different stochastic models arising in mathematical physics, mathematical finance, etc.

Results of computer investigations on fractal and geometric structures (cellular and clustering effects) characteristic for solutions of 1 and 2 dimensional Burgers equations (nonlinear partial differential equations) with random initial data or random external forces are presented in Chapter 8.

Some theoretical results – based on author's papers and more or less of original character – are presented in Chapters 3, 4 and 6. Chapters 7 and 8 are based on author's recent papers consisting in application of computational mathematics methods to experimental investigation of several stochastic models.

In Appendices we enclose the computer program *SDE–Lm.cpp* which was employed to produce several graphical examples contained in this work. Running the program one can solve approximately stochastic differential equations driven by the α–stable Lévy motion process. Another program (*Burgers.cpp*) simulates the time evolution of passive tracer densities in Burgers flows of different kind.

This work should be regarded as a continuation of or supplement to the following fundamental monograph:

JANICKI, A. and WERON, A. (1994). *Simulation and Chaotic Behavior of α–stable Stochastic Processes*, Marcel Dekker, New

York,

which can serve as a source of basic information on topics which are only briefly recalled here. Most of the results (including two original topics of this monograph: computer simulation of stable processes and their chaotic behavior) are presented with details there.

Here we present mainly the new facts and results of experiments performed after preparation of the book.

The mathematical content, all figures and computer programs providing them, and the LaTeX source code of the whole text, i.e. everything in the book, were produced by the author himself. There is no one to blame for possible occasional mistakes but the author.

I am happy to acknowledge my indebtedness to Professor Aleksander Weron, who introduced me to and has governed my paths through these new and fascinating fields of numerical and computational probability for several years.

I want to thank Professor W. A. Woyczynski for inviting me to CWRU in Cleveland, Ohio for the academic year 1994/95, and for guiding my steps in computer investigation of geometric and statistical properties of stochastic Burgers flows.

Finally, I want to express my gratitude to my colleagues P. Kokoszka, Z. Michna, and K. Podgórski, younger co–authors of a few papers.

Aleksander Janicki

Chapter 1

Preliminaries

1.1 Introduction

Modern theory of stochastic differential equations relays on the stochastic integration theory, which originated in the early work of Wiener. Stochastic integrals with respect to *Brownian motion* were defined by Itô (1944). Doob (1953) proposed a general integral with respect to L^2–martingales. On the basis of the Doob–Meyer decomposition theorem, Kunita and Watanabe (1967) further developed the theory of this integral. Doleans–Dade and Meyer (1970) extended the definition of the stochastic integral to all local martingales and subsequently to semimartingales. The natural role of semimartingales was made evident thanks to the contribution of Dellacherie (1980) and Bichteler (1981), who established that semimartingales are the most general class of integrators for which one can have a reasonable definition of stochastic integral against predictable integrands. Basic facts concerning some aspects of stochastic analysis and the theory of stochastic integrals of different types can be found, for example, in Elliott (1982), Jacod and Shiryaev (1987), Karatzas and Shreve (1988), Revuz and Yor (1991), or Kwapień and Woyczynski (1992).

Our main tool of description of stochastic models are stochastic integrals with respect to a standard *α–stable Lévy motion*. The α–stable Lévy motion together with the *Poisson process* and Brownian motion are the most important examples of *Lévy processes*, which together provide the most important examples of *Markov processes* as well as of *semimartingales*.

Thus, on the one hand, the class of SDEs in which we are interested is much broader than the class of SDEs with random measures defined by Gaussian processes and, on the other, it is contained in the class of SDEs driven by *infinitely divisible measures*, which itself is contained in the class of semimartingales.

1.2 Stochastic Differential Equations

Let a usual complete probability space $(\Omega, \mathcal{F}, P)$, together with a filtration $\{\mathcal{F}_t\}$ be given. A vast class of $\mathbb{R}^n$–valued stochastic diffusions $\{X(t) : t \geq 0\}$ in

$(\Omega, \mathcal{F}, P, \{\mathcal{F}_t\})$ driven by α–stable stochastic measures induced by a class of α–stable Lévy motion processes $\{L_{\alpha,\beta}(t) : t \geq 0\}$, with given drift and dispersion coefficients, can be described by the following stochastic differential equation

$$d\, X(t) = a(t, X(t-))\, dt + b(t, X(t-))\, dL_{\alpha,\beta}(t); \quad t > 0, \quad X(0) = X_0,$$

that can be rewritten in the following integral form

$$X(t) = X_0 + \int_0^t a(s, X(s-))\, ds + \int_0^t b(s, X(s-))\, dL_{\alpha,\beta}(s). \tag{1.2.1}$$

Several examples of application of such SDEs to stochastic modeling can be found in Janicki and Weron (1994a), (1994b).

This class includes commonly well understood an $\mathbb{R}^n$–valued Itô diffusion process $\{X(t) : t \geq 0\}$ driven by the Wiener process (Brownian motion), with given drift and dispersion coefficients, i.e., a process which can be obtained as a solution of the following stochastic differential equation

$$X(t) = X_0 + \int_0^t a(s, X(s))\, ds + \int_0^t b(s, X(s))\, dB(s), \tag{1.2.2}$$

where $\{B(t)\}$ stands for the standard Brownian motion process. There is a huge amount of possible applications of such SDEs in stochastic modeling (see, e.g., Gardiner (1983), Gardner (1986) or Sobczyk (1991)).

The theory of stochastic differential equations driven by Gaussian random measures (see Arnold (1974) or Karatzas and Shreve (1988)) has been developed for a long time.

It is clear that SDEs involving stochastic integrals with a standard α–stable Lévy motion as an integrator include the above equation as a special case.

Still more general are the so–called stochastic differential equations with jumps. In the simplest case of real–valued process the problem is to find the solution $\{X(t) : t \in [0, \infty)\}$ of the equation

$$X(t) = X_0 + \int_0^t \int_{\mathbb{R}\setminus\{0\}} f(X(s-), x)\; N(ds, dx), \tag{1.2.3}$$

where $N(ds, dx)$ is a Poisson measure of a suitable point process with a given intensity measure $ds\; d\nu(x)$ (see, e.g., Ikeda and Watanabe (1981)). All types of stochastic equations described above are special cases of general SDEs driven by semimartingales, i.e., equations of the form

$$X(t) = X_0 + \int_0^t f(X(s-))\; dY(s), \tag{1.2.4}$$

where $\{Y(t)\}$ stands for a given semimartingale process. There is a vast literature concerning this topic (for theorems on existence, uniqueness, regularity of solutions, etc., one can consult, e.g., Elliot (1982) or Protter (1990) and the bibliography therein).

1.3 Approximation and Computer Simulation of SDEs

Numerical Probability and *Computational Mathematics* are two main fields of interest in this monograph. Numerical probability contains as a substantial part methods of discretization of SDEs driven by Brownian motion, i.e., stochastic equations in the form (1.2.2), and the theory of convergence of such methods. A path of development of such methods, which can be understood as stochastic versions of well developed numerical methods of approximation of finite systems of ordinary differential equations, can be followed through the following (far from completeness) sequence of papers: Maruyama (1955), Milstein (1974), (1978), Yamada (1976), Kloeden and Pearson (1977), Clark and Cameron (1980), Platen (1980), Rootzen (1980), Talay (1983), Pardoux and Talay (1985). A fundamental monograph by Kloeden and Platen (1992) summarizes all recent developments on approximate methods solving equation (1.2.2). However at the end of a brief survey of stochastic numerical methods it contains the following

> *We conclude this brief survey with the remark that the theoretical understanding and practical application of numerical methods for stochastic differential equations are still in their infancy.*
>
> Kloeden and Platen (1992), p. XXXV

Book by Janicki and Weron (1994a) is the first available work which deals with a significantly more general problem of computer constructions of approximate solutions of SDEs driven by a standard α–stable Lévy motion processes, i.e., written in the form (1.2.1). Some basic ideas on convergence of numerical methods can be derived from the literature on the *stability theory* of stochastic integrals and SDEs with jumps (see, e.g., Jakubowski, Mémin and Pages (1989), Słomiński (1989), Kasahara, Yamada (1991), Kurtz and Protter (1991)). Here we present a new version of a theorem on weak convergence in the Skorokhod topology in the space $\mathbb{D}[0,T,\mathbb{R})$ of numerical approximations of SDE (1.2.1) based on summation of independent increments of appropriate processes inducing random measures (see Janicki, Michna and Weron (1996)), but there are still many open questions concerning this problem. Also of interest is a theorem on the rate of convergence of approximations of stochastic integrals with α–stable random measures represented by infinite random series (see Janicki (1996a)).

One of basic new ideas in Janicki and Weron (1994a) was to represent the discrete time processes approximating stochastic processes with continuous time by appropriately constructed finite sets of random samples derived from random samples approximating underling random measures. Such approach provides us with estimators of densities and quantiles of these processes on finite sets of values of discretized time and with useful quantitative information on time evolution of investigated stochastic processes. This leads to another set of open theoretical problems.

Another achievement of Janicki and Weron (1994a) was a significant progress (in comparison, for example, with rather elementary, not to say naive, methods utilized in the parallelly prepared books by Bouleau and Lépingle (1994), Kloeden, Platen and Schurz (1994), or Samorodnitsky and Taqqu (1994)) in development of methods of visualization of stochastic processes of different kinds based on application of modern computer techniques. There is a rapidly growing number of works (particularly in the fields of *Computational Physics* and *Mathematical Economy*) where the proper use of such approach (computer graphics combined with numerical and statistical methods) provides useful and sometimes surprisingly interesting information helping for better understanding of phenomena that are of complicated, chaotic or stochastic nature (see Janicki (1995b), Janicki and Weron (1994a), (1995a), (1995b), Janicki, Popova, Ritchken and Woyczynski (1996)). For other developments in physics consult, e.g., a collection of papers edited by Garbaczewski, Wolf and Weron (1995).

In this monograph we present several new examples of this kind. In our approach we feel strongly inspired by the work of S. M. Ulam, former Stefan Banach's favorite student, who was one of the first enthusiasts of application of computer methods not only to scientific calculations or to the construction of mathematical models of physical phenomena but even to the investigation of new universal laws of nature; consult e.g., Stein and Ulam (1963) or Ulam (1980).

This last remark defines, more or less, what we mean by Computational Mathematics.

1.4 Burgers Flows

Chapter 8 of this work, based on papers by Janicki (1996b), (1996c), Janicki and Woyczynski (1997), Janicki, Surgailis and Woyczynski (1995), is devoted to the computer study of geometric structures of solution of Burgers equations with random initial conditions or random potential of external forces. These equations are applicable in constructions of adhesion models of an expanding Universe, when it is reasonable to assume that the time evolution of multidimensional velocity fields $\mathbf{v} = \mathbf{v}(t, \mathbf{x})$ can be described by the relation

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}, \nabla)\mathbf{v} = \mu \Delta \mathbf{v} - 2\mu \nabla \Phi, \tag{1.4.1}$$

and when the density of matter $\rho = \rho(t, \mathbf{x})$ satisfies the continuity equation

$$\frac{\partial \rho}{\partial t} + \mathbf{div}(\rho \mathbf{v}) = 0, \tag{1.4.2}$$

for $(t, \mathbf{x}) \in (0, \infty) \times \mathbb{R}^d$. (Assymptotic behavior when $||\mathbf{x}|| \to \infty$ and proper initial conditions are discussed in Chapter 8.)