On the Number of α-Pivotal Players

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We show that bounds like those of Al-Najjar and Smorodinsky (J. Econ. Theory, 2000) as well as of Gradwohl et al. (Math. Oper. Res., 2009) on the number of α-pivotal agents can be obtained by decomposition of variance. All these bounds have a similar asymptotic behaviour, up to constant factors. Our bound is weaker than that of Al-Najjar and Smorodinsky, but we require only pairwise independent—rather than independent—types. Our result strengthens the bound of Gradwohl et al.

Keywords: α-pivotal agent, influence, direct mechanism, decomposition of variance.

JEL Classification: D62, D89.

1 Introduction

In a mechanism design problem, an agent is called α-pivotal with respect to some collective outcome if a variation in the agent's type can lead to a change in the expected outcome of at least α. Often, α-pivotality leads to necessary conditions for a mechanism to be incentive-compatible or individually rational. A participation fee, for instance, may make α-pivotality a precondition for voluntary participation, as an agent will want to pay the fee only if he can influence the outcome sufficiently.

Al-Najjar and Smorodinsky (2000a) provide an upper bound on the number of α-pivotal agents if the outcome is bounded, the agents' types are independent, and the type space

is finite. The upper bound depends on the distribution of types as well as on α, but is independent from the number of agents. This result has several interesting applications: It allows the derivation of upper bounds for the probability that a public project is realized (Al-Najjar and Smorodinsky, 2000a; Neeman, 2004), or for the size of a public project (Al-Najjar and Smorodinsky, 2000b; Birulin, 2006). Al-Najjar (2001) uses it in the analysis of authority relationships. Al-Najjar and Smorodinsky (2007) prove the efficiency of competitive mechanisms if the number of traders is sufficiently large. Al-Najjar and Smorodinsky (2001) derive an upper bound for the number of players not playing the short-term best response in a repeated game, and Gerardi and Yariv (2008) analyse how to design an optimal mechanism for information acquisition through a committee. Influence is an important issue in agenda-setting and voting models, as, for instance, analysed by Gersbach (2009), when participation in the political process is costly.

Al-Najjar and Smorodinsky (2000a) demonstrate their result by explicitly constructing a mechanism in which the number of α-pivotal agents is maximal. This mechanism is a majority voting. The topic was taken up recently by Gradwohl et al. (2009), who generalize the results of Al-Najjar and Smorodinsky in various directions—in particular, they introduce the notion of (p, α)-pivotality, relax the assumption of independent types to pairwise independence, and consider the influence of coalitions. For proving their result, the authors consider binary type spaces first and then reduce the general case to this special case.

The purpose of the present note is to show that similar upper bounds for the number of α-pivotal or (p, α)-pivotal players can be reached in a direct way by a very simple argument based on decomposition of variance. Our proof highlights the role of Al-Najjar and Smorodinsky's assumption that the type space is finite. The method we use is closely related to an argument in Appendix 1 of Mailath and Postlewaite (1990, p. 364), where the idea appears as "Bessel's inequality". Our bound on the number of α-pivotal players will turn out to be less sharp than the one by Al-Najjar and Smorodinsky, but much easier to compute. It displays a similar asymptotic behaviour, up to a constant factor. Our result for (p, α)-pivotality is somewhat sharper than that of Gradwohl et al.

The present paper is structured as follows: In Section 2 we develop the main argument. In Section 3 we derive our upper bound for the number of α-pivotal agents. In Section 4 we compare this bound to the result of Al-Najjar and Smorodinsky (2000a). In Section 5 we refer to Gradwohl et al. (2009) and consider (p, α)-pivotality.

Throughout the paper, we assume that all random variables are defined on some probability space, which we do not mention explicitly. The probability measure is denoted by P.

2 Decomposition of Variance

The following proposition is our central argument.

Proposition 1. *Let X be a real-valued random variable with finite variance, and suppose the random variables $T_1, \ldots, T_N$ are pairwise independent. Then,*[1]

$$\operatorname{Var} X \geq \sum_{i=1}^{N} \operatorname{Var} \operatorname{E}(X \mid T_i).$$

Proof. For $i = 1, \ldots, N$, let $Y_i := \operatorname{E}(X \mid T_i)$, and let $Z := X - \sum_{i=1}^{N} Y_i$. Since $X = Z + \sum_{i=1}^{N} Y_i$, we have

$$\operatorname{Var} X = \sum_{i=1}^{N} \operatorname{Var} Y_i + \operatorname{Var} Z + 2 \sum_{1 \leq i < j \leq N} \operatorname{Cov}(Y_i, Y_j) + 2 \sum_{i=1}^{N} \operatorname{Cov}(Y_i, Z). \qquad (1)$$

For $i \neq j$, the random variables Y_i and Y_j are independent because T_i and T_j are independent; hence $\operatorname{Cov}(Y_i, Y_j) = 0$. By the Law of Iterated Expectations, we have $\operatorname{E} Y_i = \operatorname{E} X$ for all $i = 1, \ldots, N$. Together with the T_i-measurability of Y_i, this yields

$$\operatorname{Cov}(Y_i, Z) = \operatorname{Cov}(Y_i, X) - \sum_{j=1}^{N} \operatorname{Cov}(Y_i, Y_j) = \operatorname{Cov}(Y_i, X) - \operatorname{Cov}(Y_i, Y_i)$$

$$= \operatorname{E}\big[Y_i \cdot X\big] - \operatorname{E} Y_i \cdot \operatorname{E} X - \operatorname{Var} Y_i = \operatorname{E}\big[\operatorname{E}(Y_i \cdot X \mid T_i)\big] - (\operatorname{E} Y_i)^2 - \operatorname{Var} Y_i$$

$$= \operatorname{E}\big[Y_i \cdot \operatorname{E}(X \mid T_i)\big] - (\operatorname{E} Y_i)^2 - \operatorname{Var} Y_i = \operatorname{Var} Y_i - \operatorname{Var} Y_i = 0.$$

As $\operatorname{Var} Z \geq 0$, the assertion now follows from Equation (1). $\qquad \square$

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[1] Note that for each i, the conditional expectation $\operatorname{E}(X \mid T_i)$ is a random variable; it can be seen as a function of T_i.

3 The Bound for α-Pivotality

We adopt the setup of Al-Najjar and Smorodinsky (2000a). We consider a set $\{1, \ldots, N\}$ of agents. The type of agent i is given by the random variable T_i. We assume the random variables T_i to be pairwise independent. Further, we assume:

Finiteness Assumption. *For each i, the support of T_i, denoted by $\mathbf{T}_i$, is a finite set.*

Under this assumption, we define

$$\varepsilon := \min_{i=1,\ldots,N} \ \min_{t \in \mathbf{T}_i} \ P(T_i = t),$$

and note that $\varepsilon > 0$.

The random variable X represents some collective outcome.[2] We assume that X has finite variance. The following definition is due to Al-Najjar and Smorodinsky (2000a, p. 323):

Definition 1. *Suppose the Finiteness Assumption holds. Let $\alpha > 0$. We say that $i \in \{1, \ldots, N\}$ is α-pivotal for X if*

$$\max_{t \in \mathbf{T}_i} \mathrm{E}(X \mid T_i = t) - \min_{t \in \mathbf{T}_i} \mathrm{E}(X \mid T_i = t) \geq \alpha. \tag{2}$$

The term on the left-hand side of the inequality is called the influence *of agent i.*

To illustrate this definition, we give an interpretation for the case of a direct mechanism if the agents are risk-neutral, types are independent, and each agent's type is private information to this agent. Let X be the outcome of the mechanism if the agents truly report their types. The outcome X need not be a function of $T_1, \ldots, T_N$; our arguments hold as long as agent i's expected utility from reporting type $t_i \in \mathbf{T}_i$ is given by $\mathrm{E}(X \mid T_i = t_i)$. Then, the quantity on the left-hand side of Inequality (2) is an upper bound for what agent i can gain from misreporting his type.[3] This interpretation underlies most of the examples cited in the Introduction.

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[2] Al-Najjar and Smorodinsky (2000a, p. 321) give the following examples: "the level of pollution, output of team production, a principal's reward, etc., or the probability of a binary outcome, e.g., the probability that a public project is undertaken."

[3] If types are not independent, one has to be very careful with this interpretation of α-pivotality. Consider, for instance, $N > 1$ agents who are either all of type 0 or all of type 1. The outcome X shall be zero if the agents are of type 0 and one if the agents are of type 1. In this setting, the influence of each agent is one, but, with the mechanism appropriately designed, what an agent could gain from misreporting his type could be strictly larger than one, since a single agent's misreporting

As a consequence of the Finiteness Assumption, α-pivotality transforms into a lower bound for the variance of the conditional expectation $\mathrm{E}(X \mid T_i)$.

Proposition 2. *If i is α-pivotal, then* $\operatorname{Var} \mathrm{E}(X \mid T_i) \geq \frac{1}{2}\varepsilon\alpha^2$.

Proof. For each i, the conditional expectation $\mathrm{E}(X \mid T_i)$ is a random variable; it takes each of the values $a_i := \max_{t \in \mathbf{T}_i} \mathrm{E}(X \mid T_i = t)$ and $b_i := \min_{t \in \mathbf{T}_i} \mathrm{E}(X \mid T_i = t)$ with a probability of at least ε. Taking into account that $b_i \leq \mathrm{E}\, X \leq a_i$ and that i is α-pivotal, we reach

$$\operatorname{Var} \mathrm{E}(X \mid T_i) = \mathrm{E}\left[\left(\mathrm{E}(X \mid T_i) - \mathrm{E}\, X\right)^2\right] \geq \varepsilon\left(a_i - \mathrm{E}\, X\right)^2 + \varepsilon\left(b_i - \mathrm{E}\, X\right)^2$$

$$\geq \varepsilon\left(a_i - \tfrac{1}{2}(a_i + b_i)\right)^2 + \varepsilon\left(b_i - \tfrac{1}{2}(a_i + b_i)\right)^2 = \tfrac{1}{2}\varepsilon(a_i - b_i)^2 \geq \tfrac{1}{2}\varepsilon\alpha^2.$$

□

From Propositions 1 and 2 we now obtain the desired upper bound for the number of α-pivotal agents:

Theorem 1.
$$\#\left\{i \in \{1, \ldots, N\} \;\middle|\; i \text{ is } \alpha\text{-pivotal}\right\} \leq \frac{2\operatorname{Var} X}{\varepsilon\alpha^2}\,.$$

□

4 Comparison

We compare our bound to the one of Al-Najjar and Smorodinsky (2000a). They define K_α^* to be the largest[4] integer K satisfying $R(\varepsilon, K) \geq \alpha$, with $R(\varepsilon, K)$ being a player's influence in a majority decision of K agents, where every agent votes "Yes" with a probability of ε, "No" with a probability of ε, and abstains from voting with a probability

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leads to a reported strategy profile that is not within the support of the type distribution. Of course, in this example, the agents' types are not even pairwise independent, so our analysis does not apply, anyway. The same problem of interpretation, however, would appear in an example of Gradwohl et al. (2009, Sec. 4.1., p. 979), to which our analysis *does* apply. In their example, in which there are $N = 2^k - 1$ agents ($k \geq 2$), types are identically Bernoulli-distributed and pairwise independent, but the support concentrates on the zero vector as well as on N other strategy profiles, in each of which exactly $(N+1)/2$ agents are of type 1. Again, a single agent's misreporting would lead to a strategy profile outside the support, which means that the influence introduced in Definition 1 need not be an upper bound for what an agent can gain from misreporting his type.

[4]Al-Najjar and Smorodinsky (2000a) define it to be the *smallest* integer, which is obviously a mistake.

of $1 - 2\varepsilon$. The authors prove that if the range of X is a subset of $[0; 1]$, the number of α-pivotal players is bounded by K_α^*, and that in a symmetric environment this bound is sharp. Using Stirling's formula, they derive the asymptotics[5]

$$R(\varepsilon, K) \simeq \frac{1}{\sqrt{\pi \varepsilon K}} \quad \text{for } K \to \infty. \tag{3}$$

This transforms into the following asymptotics for K_α^*:

Proposition 3.

$$K_\alpha^* \simeq \frac{1}{\pi \varepsilon \alpha^2} \quad \text{for } \alpha \to 0. \tag{4}$$

Proof. Let $\tilde{K}_\alpha := 1/(\pi \varepsilon \alpha^2)$. By the results of Al-Najjar and Smorodinsky (2000a), we have $K_\alpha^* \to \infty$ for $\alpha \to 0$; hence, by (3),

$$\liminf_{\alpha \to 0} \frac{\tilde{K}_\alpha}{K_\alpha^*} = \liminf_{\alpha \to 0} \frac{1}{\alpha^2 \pi \varepsilon K_\alpha^*} \geq \liminf_{\alpha \to 0} \left(\frac{1}{R(\varepsilon, K_\alpha^*)\sqrt{\pi \varepsilon K_\alpha^*}} \right)^2 = 1$$

and, as $R(\varepsilon, K_\alpha^* + 1) < \alpha$ by the definition of K_α^*,

$$\limsup_{\alpha \to 0} \frac{\tilde{K}_\alpha}{K_\alpha^* + 1} = \limsup_{\alpha \to 0} \frac{1}{\alpha^2 \pi \varepsilon (K_\alpha^* + 1)} \leq \limsup_{\alpha \to 0} \left(\frac{1}{R(\varepsilon, K_\alpha^* + 1)\sqrt{\pi \varepsilon (K_\alpha^* + 1)}} \right)^2 = 1.$$

This yields $\tilde{K}_\alpha / K_\alpha^* \to 1$ for $\alpha \to 0$. $\qquad \square$

In order to compare this to the bound from Theorem 1, we first note that by the following—rather trivial—observation, $\text{range}(X) \subseteq [0; 1]$ establishes a constraint on $\text{Var } X$.

Proposition 4. *Let Z be a real-valued random variable such that* $\text{range}(Z) \subseteq [z_1; z_2]$ *for some $z_1, z_2 \in \mathbb{R}$, $z_1 \leq z_2$. Then,* $\text{Var } Z \leq \frac{1}{4}(z_2 - z_1)^2$.

Proof. We only have to consider the case $z_1 < z_2$. Let $\lambda := (z_2 - z_1)/2$, $z^* := (z_1 + z_2)/2$, and $\tilde{Z} := (Z - z^*)/\lambda$. Then,

$$\text{Var } Z = \text{Var}(Z - z^*) = \lambda^2 \text{Var } \tilde{Z} = \lambda^2 \left(\text{E}(\tilde{Z}^2) - (\text{E } \tilde{Z})^2 \right) \leq \lambda^2 \text{E}(\tilde{Z}^2) \leq \lambda^2,$$

where the last inequality follows from $|\tilde{Z}| \leq 1$. $\qquad \square$

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[5] Two functions $f(x)$ and $g(x)$, with $g(x) \neq 0$ for all x, are defined to be *asymptotically equivalent* for $x \to x_0$ (notation: $f(x) \simeq g(x)$ for $x \to x_0$) if $f(x)/g(x) \to 1$ for $x \to x_0$.

By Proposition 4, $\text{range}(X) \subseteq [0; 1]$ implies $\text{Var}\, X \leq \frac{1}{4}$; hence Theorem 1 yields an upper bound of

$$\frac{1}{2\varepsilon\alpha^2} \,,$$

which differs from the right-hand side of (4) only by a constant factor of $\pi/2$. As the fact that this factor is strictly larger than 1 suggests, and as an explicit calculation of K_α^* for some examples shows, the upper bound $1/(2\varepsilon\alpha^2)$ is not sharp.

The difference in the upper bounds *might* be due to the fact that we require only pairwise independence, as compared to the independence assumption of Al-Najjar and Smorodinsky (2000a). It remains, however, an open question whether it is possible to construct a model with pairwise independent, but not independent types in which the number of α-pivotal agents is strictly larger than K_α^*.

5 The Bound for (p, α)-Pivotality

Gradwohl et al. (2009, p. 972) modify the concept of α-pivotality by defining (p, α)-pivotality. We are going to use this notion in the following sense:

Definition 2. [6] *Let $\alpha > 0$ and $p \in [0; 1]$. A player i is called (p, α)-pivotal for X if*

$$P\Big(\big|\text{E}(X \mid T_i) - \text{E}\, X\big| \geq \alpha\Big) \geq p.$$

This definition is meaningful even if the supports of the type variables T_i are not finite; hence we can drop the Finiteness Assumption. Like α-pivotality, (p, α)-pivotality enables us to bound the variances of the conditional expectations $\text{E}(X \mid T_i)$ from below, so we can replicate our arguments from Section 3. The following proposition is the analogue to Proposition 2.

Proposition 5. *If player i is (p, α)-pivotal, then $\text{Var}\,\text{E}(X \mid T_i) \geq p\alpha^2$.*

Proof. This immediately follows from Chebyshëv's inequality, which says that

$$P\Big(\big|\text{E}(X \mid T_i) - \text{E}\, X\big| \geq \alpha\Big) \leq \frac{\text{Var}\,\text{E}(X \mid T_i)}{\alpha^2} \,.$$

$\square$

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[6] Gradwohl et al. (2009) use strict inequalities ("$>$" instead of "$\geq$") in their definition. For the present paper, the given—weaker—definition of (p, α)-pivotality is sufficient.

Propositions 1 and 5 yield an upper bound for the number of (p, α)-pivotal agents:

Theorem 2.
$$\#\big\{i \in \{1, \ldots, N\} \;\big|\; i \text{ is } (p, \alpha)\text{-pivotal}\big\} \leq \frac{\operatorname{Var} X}{p\alpha^2} \,.$$

$\square$

For a comparison of Theorems 1 and 2, suppose that the Finiteness Assumption holds. Since
$$\min_{t \in \mathbf{T}_i} \operatorname{E}(X \mid T_i = t) \leq \operatorname{E} X \leq \max_{t \in \mathbf{T}_i} \operatorname{E}(X \mid T_i = t),$$

and since thus α-pivotality implies $(\varepsilon, \alpha/2)$-pivotality, Theorem 2 yields

$$\#\big\{i \in \{1, \ldots, N\} \;\big|\; i \text{ is } \alpha\text{-pivotal}\big\} \leq \frac{4 \operatorname{Var} X}{\varepsilon\alpha^2}.$$

This is by factor 2 worse than the bound given by Theorem 1.

Theorem 2 corresponds to Theorem 2.1 in Gradwohl et al. (2009), which says that if $\operatorname{range}(X) \subseteq [-1; 1]$, the number of (p, α)-pivotal players is not greater than $8/(p\alpha^2)$. By Proposition 4, $\operatorname{range}(X) \subseteq [-1; 1]$ implies $\operatorname{Var} X \leq 1$; hence Theorem 2 yields a bound of $1/(p\alpha^2)$. Theorem 2 is thus stronger than Theorem 2.1 of Gradwohl et al. This strengthening can be transferred to Theorem 2.2 of Gradwohl et al. by building the proof on the above Theorem 2 instead of Theorem 2.1.

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