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Endogenous preferences in games

with type indeterminate players

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Endogenous Preferences in Games with Type Indeterminate Players

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Abstract

The Type Indeterminacy model is a theoretical framework that uses some elements of quantum formalism to model the constructive preference perspective suggested by Kahneman and Tversky. In this paper we extend the TI-model from simple to strategic decision-making and show that TI-games open a new field of strategic interaction. We first establish an equivalence result between static games of incomplete information and static TI-games. We next develop a new solution concept for non-commuting dynamic TI-games. The updating rule captures the novelty brought about by Type Indeterminacy namely that in addition to affecting information and payoffs, the action of a player impacts on the profile of types. We provide an example showing that TI-game predictions cannot be obtained as Bayes Nash equilibrium of the corresponding classical game.

Keywords: type indeterminacy, games, endogeneous preferences

1. Introduction

This paper belongs to a very recent and rapidly growing literature where formal tools of Quantum Mechanics are proposed to explain a variety of behavioral anomalies in social sciences and in psychology (see e.g., Deutsch (1999), Busemeyer et al. (2006, 2007, 2008), Danilov et al. (2008), Franco (2007), Danilov et al. (2008), Lambert-Mogiliansky et al. (2009)). To many people it may appear unmotivated or artificial to turn to Quantum mechanics when investigating human

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behavioral phenomena. However, the founders of QM, including Bohr (1993) and Heisenberg (2000) early recognized the similarities between the two fields. In particular Bohr was influenced by the psychology and philosophy of knowledge of Harald Höffding. The similarity stems from the fact that in both fields *the object of investigation cannot (always) be separated from the process of investigation.* Quantum Mechanics and in particular its mathematical formalism was developed to respond to that epistemological challenge(see the introduction in Bitbol (2009) for a enlightening presentation).

The use of quantum formalism in game theory was initiated by Eisert et al. (1999) who study how the extension of classical moves to quantum ones can affect the analysis of a game. Another example is La Mura (2005) who investigates correlated equilibria with quantum signals in classical games.[1] Our approach is different from the so-called quantum game approach. It is based on the idea that players' preferences (types) (rather than the strategies they can choose) can feature non-classical (quantum) properties. This idea is formalized in the Type Indeterminacy (TI) model of decision-making introduced by Lambert-Mogiliansky, Zamir and Zwirn (2009).

A main interest with TI-game is that the TI-hypothesis extends the field of strategic interactions. The chosen actions impact not only on the payoffs of other players but also on the profile of types of the players i.e., who the players *are*. In a TI-model, players do not have a deterministic, exogenously given, type (preferences). The

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[1]Whether and when the use of quantum strategies (or strategies using quantum signals) can bring something truly novel to game theory has been discussed in Levine (2005) and in Brandenburger (2010).

types change along the game together with the decisions made in the Game Situations[2] (which are modelled as measurements of the type). The players' type are endogenous to the game.

This paper follows an introduction to TI-games in Busemeyer et al. (2009) where we show how the TI approach could provide an explanation to why cheap-talk promises matter appealing to the quantum indeterminacy of players' type. In the present paper we go a step further by investigating how players can "exploit" the type indeterminacy of their opponent. More precisely, we want to model how people can influence a partner or an opponent with respect to what she actually wants to do, i.e., with respect to her taste or preferences (see Feldman 1988). In TI-games, preferences are intrinsically indeterminate. A pre-play or a node from which to move may therefore either increase or decrease the ex-ante probability for a specific move in the future. Therefore, TI-games provide an argument why a player would try to induce or refrain from inducing his opponent to face a particular node. She may want to "prepare" his opponent so he plays as she desires.

This idea is captured in the following example that we investigate in details in Section 3. Alice wants Bob to agree to cooperate with her in a new project. Bob is indeterminate with respect to his willingness to engage in this project which appeals to open-mindedness. Bob is also indeterminate with respect to his taste for personal challenges.[3] Now, Alice who is Bob's boss handles over legal cases for him to prepare. She has the choice between two tasks: either a standard dispute or a more intricate case. The standard dispute is best handled routinely. The intricate case can be treated as a routine job too. But Bob can also adopt a non-standard inventive approach which is personally challenging. So this tasks forces Bob to determine himself with respect to his taste for personal challenges. Alice would like him to handle the intricate case but she understands that this may affect his attitude toward the project. She knows that indeterminacy with respect to personal challenges has a stimulating effect with respect to his acceptance to cooperate. Therefore because she mostly cares about the project and although she would clearly prefer

him to prepare the intricate case, she asks him to handle a simple standard case. Her choice reflects a concern to "prepare" him so as to increase the chance he accepts to cooperate in the project.

From a formal point of view the one single novelty compared with the standard approach, is that we substitute the Harsanyi type space with a Hilbert space of types. We find that much of conventional game theory can be maintained. The first novel results appear in multi-stage non-commuting games and they are linked to updating. We formulate an updating rule consistent with the algebraic structure of the type space of TI-games. We show that this rule gives new content (beyond the informational one) to pooling respectively separating behavior. The intuition is that when the opponent's best-reply to an action implies some pooling of his eigentypes, some indeterminacy is preserved and the probabilities for his next-following choices may be marked by interference effects. These interference effects are absent when the eiegentypes separate. We define a TI-Nash equilibrium and demonstrate in an example that the set of Bayes Nash equilibria and TI-Nash equilibria do not coincide.

The paper is organized as follows. In Section 2 we introduce static games of maximal information and establish an equivalence result. In Section 3 we move to dynamic TI-games and investigate our lead example. We formulate the concept of TI Nash equilibrium and show how it relates to the Bayes Nash equilibrium concept. We end with some concluding remarks.

2. Static TI-games of maximal information

In the TI-model a simple "decision situation" is represented by an *observable*[4] called a *DS*. A decision-maker is represented by his state or *type*. A type is a vector $|t_i\rangle$ in a Hilbert space. The measurement of the observable corresponds to the act of choosing. Its outcome, the chosen item, actualizes an *eigentype*[5] of the observable (or a *superposition*[6] of eigentypes if the measurement is coarse). It is information about the preferences

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[2] Game Situations are situations where the players must choose an action in a strategic context. In TI-games, they are modelled as operators.

[3] Open-mindedness and taste for personnal challenge are not the same psychological features but they are somehow related.

[4] An observable is a linear operator that operates on the state of a system.

[5] The eigentypes are the types associate with the eigenvalues of the observable i.e., the possible outcomes of the measurement of the *DS*.

[6] A superposition is a linear combination of the form $\sum \lambda_i |t_i\rangle$; $\sum \lambda_i^2 = 1$ where the t_i are possible states/types of the players.

(type) of the agent. For a detailed exposition of the TI-model see Lambert-Mogiliansky et al. (2009). How does this simple scheme change when we are dealing with strategic decision-making?

We denote by *GS* (for Game Situation) an observable that measures the type of an agent in a strategic situation, i.e., in a situation where the outcome of the choice, in terms of the agent's utility, depends on the choice of other agents as well. The interpretation of the outcome of the measurement is that the chosen action is a *best reply* against the opponents' expected action.[7] This interpretation parallels the one in the simple decision context. As in standard game theory the chosen move is information about the type of the player *modulo equilibrium reasoning*. We say that a *GS measures* a type characteristics and it changes his type i.e., its outcome actualizes some (superposition of) eigentype(s) .

Types and eigentypes
We use the term *type* as the term *quantum pure state*. A (pure) type $|t\rangle \in \mathbb{T}$[8] where $\mathbb{T}$ is a (finite dimensional) Hilbert space, is maximal information about the player i.e., about his payoff function. Generally, the (Harsanyi) type includes a player's information and beliefs. But in this paper we let the term type exclusively refer to the payoff function (or preferences). We shall be dealing with TI-games of *maximal information* where all players are represented by pure types and there is no uncertainty related to the state of the world. The only uncertainty that we consider is related to the players' type because an opponent's pure type is probabilistic information about his eigentypes.

In a TI-game we also speak about the *eigentypes* of a game M, $e_i(M) \in E(M)$, $E(M) \subset \mathbb{T}$. The term eigentype parallels the term eigenstate of a system in Physics. It is a state associated with one of the possible eigenvalues of an observable. An eigentype is thus the type associated

with one of the possible outcomes of a *GS* (or more correctly of a complete set of commuting *GS* associated with a game).[9] The eigentypes are truly *private and complete information* about the payoff functions *in a specific static game M*. Any eigentype of a player knows his own M-game payoff function but he may not know that of the other players.

The classical Harsanyi approach only uses a single concept, i.e., that of type and it is identified both with the payoff function and with the player. In any specific TI-game M, we must distinguish between the type which is identified with the player and the eigentypes (of M) which are identified with the payoff functions in game M. A helpful analogy is with multiple-selves models (see e.g., Strotz (1956) and Fudenberg and Levine, (2006)). In multiple-selves models, we are most often dealing with two "levels of identity": the short-run impulsive selves on the one side and a long-run "rational self" on the other side. In our context we have two levels as well: the level of the player (the type) and the level of the selves (the eigentypes) which are to be viewed as potential incarnations of the player *in a specific game*. In a TI-model a player is described as a superposition of (simultaneous) selves.

In a maximal information TI-game the initial type of the players are pure types (i.e., not mixed) and they are common knowledge among players. For any static game M, a type t_i can be expressed in terms of the eigentypes $|e_i(M)\rangle \in E(M) \subset \mathbb{T}$, with $|e_i(M)\rangle \perp |e_j(M)\rangle$, $i \neq j$. An eigentype is information about the value of (all) the type characteristics *relevant to a particular game*. Consider a multi-move game: M followed by N, the TI-model allows for the case when the type characteristics relevant to M respective N are "incompatible" in the sense that they cannot be revealed (actualized) simultaneously.[10] This is the source of intrinsic indeterminacy i.e., of an uncertainty not due to incomplete information. The classical Bayes-Harsanyi model corresponds to the special case of the TI-model when all *GS* commute or equivalently all type characteristics are compatible.

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[7] The notion of "actualized best-reply" is problematic however. A main issue here is that a best-reply is a response to an *expected* play. When the expected play involves subjective beliefs there may be a problem as to the observability of the preferences. This is in particular so if subjective beliefs are quantum properties. But in the context of maximal information games (see below for precise definition) probabilities are objective which secures that the actualized best-reply is well-defined.

[8] We use Dirac ket notation $|.\rangle$ to denote a vector in a Hilbert space.

[9] A *GS* corresponds to a specific strategic decision situation. A complete set of commuting *GS* provides information about a type's behavior for any possible play of any opponent. It is complete description of his preferences *in this game* and we identify it with a payoff function.

[10] The tensor product set $E(M) \otimes E(N)$ does not exist when M and N are incompatible.

Assumption 1

In a TI-game all strategic reasoning is done by the eigentypes of the players.

This key assumption says that the reasoning leading to the determination of the best-reply is performed at the level of the eigentypes of a player or equivalently at the level of the (simultaneous) selves.[11] Another way to express this is simply to assume that the players are able to reason from different perspectives. Note that this is not as demanding as it may at first appear. Indeed we are used in standard game theory to the assumption that players are able to put themselves "in the skin" of other players (and thus other types) to think out how those will play in order to be able to best-respond to that.

2.1 TI-Nash equilibrium in static games

Under Assumption 1, the eigentypes are the "real players" and we shall see that under this assumption a static TI-game looks very much like a classical incomplete information game and a static TI-equilibrium can be defined as a Nash equilibrium of the original two-player game expanded to the eigentypes of each player.

Let A_j be a finite set of actions available to player $j = 1, 2$. Each player is represented by his type $|t_j\rangle \in \mathbb{T}$. For any game situation M we have $E_j(M) \subset \mathbb{T}$, where $E_j(M) = \left\{ \left| e_1^j(M) \right\rangle, ..., \left| e_k^j(M) \right\rangle \right\}$ is the set of eigentypes of player j in GS M. In the static context of this section we can delete the qualifier in parenthesis and write $\left| e_1^j \right\rangle$ and E_j. A pure strategy for player 1 is a function $s_1 \in S_1$, $s_1 : E_1 \rightarrow A_1$.

The initial type vector of player $j = 1$ can be expressed in terms of the eigentypes of M:

$$|t_1\rangle = \sum_{i=1}^{k} \lambda_i \left| e_i^1 \right\rangle, \quad \sum_{1}^{k} \lambda_i^2 = 1. \qquad (1)$$

The initial common knowledge beliefs about the eigentypes are given by the types $|t_j\rangle$ according to Born's rule

$$prob\left(e_1^i \middle| |t_1\rangle \right) = \lambda_i^2.$$

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[11] We could think of the players as being involved in some form of parallel reasoning: all the active (with non-zero coefficient of superposition) eigentypes perform their own strategic thinking.

We call e_1^i a *potential* eigentype of player 1 iff $\lambda_i > 0$.[12]

In the Bayes-Harsanyi model, Nature moves first and selects the type of each player who is privately informed about it.[13] In a TI-game uncertainty is (partially) resolved by the measurement i.e., the actual act of playing. So before playing, the player does not know his own payoff function (i.e., his eigentype) only his initial (superposed) type . However, each one of his selves (we use the terms self and eigentype interchangeably) knows his own payoff function. The potential selves of a player all have the same information about the opponent's type. Now if the selves know the strategy of the eigentypes of their opponent,[14] they can compute their expected payoff using the information encapsulated in the (superposed) initial type of his opponent.

Definition

A pure strategy TI-Nash equilibrium of a two-player static game M with initial types $|t_1\rangle = \sum \lambda_i \left| e_i^1 \right\rangle$ and $|t_2\rangle = \sum \gamma_i \left| e_i^2 \right\rangle$ is
i. A profile of pure strategies (s_1^*, s_2^*) with

$$s_1^*\left(e_i^1 \right) = \arg \max_{s_1' \in S_1} \sum_{e_i^2; \gamma_i > 0} \gamma_i^2 u_i \left(s_1', s_2^*\left(e_i^2 \right), \left(e_i^1, e_i^2 \right) \right)$$

$$(2)$$

for all $e_i^1; \lambda_i > 0$ $i = 1, ..k$ and similarly for player 2.
ii. A corresponding profile of resulting types (t_1', t_2'),

$$|t_1' | a_i\rangle = \sum_{e_i; s_1^*\left(e_i^1 \right) = a_i^1} \lambda_i' \left| e_i^1 \right\rangle$$

where $\lambda_i' = \frac{\lambda_i}{\sqrt{\sum_{j \neq i} \lambda_j^2 \left(s_1^*\left(e_j^1 \right) = a_i^1 \right)}}$ and a_i is the action played by player 1. Similarly for $\left| t_2' | a_i^2 \right\rangle$.

The first part (i) says that each of the potential eigentypes of each player maximizes his expected utility given the (superposed) type of his

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[12] We note that this is equivalent to an incomplete information representation with player $2's$ initial beliefs about 1 given by

$$p\left(e_i^1 \middle| e_j^2 \right) = \lambda_i^2 \text{ for all } e_j \in E_2.$$

[13] Se Fudenberg and Tirole (1991) for a definition of incomplete information games.

[14] That is they can compute the equilibrium behavior of all other selves.

opponent and the strategies played by the opponent's potential eigentypes. It is very similar to the definition of a Bayesian equilibrium strategy profile except that the probabilities for the opponent's eigentypes are given by the initial superposed type instead of a joint probability distribution.

The second part of the definition (ii) captures the fact that in a TI-game the players' type is modified by their play. The rule governing the change in the type is given by the von Neuman-Luder's projection postulate. As well known for one single measurement, it is equivalent to Bayesian updating i.e., *within* the set $E_j(M)$.

Proposition 1 *A pure strategy TI-Nash equilibrium profile of a static maximal information game M with eigentypes $(e_1,..,e_k)$ and initial types $\left| t_i^j \right\rangle = \sum_{i=1}^k \lambda_i^j \left| e_i^j \right\rangle$, $j = 1, 2$, is equivalent to a Bayesian pure strategy Nash equilibrium profile of a game with type space $E = \{e_1,..,e_k\}$ and common beliefs given by the distributions $p(e_i | t_1) = \left(\lambda_i^1 \right)^2$, $p\left(e_i^2 \big| t_2 \right) = \left(\lambda_i^2 \right)^2$.*

The proof follows immediately from the definitions.

3. Multi-stage TI-game

When it comes to games composed of more than one step for at least one player, the crucial issue for TI-games is whether the corresponding GS commute with each other or not.

Commutativity of GS We say that two GS M and N commute if they share a common set of eigentypes $E = E(M) \otimes E(N)$. This is the standard definition of commuting observables.

Definition

A commuting multi-stage TI-game is such that for each player all the GS he may face (in and out of the equilibrium path) commute with each other. Otherwise we say that the multi-stage game is non-commuting.

If there is no observation between two commuting moves we can merge the two GS into one compound GS with outcome set $(a_i(M), a_i(N))$ and the static TI-equilibrium concept applies.

3.1 Multi-move games with observed actions

We noted above that commuting multi-stage TI-games without observation are not distinguishable from static TI-game both of which are equivalent to static incomplete information games.

This result extends further to commuting dynamic games with observed action.

Result

Commuting dynamic TI-games are not distinguishable from classical games of incomplete information in terms of equilibrium predictions.

This result follows directly from a general result proving the equivalence between the quantum and the classical models with respect to the predictions in a context where all measurements are commuting (see e.g., Danilov et al. (2008) for a derivation of this result in a Social Sciences context). The intuition is that the type space representing commuting type characteristics in the TI-context has a standard Boolean algebraic structure.

Non-commuting multi-stage TI-games: strategic manipulation of players' type We are interested in multi-stage game with observed actions where in each period t the players simultaneously choose their action which are revealed at the end of the period.[15] A simple case of a multi-move game with observation is in next-following example which captures the story we gave in the introduction.

Example We have 2 players, Alice and Bob and the following sequence of moves:

stage 1

- Alice chooses between Standard task (S) or Intricate task (I) and

- Bob observes Alice's choice and chooses between Standard treatment (S) and Inventive treatment (I). We refer to this game situation as GS_1.

stage 2

- Alice invites Bob to join her new project.

- Bob chooses between Accept(A) or Reject(R) the invitation which ends the game. We refer to this game situation as GS_2.

Alice is of known eigentype with preferences described below. Bob is indeterminate with respect to the eigentypes relevant to GS_1: $E^1 = \{\theta_1, \theta_2\}$ with

θ_1 : always prefers to do the Standard treatment of any task.

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[15] We adopt the convention that simultaneous move games include games where the players move in alternation, that is simply we allow for nul moves.

θ_2 : enjoys challenges and can exert inventive effort if he finds it worthwhile.[16]

Bob's initial type is

$$|t_B\rangle = \lambda_1 |\theta_1\rangle + \lambda_2 |\theta_{2B}\rangle$$

We set $\lambda_1 = \sqrt{.6}$, $\lambda_2 = \sqrt{.4}$.

We next assume that Bob's preferences between Accept and Reject depend only on Bob's type i.e., not on the history of play. GS_2 is a simple decision situation, a DS (see Lambert-Mogiliansky et al. 2009). We consider two decision types: τ_1 who is open-minded and Accepts the invitation and τ_2 who is conservative and Rejects. We assume that GS_1 is an operator that does not commute with GS_2 which means that Bob's eigentype θ_i can be expressed in terms of the τ_i:

$$|\theta_1\rangle = \alpha_1 |\tau_1\rangle + \alpha_2 |\tau_2\rangle \qquad (3)$$
$$|\theta_2\rangle = \beta_1 |\tau_1\rangle + \beta_2 |\tau_2\rangle$$

And let $\alpha_1 = \sqrt{.4}$, $\beta_1 = \sqrt{.6}, \alpha_2 = \sqrt{.6}$, $\beta_2 = -\sqrt{.4}$.

Alice's payoff depends critically on Bob's moves at both stages. In particular Alice's payoff is zero if Bob plays R at stage 2. If Bob plays A, Alice's payoff depends on her own choice as well. If she asks to handle the intricate case her payoff is 170 if Bob choose the inventive effort and is 120 if Bob handles the (intricate) case as a routine job. If she chooses the standard case, her payoff is 100 whenever Bob plays A. With this payoff structure Alice badly needs Bob to play A at stage 2 and she prefers him to handle the intricate case at stage 1.

In the TI-Nash equilibrium we exhibit Alice asks Bob to handle the Standard case even though she prefers him to hand the Intricate case. The reason is that she as an incentive to manipulate Bob's type to increase the probability for his acceptance to cooperate. More specifically Alice's realizes that if she plays S, the θ_2 of Bob will choose S and will pool with θ_1 (who always choose S). The resulting type of Bob is then the same as his initial type, indeterminacy (between θ_1 and θ_2) has not been resolved: $\left|t_B'\right\rangle = \lambda_1 |\theta_1\rangle + \lambda_2 |\theta_2\rangle = |t_B\rangle$. This implies (by Born's rule) an ex-ante probability[17] for the play of A equal to

$$prob(A) = \lambda_1^2 \alpha_1^2 + \lambda_2^2 \beta_1^2 + 2\lambda_1 \alpha_1 \lambda_2 \beta_1 = .96$$

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[16]When the case is intricate, θ_2 enjoys exerting the inventing effort.

[17]This obtains from substituting the θs in Bob's type vector using (3): $|t_B\rangle = \lambda_1 (\alpha_1 |\tau_1\rangle + \alpha_2 |\tau_2\rangle) +$
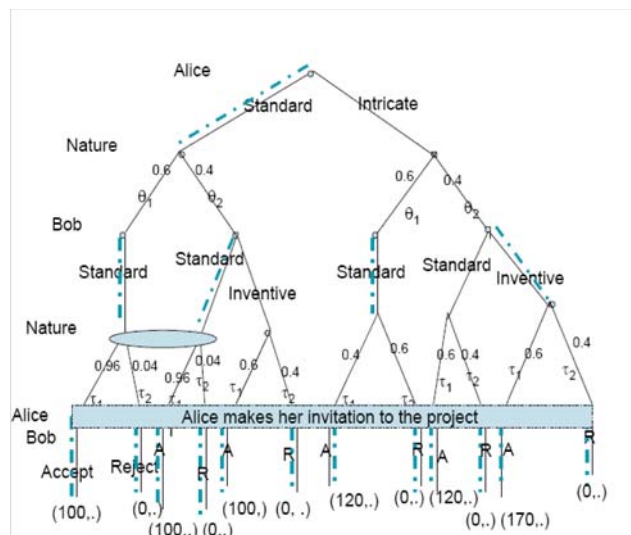

Figure 1:

This should be contrasted with the probability for A that obtains when Alice plays I. In that case Bob's θ_2 best-replies by playing I thus separating from θ_1. Bob's resulting type from his play in GS_1 is $|t_B'\rangle = |\theta_1\rangle$ with probability λ_1^2 and $|t_B'\rangle = |\theta_2\rangle$ with probability λ_2^2. This yields an ex-ante probability for A

$$Prob(A) = \lambda_1^2 \alpha_1^2 + \lambda_2^2 \beta_1^2 = (.6 \times .4) + (.4 \times .6) = .48$$

In our numerical example Alice's expected payoff is then $0.24 \times 120 + 0.24 \times 170 = 69,6$ while when she play S her payoff is $0.96 \times 100 = 96 > 69,6$. Note that in our example the interference term is positive. A psychological interpretation is as follows: both the type who likes personal challenges θ_1 and the one who dislikes them, θ_2, are open to new ideas. The first because novelties have a taste of excitement/challenge, the second because in face of a novel situation you cannot be judged so easily, i.e., it is a very noisy test of capacity, not a *personal* challenge. Therefore when both types are present in the mind of the player they interfere positively to determine his propensity to accept cooperation in the project. The game tree is in figure 1.

Note that we simplified the tree in particular we did not write out some of the branches which are never chosen. We know that Bob's θ_1 always

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$\lambda_2 (\beta_1 |\tau_1\rangle + \beta_2 |\tau_2\rangle) =$
$\quad = (\lambda_1 \alpha_1 + \lambda_2 \beta_1) |\tau_1\rangle + (\lambda_1 \alpha_2 + \lambda_2 \beta_2) |\tau_2\rangle$.

chooses Standard by definition of the eigentype so we omitted the Inventive branch. We did similarly for the τ types who both have simple strict preferences defining their choice. The doted line depicts equilibrium strategies.

As we can see in Figure 1 Nature plays twice and its 2nd move does not define the same probabilities at each node. In the corresponding (first-hand) classical model,[18] the type set of Bob is $\{\theta_1\tau_1, \theta_1\tau_2, \theta_2\tau_1, \theta_2\tau_1\}$. In the TI-game there is no single probability distribution over that type set, $prob(\theta_1\tau_1|S_{Alice}) \neq prob(\theta_1\tau_1|I_{Alice})$. This implies that Nature's 2 moves cannot be collected at the beginning of the game - before Alice' move - as usual. Indeed this is the expression of the non-commutativity of Bob's GS in terms of measurements of his type[19]. The probabilities of Nature's 2nd move depend on Bob's strategy in GS_1 (more precisely on whether his eigentypes pool or not). This also implies that in contrast with a standard game tree we cannot write out the game tree independently of the strategies played (this reminds of psychological games see Geanakoplos et al. (1989)). We collected the two nodes following Bob's play of S into an "information set" for Nature. This is to capture the fact that Bob's pooling play does not break intrinsic uncertainty (indeterminacy). The type of Bob is neither θ_1 nor θ_2 but a (the initial) superposition of the two.

To see that our TI-Nash equilibrium (TINE) is not a Bayes Nash equilibrium we must determine a probability distribution of the set $\{\theta_1\tau_1, \theta_1\tau_2, \theta_2\tau_1, \theta_2\tau_1\}$. Setting $prob(\tau_1) = .4$ and $prob(\tau_2) = .6$, we obtain that Alice's payoff when playing I with Bob best-replying as in the game tree is equal to $U_A(I) = 56$ while her payoff from S is 40. Alternatively if $prob(\tau_1) = .6$, we similarly obtain $U_A(I) = 84 > U_A(S) = 60$. Hence, given Bob's play Alice prefers to choose I if she believe Bob is classical rather than S as in the TINE.

3.2 Nash equilibrium for non-commuting TI-games

We next provide a general definition of the TINE, a concept of Nash equilibrium that is standard in all respect but the updating rule. Instead of the

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[18]The corresponding first-hand classical model is defined as the TI-game with the restriction that all type charcteristics of each player are compatible with each other. It is identical to the TI-game in all other respects.

[19]They cannot be merged into one single measurement

usual Bayesian updating consistent with classical uncertainty we shall have an updating rule that is consistent with quantum indeterminacy, we call it QI-updating.

QI-updating in non-commuting multi-move games

QI-updating is made of two components. The first is the von-Neuman-Luder's projection postulate. It captures the modification of the type vector corresponding to the observed action. When all GS commute this is sufficient and it is equivalent to Bayesian updating. When the next following GS does not commute with the preceding one, a second component is required. The resulting type vector must be written in terms of the eigenvectors of the next-coming GS. This operation corresponds to a change of basis. The QI-updating rule can be presented when considering one single player who chooses two actions in two successive non commuting GS.

Let the first GS be called A with a corresponding set of actions $\{a_1, ..., a_n\}$. It is a measurement of type characteristics $E\left(h^0\right) = \left\{e_1\left(h^0\right), ..., e_m\left(h^0\right)\right\}$ where h^0 is history at time 0. The second GS be called B with a corresponding set of actions $\{b_1, ..., b_n\}$ it is a measurement of type characteristics $E(h^1) = \left\{e_1\left(h^1\right), ..., e_m\left(h^1\right)\right\}$. The assumption is that $E\left(h^0\right)$ and $E\left(h^1\right)$ are two incompatible type characteristics or equivalently A and B are non-commuting measurements.

Step 1

Let $|t\rangle = \sum_i \lambda_i \left|e_i\left(h^0\right)\right\rangle$ be the common knowledge type vector of our player. The initial beliefs[20]

$$B0: \ \mu\left(e_i\left(h^0\right)\right) = \lambda_i^2, \qquad (4)$$

Assume we observe that our player chooses action a_1. Let $s^*\left(h_1\right)$ be our player's equilibrium strategy. The type vector $\left|t\left(h^1\right)\right\rangle$, resulting from history h^1 is, in terms of $E\left(h^0\right)$ eigenvectors :

$$\left|t\left(h^1\right)\right\rangle = \sum_i \lambda_i' \left|e_i\left(h^0\right)\right\rangle \qquad (5)$$

where the sum is taken over i such that $s^*\left(e_i^1\left(h^0\right)\right) = a_1$ and $\lambda_i' = \frac{\lambda_i}{\sqrt{\sum_k \lambda_k^2\left(s_1^*\left(e_k^1(h^0)\right)=a_1^1\right)}}$. The updated beliefs in term of the eigentypes of $E\left(h^0\right)$ are

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[20]The beliefs are shared by all eigentypes of the other players.

$$B1: \ \mu\left(e_i^1\left(h^0\right)\middle|a_1\right) = \lambda_i'^2.^{21} \tag{6}$$

Step 2

We must now express $\left|t\left(h^1\right)\right\rangle$ in terms of the eigentypes of $E\left(h^1\right).^{22}$ The translation is performed using a basis transformation matrix with elements $\delta_{ij} = \left\langle e_j\left(h^1\right)\middle|e_i\left(h^0\right)\right\rangle$. Collecting the terms, we can write

$$\left|t\left(h^1\right)\right\rangle = \sum_j\left(\sum_i\lambda_i'\delta_{ij}\right)\left|e_j\left(h^1\right)\right\rangle$$

The probability for eigentype $e^1\left(h^1\right)$ at date $t = 1$ is

$$B2: \mu\left(e_1\left(h^1\right)\middle|a_1\right) = \left(\sum_i\lambda_i'\delta_{1i}\right)^2 \tag{7}$$

This is the **crucial formula** that captures the key distinction between the classical and the quantum approach. B2 is *not* a conditional probability formula where the δ_{ij}^2 are statistical correlations between the eigentypes at the two stages. The player is a non-separable system with respect to type characteristics $E\left(h^0\right)$ and $E\left(h^1\right).^{23}$ As a consequence, the updated beliefs are given by the square of a sum (implying cross terms) and not the sum of squares.

To see that this makes a difference, recall our example. When Bob's $\theta-$types separate the probability for type τ_1 (for Accept) is 0.48 while when Bob's $\theta-$types pool it jumps to .96 - thanks to the positive interference effects.

Thus, when considering a move a player must account not only for the best-reply of his opponent as usual but also for the induced resulting type of the opponent. More precisely, type indeterminacy gives a new strategic content to pooling respectively separating moves, a content that goes beyond the informational one. When some

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[21] $\mu\left(e_i\left(h_0\right)\middle|a_1\right)$ is any probability if $\sqrt{\sum_k\lambda_k^2\left(s_1^*\left(e^1\left(h_0\right)\right) = a_1\right)} = 0.$

[22] The potential eigentypes of each player must reason using the expectation about the opponent's play. That expected play is computed from the best-replies of the opponent's $E\left(h^1\right)$ −eigentypes and from their relative probability weights in the type vector $\left|t\left(h^1\right)\right\rangle.$

[23] This means that the tensor product set $E\left(h_0\right)\otimes E\left(h_1\right)$ does not exist.

eigentypes of a player pool, that player remains indeterminate (superposed) with respect to those eigentypes. This preserved indeterminacy implies that in the next following (non-commuting) *GS*, the superposed eigentypes *may interact with each other* producing interference effects that affect the probabilities for future actions. Property B2 of the $\mu()$ function secures that the players takes into account the impact of a play on the resulting profile of types.

For each player i, history h^t, eigentype $e_j^i\left(h^t\right)$ and alternative strategy s'

$$\begin{aligned}P \ : \ &u_i\left(s\middle|h^t, e_j^i\left(h^t\right), \mu\left(., h^t\right)\right) \geq \\ &u_i\left(s'\middle|h^t, e_j^i\left(h^t\right), \mu\left(., h^t\right)\right)\end{aligned}$$

Definition 1 *A TI-Nash equilibrium of a multi-stage game is a pair (s, μ) that satisfies conditions P and B0-B2 above.*

From our example we know a TI Nash equilibrium is not necessary a Bayes Nash equilibrium of the corresponding first-hand classical game where the corresponding classical game where the type characteristics θ and τ are compatible with each other. The game is the same in all other respects.[24]

4. Concluding remarks

This paper constitutes a first step in the development of a theory of games with type indeterminate players. Compared with conventional game theory the TI approach amounts to substituting the standard Harsanyi type space for a Hilbert space. We show that for this has no implication for the analysis of static games. In contrast in a multi-move context, we must define an updating rule consistent with the algebraic structure of our type space. We show that for non-commuting TI-games, it implies that players can manipulate each others' type thereby extending the field of strategic interaction. Using the new updating rule we define an equilibrium concept similar to the Bayes Nash equilibrium. We call it TI Nash-equilibrium. We provide an example showing how the two concepts differ.

We have learned that TI-games may bring forth new results in the context of multi-stage games or

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[24] We are currently working on establishing the conditions when the TI-game's predictions cannot not be reproduced even when considering a larger class of classical games i.e., including hidden variables.

when a game is preceded by some form of "pre-play". We conjecture that the Type Indeterminacy approach may bring new light on the a variety of issues including: players' choice of selection principle in multiple equilibria situation, the selection of a reference point or path-dependency.

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