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estimation of sequential English auctions**

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identification, nonparametric estimation**



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# Identification and estimation of sequential English auctions\*

Laurent Lamy<sup>†</sup>

## Abstract

Brendstrup (2007) and Brendstrup and Paarsch (2006) claim that sequential English auction models with multi-unit demand can be identified from the distribution of the last stage winning price and without any assumption on bidding behavior in the earliest stages. We show that their identification strategy is not correct and that non-identification occurs even if equilibrium behavior is assumed in the earliest stages. For two-stage sequential auctions, an estimation procedure that has an equilibrium foundation and that uses the winning price at both stages is developed and supported by Monte Carlo experiments. Identification under general affiliated multi-unit demand schemes is also investigated.

*Keywords:* Sequential auctions, nonparametric identification, nonparametric estimation

*JEL classification:* C14, D44

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# 1 Introduction

The derivation of an equilibrium in sequential auctions with multi-unit demand is known to be untractable without very stringent conditions. The most general treatment is from Gale and Stegeman (2001) where the authors completely characterize a unique equilibrium allocation in a complete information framework with two buyers. Incomplete information adds new caveats. First, two bidders may become asymmetrically informed about the valuations of a third opponent. That is the reason why the information disclosure rules of previous bids are crucial in those game-theoretical analysis, even for sequential auctions with unit-demand as in Milgrom and Weber (2000). Second, with multi-unit demand, equilibrium derivations with multi-dimensional signals are hardly tractable. In particular, Katzman's (1999) general treatment of two-stage sequential auctions with multi-unit demand in incomplete information is limited to sequential English or second price auctions (where the second stage is thus dominant strategy solvable) and to equilibria with 'separable bid functions' where each bidder bases his first-stage bid solely on either his high or his low valuation and mostly to symmetric environments. Furthermore, endogenous valuations may arise if there are more than three bidders (or with two bidders and binding reserve prices): the valuation of a bidder may depend on the identity of the winner he anticipates if he loses the auction, which opens the door for strategic nonparticipation à la Jehiel and Moldovanu (1996) and/or multiple equilibria.

The lack of established theoretical benchmarks for sequential auctions seems to leave little room for a structural approach. However, in the independent private-values paradigm with decreasing marginal utilities, Brendstrup (2007) and Brendstrup and Paarsch (2006), henceforth B&BP, propose a strategy that relies solely on the fact that bidding up to one's remaining valuation is a weakly dominant strategy for the bidders at the last stage of the game if this last stage is an English auction.<sup>1</sup> More precisely, the unique assumption they impose on their sequential auction model is that the winning price of the last stage corresponds to the second-highest valuation of the remaining units. Then they claim that the model is identified only through the distribution of the winning price at the last stage and the identity of the winner

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<sup>1</sup>By imposing a specific demand-generation scheme for bidders' valuations that guarantees a kind of stationarity, Donald et al. (2006) are able to exploit the winning bids at all stages in a structural way in sequential English auctions.

conditional on a given state, i.e. a given set of winners in all but the last stage of the auction. In the case of symmetric bidders, Brendstrup (2007) proposes a related nonparametric estimation procedure while Brendstrup and Paarsch (2006) propose a semi-nonparametric estimation procedure in the more general case with asymmetric bidders. Those works correctly recognized that, even if bidders are symmetric ex ante, the outcomes of the early auctions lead to endogenous asymmetry among bidder in later auctions. Nevertheless, their derivations do not account for a selection bias: it does not fully handle all the informational content embraced by the number of units obtained by the bidders in the earliest stages of the auction, in particular, the one resulting from the strategic nature of the previous interactions between bidders. In other words, for a given set of primitives, the distribution of the winning price at the last stage does not solely depend on the number of units assigned to the different bidders in the earliest stages but also crucially on the way bidders bid in the earliest stages.<sup>2</sup> We limit formal analysis to the case of two-stage sequential English auctions with symmetric bidders.

This paper is organized as follows. In section 2, we present the model and the different bidding heuristics we will consider. Section 3 is devoted to identification. Without any specific assumption on the bidding heuristic, we show that the model is not identified. Furthermore, with two bidders, we show that the model is not identified even if equilibrium behavior is imposed in the earliest stages. B&BP's identification and estimation procedures are valid under a bidding heuristic where bidders bid randomly, i.e. independently of their private values, in the first stage, a bidding heuristic that is not an equilibrium. On the contrary, the paper then mainly focus on the equilibrium where the bidding function in the first stage depends solely on bidders' high valuations: such an equilibrium always exists in Katzman's (1999) framework but also if the underlying important symmetry assumption -that prevails in both Katzman (1999) and B&BP- on the generation of multi-unit demand valuations is relaxed. In section 4, we do not solely adapt Brendstrup's (2007) nonparametric procedure that is based only on the last stage winning price but we propose a nonparametric estimation procedure that also uses the winning price at the first stage. Section 5 summarizes results of some Monte Carlo experiments. Section 6 is mainly devoted to a generalization of B&BP's model that relax the symmetry

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<sup>2</sup>The same 'selection bias' issue arises also in Brendstrup's (2006) analysis of sequential English auctions with heterogeneous objects with synergies.

assumption on the different draws of a given bidder: the extension involves a generalized form of multi-unit demand that covers not solely B&BP’s framework but also unit-demand as special cases. We then prove identification from the distribution of winning prices at both stages. We conclude in section 7. Technical proofs are relegated in the Appendix.

## 2 The model

We consider Brendstrup’s (2007) model of sequential English auctions with multi-unit demand under the symmetric independent private-values paradigm. We limit our analysis to two-stage auctions which correspond to the environment investigated theoretically by Katzman (1999) under risk neutrality. We make thus the following assumptions:

- A1.** The auction consists of 2 stages, at each stage of which an identical indivisible object is sold.
- A2.** There are  $n \geq 2$  potential bidders bidding on both units.
- A3.** The valuations of potential bidder  $i$  are 2 independent draws from an atomless cumulative distribution function  $F(x)$  on  $[\underline{x}, \bar{x}]$ , which is three times differentiable on  $(\underline{x}, \bar{x})$  and has probability density function  $f(x) > 0$  for all  $x \in (\underline{x}, \bar{x})$ .<sup>3</sup>
- A4.** The draws of potential bidders are mutually independent.
- A5.** The transaction price (winning price) in the last stage is the second-highest valuation of the remaining unit.
- A6.** A sequence of identical auctions is observed.

In B&BP, no assumption is made on the bidding behavior in all but the last stage of the auction. The unique assumption on the way bidders are playing the sequential auction game is that the winning price at the last stage corresponds exactly to the second highest of the valuations for this final unit. However, as it will be argued in section 3, the econometrician can not circumvent the issue of modeling the bidding behavior in the earliest stages of the auction. Below we introduce three kinds of “bidding heuristics” at the first stage.

**Bidding heuristic R:** Bidders are bidding ‘randomly’: their bid functions in the first stage do not depend on their valuations.

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<sup>3</sup>The conditions on the smoothness of  $F$  matter only for the estimation section. For our identification results, they can be dropped.

**Bidding heuristic M1:** Bidders are using a common bidding function that is based solely on their own high valuations and that is strictly increasing.

**Bidding heuristic M2:** Bidders are using a common bidding function that is based solely on their own low valuations and that is strictly increasing.

Under our bidding heuristics, note that we do not enter into the details of the bidding function. However, we emphasize that we assume that bidders are using the same bidding function under M1 and M2. The bidding heuristics M1 and M2 have an equilibrium foundation under standard additional restrictions as shown by Katzman (1999). On the contrary, it is straightforward to check that bidders playing according to heuristic R is incompatible with any equilibrium behavior under any standard restriction as, e.g., our subsequent assumption A7. Nevertheless, this benchmark is useful since B&BP's analysis remains valid under this heuristic.

**Remark** Contrary to B&BP, we do not assume that the identities of the winners of the previous stages are observed. This information does not matter here because first we limit our analysis to two-stage sequential auctions, second bidders are ex ante symmetric and third we consider only 'symmetric bidding heuristics'.

### 3 (Non-)Identification

In this section, we show how to identify  $F$  from  $G_2$ , the cumulative distribution function (CDF) of the winning price in the last stage, for the simple bidding heuristics we have proposed.

Consider first heuristic R where the winning or losing status in the first stage does not convey any information on the valuations of the bidders. Then the CDF of the valuation for the second unit for the winning bidder corresponds to the lowest draw from a sample of 2 independently and identically draws from the CDF  $F$  and is thus given by  $F_{w,2}(x) = 2F(x) - F^2(x)$ . For a losing bidder, the valuation for the second unit corresponds to the highest draw from a sample of 2 independently and identically draws from the CDF  $F$  and is thus given by  $F_{l,2}(x) = F^2(x)$ . Those are special cases of the more general bijection formula between the distribution of the  $l$ th largest order-statistic from a sample of  $m$  independently and identically distributed draws and the distribution  $F(x)$  of the underlying draws, which has the form

$$F_l^m(x) = \frac{m!}{(m-l)!(l-1)!} \int_0^{F(x)} v^{m-l}(1-v)^{l-1} dv = \phi_{l,m}(F(x)). \quad (1)$$

This formula is the first crucial technical step in B&BP's analysis that allows to trace back bidders' valuation distributions from their bidding CDFs at the last stage conditional on a given number of units won in the earliest stages for any number of stages. Furthermore, under heuristic R, assumption A4 guarantees that bidders' valuations for the second unit are drawn independently. The winning price at the last stage corresponds then to the second order statistic among  $n$  independently distributed CDFs, one being distributed according to  $F_{w,2}$  while the  $n-1$  remaining ones according to  $F_{l,2}$ . From Balakrishnan and Rao (1998), the CDF  $G_2$  is thus given by

$$G_2(x) = \frac{1}{(n-2)!} \int_{\underline{x}}^x \text{Perm} \begin{vmatrix} F_{l,2}(v) & \dots & F_{l,2}(v) & F_{w,2}(v) \\ \vdots & \ddots & \vdots & \vdots \\ F_{l,2}(v) & \dots & F_{l,2}(v) & F_{w,2}(v) \\ f_{l,2}(v) & \dots & f_{l,2}(v) & f_{w,2}(v) \\ (1-F_{l,2}(v)) & \dots & (1-F_{l,2}(v)) & (1-F_{w,2}(v)) \end{vmatrix} dv, \quad (2)$$

where Perm denotes the Permanent operator that is applied here to a  $n \times n$  matrix.<sup>4</sup> This is the second crucial technical step in B&BP's analysis that links the observed winning price distribution and bidders' valuation distributions for the second unit. In our two-stage sequential auction framework where losing bidders are symmetric, expression (2) simplifies to:

$$G_2(x) = \int_{\underline{x}}^x (n-1)[F_{l,2}(v)]^{n-3} \{ (n-2)F_{w,2}(v)f_{l,2}(v)[1-F_{l,2}(v)] + f_{w,2}(v)[F_{l,2}(v)][1-F_{l,2}(v)] + [1-F_{w,2}(v)][F_{l,2}(v)]f_{l,2}(v) \} dv.$$

After some calculation, it reduces to  $G_2(x) = \Psi_R[F(x)]$  where  $\Psi_R$  is the polynomial:

$$\Psi_R[X] = 2(n-1)X^{(2n-3)} - (n-2)X^{(2n-2)} - 2(n-1)X^{(2n-1)} + (n-1)X^{2n}.$$

<sup>4</sup>For a  $n \times n$  matrix  $A = (a_{ij})_{1 \leq i, j \leq n}$ , the Permanent of  $A$  is given by  $\text{Perm}A = \sum_{\sigma \in \Sigma_n} \prod_i^n a_{\sigma(i), i}$ , where  $\Sigma_n$  is the set of permutation of  $\{1, \dots, n\}$ .

On the contrary, under heuristic M1, the winning or losing status in the first stage does convey information with respect to the valuations of the bidders such that those technical steps that are relying of the independence of bidders' valuations draws can not be directly applied as in B&BP.<sup>5</sup> Consider heuristic M1 and now work conditional on the highest high valuation among all bidders, a variable which is denoted by  $u$ . Conditional on  $u$ , the CDF of the valuation for the second unit is given by  $F_{w,2}(x|u) = \min\{F(x)/F(u), 1\}$  for the winning bidder that has won the first unit in the first stage and  $F_{l,2}(x|u) = \min\{F^2(x)/F^2(u), 1\}$  for losing bidders that have not obtained the first unit. Conditional on  $u$  the  $n$  valuations for the second unit are distributed independently which allows to apply (2) which leads to:

$$G_2(x|u) = \begin{cases} (n-1)\frac{F^{2n-3}(x)}{F^{2n-3}(u)} + \frac{F^{2n-2}(x)}{F^{2n-2}(u)} - (n-1)\frac{F^{2n-1}(x)}{F^{2n-1}(u)} & \text{if } x \leq u \\ 1 & \text{if } x > u \end{cases}.$$

After integrating with respect to the variable  $u$  which is distributed according to  $F^{2n}$  and after some calculation, it reduces to  $G_2(x) = \Psi_{M1}[F(x)]$  where  $\Psi_{M1}$  is the polynomial:

$$\Psi_{M1}[X] = \frac{2n(n-1)}{3}X^{(2n-3)} + nX^{(2n-2)} - 2n(n-1)X^{(2n-1)} + \frac{(n-1)(4n-3)}{3}X^{2n}.$$

**Remark** Under heuristic M1, the distributions  $F_{w,2}$  and  $F_{l,2}$  do not correspond to  $\phi_{2,2}(F)$  and  $\phi_{1,2}(F)$  their counterparts under heuristic R, contrary to what B&BP have claimed. The integration of  $F_{w,2}(x|u)$  and  $F_{l,2}(x|u)$  with respect to  $u$  leads to

$$F_{w,2}(x) = \underbrace{[2F(x) - F^2(x)]}_{\text{B\&BP's term: } \phi_{2,2}(F(x))} + \underbrace{\frac{F(x)(1-F(x))}{2n-1} \left[ F(x) \sum_{i=1}^{2n-2} F^{i-1}(x) - (2n-2) \right]}_{\leq 0, \text{negative bias}}$$

$$\text{and } F_{l,2}(x) = \underbrace{F^2(x)}_{\text{B\&BP's term: } \phi_{1,2}(F(x))} + \underbrace{\frac{2}{2n-2} [F^2(x) - F^{2n}(x)]}_{\geq 0, \text{positive bias}}.$$

<sup>5</sup>Those insights are also valid for heuristic M2 whose analysis here will be mainly limited to the case  $n = 2$  where it has an equilibrium foundation.



The above exact formulas confirm the intuition that a bidder who wins [loses] the first unit of the auction sequence is more likely to have a high low valuation [a low high valuation] compared to the corresponding ex ante distributions that have been considered in B&BP. Note also that we can not plug the expression of  $F_{w,2}(x)$  and  $F_{l,2}(x)$  into the expression (2) since the valuations for the second unit are correlated: it is only conditional on  $u$  that they are independent.

**Proposition 3.1** *Under heuristic  $i \in \{R, M1\}$ , we have  $G_2(x) = \Psi_i[F(x)]$  where  $\Psi_i$  is a known and strictly increasing polynomial function from  $[0, 1]$  to  $[0, 1]$  and such that  $\Psi_i^{-1}$  is differentiable on  $(0, 1)$ .*

$$\begin{aligned}\Psi_R[X] &= 2(n-1)X^{(2n-3)} - (n-2)X^{(2n-2)} - 2(n-1)X^{(2n-1)} + (n-1)X^{2n} \\ \Psi_{M1}[X] &= \frac{2n(n-1)}{3}X^{(2n-3)} + nX^{(2n-2)} - 2n(n-1)X^{(2n-1)} + \frac{(n-1)(4n-3)}{3}X^{2n}\end{aligned}$$

Moreover,  $\Psi_R(x) > \Psi_{M1}(x)$  on  $(0, 1)$  for  $n = 2$  while  $\Psi_R(x) < \Psi_{M1}(x)$  on  $(0, 1)$  for  $n \geq 3$ .

If the econometrician is prepared to assume that bidders are bidding according to one of the heuristic  $i \in \{R, M1\}$ , then, exactly as in B&BP, proposition 3.1 guarantees that the distribution of winning bids at the second stage enables identification of the distribution of valuations through the mapping:  $F(x) = \Psi_i^{-1}[G_2(x)]$  and a nonparametric procedure as in Brendstrup (2007) can be developed. Nevertheless, another corollary of proposition 3.1 is a non-identification result: without any assumption on the bidding behavior on the first stage, the distribution  $F(\cdot)$  is not identified from the distribution of the winning price of the last stage. Any atomless CDF  $G_2$  of the winning price at the last stage such that, on the interior of the bidding support,  $G_2$  is three times differentiable and the corresponding PDF  $g_2$  is strictly positive can be viewed as resulting either from  $F_R(x) = \Psi_R^{-1}[G_2(x)]$  or from  $F_{M1}(x) = \Psi_{M1}^{-1}[G_2(x)]$  where  $F_i$  is actually a CDF satisfying assumption A3 and such that the CDFs  $F_i$ ,  $i \in \{R, M1\}$ , are distinct.

**Corollary 3.2 (General non-identification)** *Under assumptions A1-A6,  $F(\cdot)$  is not identified from the transaction price of the last stage.*

B&BP do not model the behavior of the bidders in the earliest stages of the auction. In particular, bidders' information and beliefs are not modeled (it covers

both complete and incomplete information environments), bidders' preferences are not fully specified (the implicit dominant strategy assumption in A5 covers any kind of risk aversion) and even equilibrium behavior is not assumed in the earliest stages. We now ask whether the negative result in corollary 3.2 still holds under standard equilibrium restrictions. Next assumption corresponds exactly to Katzman's (1999) framework.

**A7.** Valuations are private information, bidders are risk neutral and are playing according to Bayes Nash equilibrium at the first stage.

Katzman (1999) shows that, for  $n = 2$  and for any CDF  $F(\cdot)$ , there exists equilibria that are consistent with either heuristic M1 or M2 while remaining consistent with assumption A7. Similarly to what we have done under heuristic M1, we now show that any CDF for the winning price at the last stage can be viewed as resulting from an equilibrium under heuristic M2. Consider heuristic M2 and now work conditional on the highest low valuation among all bidders, a variable which is denoted by  $t$ . Conditional on  $t$ , the CDF of the valuation for the second unit is given by  $F_{w,2}(x|t) = \mathbf{1}[x \geq t]$  for the winning bidder that has won the first unit in the first stage and  $F_{l,2}(x|t) = F^2(x)/(2F(t) - F^2(t))$  if  $x \leq t$ ,  $F_{l,2}(x|t) = (2F(x)F(t) - F^2(t))/(2F(t) - F^2(t))$  if  $x > t$  for losing bidders that have not obtained the first unit. Conditional on  $t$  the  $n$  valuations for the second unit are distributed independently which allows to apply (2) and, for  $n = 2$ , it leads to:

$$G_2(x|t) = F_{w,2}(x|t) + F_{l,2}(x|t) - F_{w,2}(x|t) \cdot F_{l,2}(x|t) = \begin{cases} \frac{F^2(x)}{2F(t) - F^2(t)} & \text{if } x < t \\ 1 & \text{if } x \geq t \end{cases}.$$

Remark that the CDF  $G_2(\cdot|t)$  has an atom at  $x = t$ . The integration with respect to the variable  $t$  which is distributed according to  $(2F - F^2)^2$  leads to  $G_2(x) = \Psi_{M2}[F(x)]$  where  $\Psi_{M2}$  is the polynomial

$$\Psi_{M2}[X] = 6X^2 - 8X^3 + 3X^4.$$

Then the same logic that leads to corollary 3.2 leads to the following non-identification result.

**Corollary 3.3 (Non-identification under equilibrium behavior)** *Under assumptions A1-A7 and for  $n = 2$ ,  $F(\cdot)$  is not identified from the transaction price of the*

last stage.

Corollary 3.3 is limited to  $n = 2$  where we use an established equilibrium multiplicity result. As emphasized in the introduction, there is a lack of knowledge on the theoretical side under more general setups. Even for  $n = 2$  and under A7, the full equilibrium set is not known: recall that Katzman's (1999) analysis is limited to 'separable' strategies that depend solely on either the low or the high valuation.<sup>6</sup>

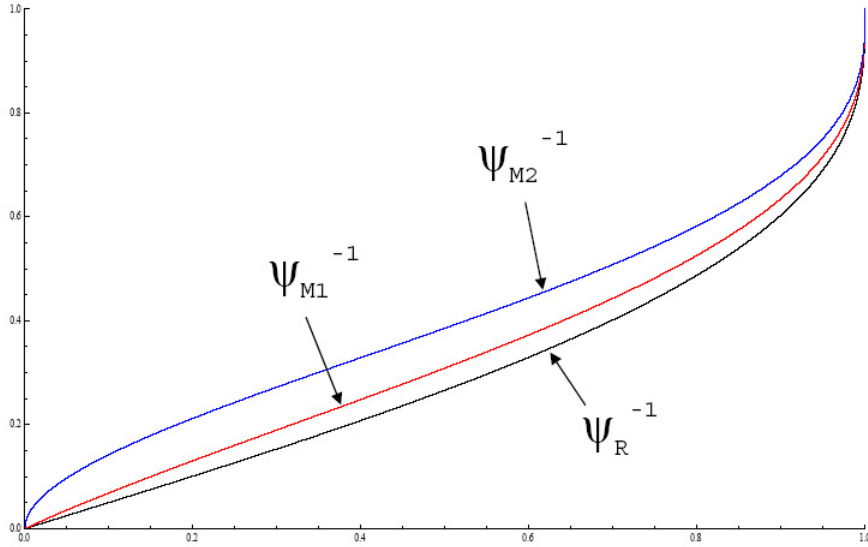


Figure 1:  $F(x)$  as a function of  $G_2(x)$ .

We can revisit Example 1. in Brendstrup (2007): the two bidders and two units case.  $F(\cdot)$  is uniquely characterized as an implicit function by the equation  $G_2(x) = \Psi_i[F(x)]$  for the different heuristics  $i \in \{R, M1, M2\}$ . In Figure 1 the functions  $\Psi_i^{-1}[X]$ ,  $i \in \{R, M1, M2\}$  are depicted, equivalently it gives the expression of  $F(x)$  as a function of  $G_2(x)$  for our different bidding heuristics. The differences between two curves  $i$  and  $j$  corresponds then to the bias when one assumes a wrong heuristic  $i$  while the true bidding heuristic is  $j$ . The graphs show that the bias is especially important between  $\Psi_R^{-1}$  and  $\Psi_{M2}^{-1}$ . If one assumes heuristic R, as it is implicitly the case in B&BP, while the true bidding heuristic is either M1 or M2, then the CDF  $F$  is underestimated according to first order stochastic dominance. The bias is greater than 10% for more than one third of the support in the case of heuristic M2. Note that the sign of the misspecification bias if one assumes heuristic

<sup>6</sup>Another source of non-identification would emerge if we do not assume an 'incomplete information' structure (as under A7) but allow also bidding under complete information. Then a similar non-identification result as corollary 3.3 could be derived for any number of bidders while still restricting attention to bidding behaviors that are Nash equilibria with risk neutral bidders.

R while the true bidding heuristic is either M1 changes for  $n \geq 3$  as established by proposition 3.1.

In the rest of this paper we will consider equilibria under heuristic M1. At first glance, it seems an arbitrary selection rule. The following argument makes a strong case in favour of those equilibria in the case where there are at least three bidders.<sup>7</sup>

**Proposition 3.4** *If  $n \geq 3$ , then a symmetric equilibrium allocates the good efficiently if and only if it follows heuristic M1.*

In general, welfare maximization is not a popular selection rule in game theory. However, in assignment problems, equilibria that guarantee allocative efficiency have a special foundation: they do not depend on the existence of resale opportunities after the assignment from the auction stages and also do not depend on the way to model them if any, since there is then no room for mutually profitable sales. On the contrary, allocative inefficiencies imply the existence of mutually profitable sales between an auction winner and a bidder that loses one of the auctions. As emphasized by Hafalir and Krishna (2008) in the case of one good for sale, the equilibrium bid functions depend crucially on how the market power is distributed at the resale stage. In a nutshell, with at least three bidders, equilibria under heuristic M1 are the only symmetric equilibria that are robust to the details of the aftermarket rules. Furthermore, we conjecture that the ‘ratchet effect’ associated to resale opportunities would preclude the existence of strictly monotone equilibria as they are precluding pure separating equilibria in the case of one object for sale (see Lebrun (2010)). In any cases, the way the ratchet effect works depends crucially on the disclosure rules about the submitted bids such that equilibria under heuristic M1 are the only equilibria that are not subject to the ratchet effect and then robust to the details of those rules. Note in particular that, with resale and beyond heuristic M1, then it is no longer a weakly dominant strategy for a given bidder to bid up to his true valuation for the last unit: the ratchet effect will typically prevent assumption A5, i.e. the simple characterization of the equilibrium strategies at the last stage that was the starting point of B&BP’s analysis.

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<sup>7</sup>Equilibria under heuristic M1 have also a special appeal if information is costly as in Compte and Jehiel (2007): bidders have no incentives to learn before the first stage their low valuation if they anticipate that such an equilibrium is played and that they will have the opportunity to learn this valuation between the two auction stages.

## 4 Estimation

In this section, assumption A7 is replaced by the following additional assumption. **A8.** Valuations are private information, bidders are risk neutral and are playing according to the Bayes Nash equilibrium that is consistent with heuristic M1 at the first stage.

In this section, we set up the estimation method. We do more than simply fixing Brendstrup's (2007) procedure to account for the selecting bias that arises under heuristic M1 with respect to heuristic R but we propose a nonparametric estimation procedure that uses the first stage's bids in order to gain in term of efficiency as it will be argued in section 5.<sup>8</sup> Let  $T$  denote the total number of observations. Each observation  $t \in \{1, \dots, T\}$  consists of a pair of prices  $(B_t^1, B_t^2)$  where  $B_t^i$  corresponds to the winning price at the  $i$ th stage.

### Estimation from the first stage

From Katzman (1999), the equilibrium bid function  $\beta(\cdot)$  at the first stage under A8 is uniquely given by:

$$\beta(x) = x - \frac{\int_x^x F^{2n-3}(u)du}{F^{2n-3}(x)}. \quad (3)$$

The derivation with respect to the variable  $x$  of the above expression and the change of variable  $b = \beta(x)$  leads to the equation:

$$\beta^{-1}(b) = b + \frac{1}{2n-3} \cdot \frac{D_1(b)}{d_1(b)}, \quad (4)$$

where  $D_1$  and  $d_1$  are respectively the CDF and the PDF of the bids at the first stage. Such a reparametrization of the equilibrium equation is similar to the one that first appeared in Guerre et al. (2000) for the first price auction and that allows to express bidders' private valuations from their bids and the elasticity of their probability of winning. First it shows identification from the CDF of the winning price of the first stage since the bid distribution can be identified from the winning price CDF. This is summarized in the following corollary. Furthermore it will also give a natural nonparametric estimation path.

**Corollary 4.1** *Under A1-A6 and A8,  $F(\cdot)$  is identified from the winning price of*

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<sup>8</sup>A similar two step procedure could also be proposed for the equilibrium that is consistent with heuristic M2 when  $n = 2$ .

the first stage.

Let  $G_1$  and  $g_1$  denote respectively the CDF and the PDF of the winning price at the first stage which can be estimated respectively by its empirical distribution and by standard kernel estimation techniques:

$$\widehat{G}_1(b) = \frac{1}{T} \sum_{t=1}^T \mathbf{1}(B_t^1 \leq b) \quad \text{and} \quad \widehat{g}_1(b) = \frac{1}{h_g T} \sum_{t=1}^T K_g\left(\frac{b - B_t^1}{h_g}\right), \quad (5)$$

where  $h_g > 0$  is a bandwidth and  $K_g(\cdot)$  is a kernel with bounded support.

The relation between the bid distribution in the first stage and the winning price distribution is given by  $D_1(b) = \phi_{n-1,n}^{-1}(G_1(b))$ . The empirical counterpart gives

$$\widehat{D}_1(b) = \phi_{n-1,n}^{-1}(\widehat{G}_1(b)) \quad \text{and} \quad \widehat{d}_1(b) = \frac{\widehat{g}_1(b)}{\phi'_{n-1,n}(\phi_{n-1,n}^{-1}(\widehat{G}_1(b)))} \quad (6)$$

Then the empirical counterpart of equation (4) can be used to build a set of ‘pseudo-valuations’ in the same vein as in Guerre et al.’s two stages estimator:

$$X_t^1 = B_t^1 + \frac{1}{2n-3} \cdot \frac{\widehat{D}_1(B_t^1)}{\widehat{d}_1(B_t^1)}. \quad (7)$$

We do not detail this point here but a trimming rule at the boundaries of the support is needed to avoid some bias in the same way as in Guerre et al. (2000). Then we use the pseudo sample  $\{X_t^1, t = 1, \dots, T\}$  to estimate nonparametrically the CDF  $F_{n-1,n}^1$  and PDF  $f_{n-1,n}^1$  of the valuation corresponding to the highest losing bidder in the first stage for the underlying CDF  $F$ :

$$\widehat{F}_{n-1,n}^1(x) = \frac{1}{T} \sum_{t=1}^T \mathbf{1}(X_t^1 \leq x) \quad \text{and} \quad \widehat{f}_{n-1,n}^1(x) = \frac{1}{h_f T} \sum_{t=1}^T K_f\left(\frac{x - X_t^1}{h_f}\right), \quad (8)$$

where  $h_f > 0$  is a bandwidth and  $K_f(\cdot)$  is a kernel with bounded support.

Since the high valuation for a given bidder is distributed according to the CDF  $[F(x)]^2$ , the relation between the high valuation of all bidders and the high valuation corresponding to the highest losing bidder in the first stage is  $F_{n-1,n}^1(x) = \phi_{n-1,n}([F(x)]^2)$ . Finally, the winning price from the first stage auction leads to a first estimator:

$$\widehat{F}^1(x) = [\phi_{n-1,n}^{-1}(\widehat{F}_{n-1,n}^1(x))]^{\frac{1}{2}} \quad \text{and} \quad \widehat{f}^1(x) = \frac{\widehat{f}_{n-1,n}^1(x)}{2[\phi_{n-1,n}^{-1}(\widehat{F}_{n-1,n}^1(x))]^{\frac{1}{2}} \phi'_{n-1,n}(\phi_{n-1,n}^{-1}(\widehat{F}_{n-1,n}^1(x)))} \quad (9)$$

The statistical properties of this estimator from the first stage's winning price, e.g. uniform consistency, can be derived exactly in the same way as in Guerre et al. (2000) due to the similarity of the estimation procedure. The unique fundamental difference with Guerre et al. comes from the fact that we do not observe the bid [pseudo valuation] distribution but only the highest losing bid [highest losing pseudo valuation] distribution which requires the uses of the transformations (6) and (9). Those transformations involve differentiable functions on  $(0, 1)$  such that the delta method applies.

#### Estimation from the second stage

Bids at the second stage can be used to give a nonparametric estimate of  $F(\cdot)$  exactly as in Brendstrup (2007) provided that we replace his function  $\Psi = \Psi_R$  with the one which corresponds to heuristic M1, i.e.  $\Psi_{M1}$ . The asymptotic statistical properties are the same as in Brendstrup (2007) since  $\Psi_{M1}^{-1}$  is differentiable on  $(0, 1)$  exactly as  $\Psi_R^{-1}$  was in his analysis and since we made the same smoothness assumptions. Let  $\widehat{F}^2(x)$  and  $\widehat{f}^2(x)$  denote the estimator of the CDF and PDF from this stage.

Finally we propose to estimate the CDF and PDF of the latent valuations by combining our estimators from both stages using a weighted least squares approach.

$$\widehat{F}(x) = \text{Arg min}_s \mathbf{g}(s)' \mathbf{W} \mathbf{g}(s) \quad \text{and} \quad \widehat{f}(x) = \text{Arg min}_s \gamma(s)' \mathbf{\Omega} \gamma(s)$$

where  $\mathbf{g}(s)$   $[\gamma(s)]$  is a 2 dimensional vector with elements  $\widehat{F}^i(s) - s$   $[\widehat{f}^i(s) - s]$  and  $\mathbf{W}$   $[\mathbf{\Omega}]$  is a weighting matrix.

Under A1-A6 and A8,  $F(\cdot)$  is identified independently either from the first stage bids or the second stage bids. A direct testable restriction is that both distributions should identify a common distribution. If  $\widehat{F}^1$  and  $\widehat{F}^2$  are not close to each other then we can suspect that the model is misspecified. Section 6 extends the model such that discrepancies between  $\widehat{F}^1$  and  $\widehat{F}^2$  would be allowed.

## 5 Monte Carlo Study

This section describes results of our Monte Carlo study in the two-bidder environment when the underlying distribution  $F(\cdot)$  that generates the data is the uniform distribution on  $[0, 1]$ . It consists of two parts. First we investigate how Bendstrup's (2007) estimation procedure poorly behaves under the bidding heuristic M2. Second, we study the small sample properties of our estimation procedure under heuristic M1 and in particular the gain from using the winning prices at both stages.

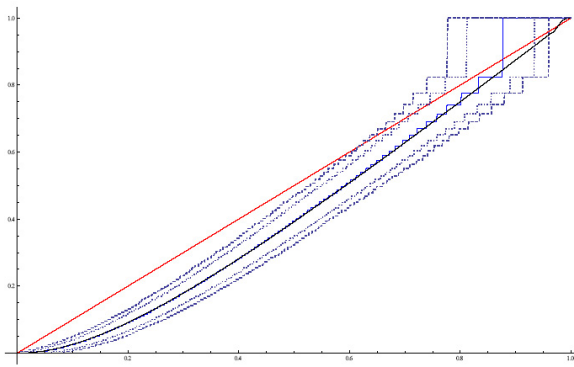


Figure 2a

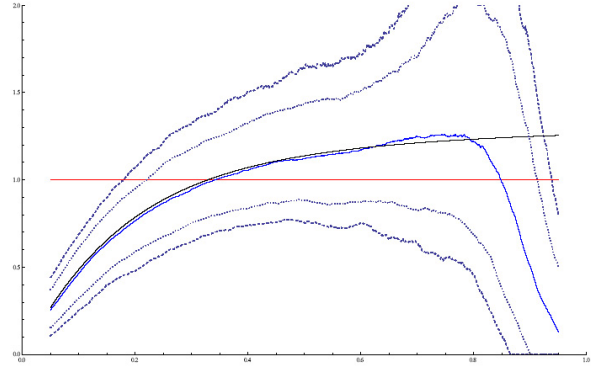


Figure 2b

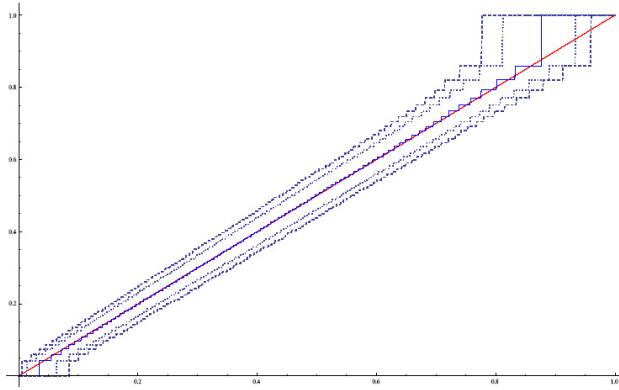


Figure 3a

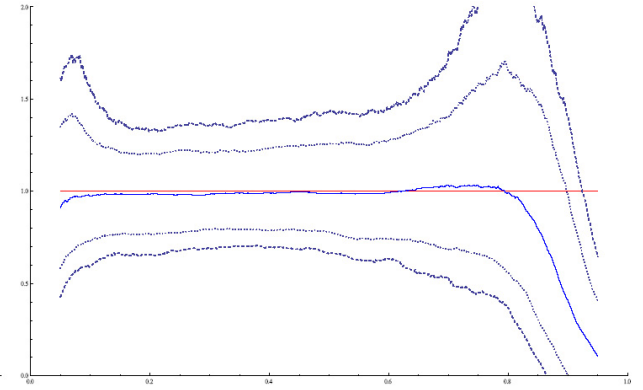


Figure 3b

The median, the 2.5, 10, 90 and 97 percentiles of  $\widehat{F}$  (Fig. 2a & 3a) and  $\widehat{f}$  (Fig. 2b & 3b) with Bendstrup's estimator (Figures 2) and our correction of Bendstrup's estimator with respect to the (correct) bidding heuristic M2 (Figures 3) are depicted in blue.

Our finite sample distributions are based on 2000 replications for a sample size of  $T = 100$ . The bandwidths and kernels are chosen in the same way as in Bendstrup (2007): kernels are given by  $K(x) = \frac{3}{4}(1 - x^2)$  for  $x \in [-1, 1]$  and 0 otherwise;



bandwidths are given by  $0.79 \cdot R \cdot T^{-1/5}$  where  $R$  is the interquantile range of the underlying data whose density is estimated. The weighting matrix  $W$  and  $\Omega$  are chosen to be the identity matrix. In Figures 2 and 3, the red curves correspond to the true CDFs or PDFs. When the estimation model is misspecified as in Figures 2, the black curve depicts the estimated CDF or PDF if the true bidding CDF were known but the B&BP identification path is used. The blue curves summarize our Monte Carlo simulations as indicated in the legend.

There are several striking features. First, Bendstrup's (2007) estimation procedure for the CDF is severely biased downwards. On the contrary, with a well-specified model, the estimator is not biased except at the bounds of the support: the problems in those areas come from the non-differentiability of the function  $\Psi_{M2}$  at the bounds. Second, the same comments hold for the PDFs. Nevertheless, we should note that the variance is very large, which makes the bias issue less outstanding (except at the lower fifth of the support). This point is not surprising from the nonparametric estimation viewpoint with only 100 points but stands in great contrast with the corresponding simulations reported by Bendstrup (2007) where the variance for the estimator of the PDF was surprisingly low.

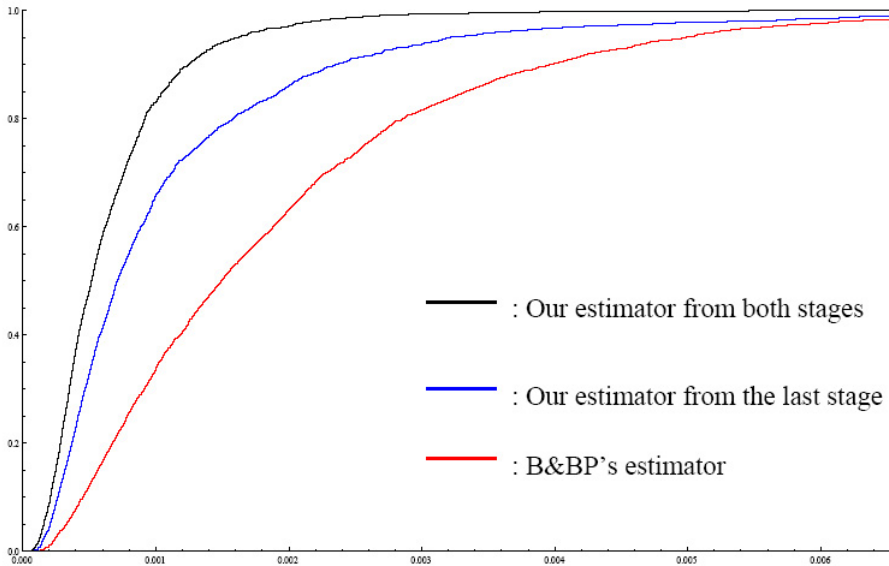


Figure 4: CDF of the mean squared error (MISE) of various estimators.

In Figure 4, we report the CDF of the mean squared error (MISE),  $MISE = \int_0^1 (\hat{F}(x) - F(x))^2 dx$ , of three estimation procedures when the data is generated from the equilibrium that is consistent with heuristic M1: first, in red, Bendstrup's (2007) estimator that is also biased, second, in blue, the analog of Bendstrup's (2007)

estimator that uses only the last stage bids and third, in black, our estimator that uses the bids from both stages. More precisely, we consider a trimmed version of the MISE where the integral is on the support  $[0.2, 0.8]$  to avoid the important nuisances that occur at the bounds. Naturally, our two estimators that are consistent with heuristic M1 clearly outperform the one that is consistent with heuristic R that is biased. More outstanding is the gain when we move to the estimation procedure that uses the winning price only at the last stage to the one that uses the winning price at both stages.

## 6 Extension

As in Katzman (1999), our analysis has been limited to risk neutral bidders and -also as in B&BP- to multi-unit demand valuations with draws that are generated independently from a unique CDF. In this section, we propose an important extension where we consider that one bidder's valuations are generated from a general affiliated distribution and we investigate whether we can still identify the model from the observation of the winning price at both stages under the assumption that bidders are playing an equilibrium that is consistent with heuristic M1. Finally, we end the section with the issue of the non-existence of an equilibrium that is consistent with heuristic M1 with risk averse bidders. It is left to the reader to check that proposition 3.4 and thus the argument in favour of the 'heuristic M1' equilibrium selection rule (when such an equilibrium exists) still hold under those extensions.

### 6.1 General affiliated multi-unit demand schemes

The sampling scheme in Katzman (1999) and B&BP and that was captured by assumption A3, relies on an important symmetry restriction: the different valuations for a given bidder come from independent draws from the same underlying CDF. In the specific case with two valuations, let  $(x_1, x_2)$  (with  $x_1 \geq x_2$ ) denote the pair of valuations for a given bidder. Let  $F1(\cdot)$  denote the CDF of the high valuation  $x_1$  and  $F2(\cdot|x_1)$  the CDF of the low valuation  $x_2$  conditional on the realization of the high valuation  $x_1$ . Under assumption A3 we have the underlying restriction  $F2(\cdot|x_1) = [F1(\cdot)/F1(x_1)]^{1/2}$ . On the contrary, we will allow general forms for  $F2(\cdot|x_1)$ . In the following, assumption A3 is thus replaced by assumption A3b:

**A3b.** The valuations of potential bidder  $i$  are a draw  $(x_a, x_b) \in [\underline{x}, \bar{x}]^2$  from the differentiable atomless CDF  $F^*(x_a, x_b)$  having probability density function  $f^*(x_a, x_b)$  which is assumed to be affiliated: the high [low] valuation is then given by  $x_1 = \max\{x_a, x_b\}$  [ $x_2 = \min\{x_a, x_b\}$ ]. Let  $F(\cdot, \cdot)$  denote the CDF of  $(x_1, x_2)$ .

This generalized model covers also the unit-demand case as a special case if  $F2(\cdot|x)$  reduces to an atom at  $\underline{x}$ , i.e.  $F2(y|x) = \mathbf{1}[y \leq \underline{x}]$  and also to the flat multi-unit demand case if  $F2(\cdot|x)$  reduces to an atom at  $x$ , i.e.  $F2(y|x) = \mathbf{1}[y \leq x]$ .

**Lemma 6.1** *For any  $x^+, x^- \in [\underline{x}, \bar{x}]$  with  $x^+ > x^-$ , the CDF  $F2(\cdot|x^+)$  dominates  $F2(\cdot|x^-)$  according to first order stochastic dominance:  $F2(y|x^+) \leq F2(y|x^-)$ , for any  $y \in [\underline{x}, \bar{x}]$ .*

**Proof** From Milgrom and Weber's (1982) basic properties on affiliation, the affiliation of the variables  $X_a, X_b$  implies the affiliation of  $X_1, X_2$  as the corresponding order-statistics which guarantees then that  $F2(y|x)$  is nondecreasing in  $x$ . **Q.E.D.**

We first show the existence of an equilibrium that is consistent with heuristic M1 if we maintain assumption A7 as in Katzman (1999). Next proposition is thus a generalization of Theorem 2 in Katzman (1999).

**Proposition 6.1** *Under A1-A7, there exists a unique equilibrium under heuristic M1: the first stage bid function  $\beta$  is given by*

$$\beta(x) = \int_{\underline{x}}^x y \frac{d[[F1(y)]^{n-2} F2(y|x)]}{[F1(x)]^{n-2}}. \quad (10)$$

**Remark** We have assumed that one bidder's valuations are affiliated in order to guarantee that the right hand side of equation (10) is strictly increasing with respect to the variable  $x$ . Indeed, this latter condition is sufficient to guarantee the existence of an equilibrium under heuristic M1 as in can be checked in the proof and our following identification result would also extend under such a milder restriction.

**Proposition 6.2** *Under A1-A6 and A8,  $F(\cdot, \cdot)$  is identified from the winning price at both stages.*

**Proof** Let  $G_{(P_1, P_2)}(\cdot, \cdot)$  denote the CDF of the winning prices at both stages where  $P_i$  corresponds to the winning price at the  $i$ th stage, which is assumed to be known. Let  $G_{P_1}(\cdot)$  denote the marginal distribution of  $P_1$  and  $G_{P_2|P_1}(\cdot|\cdot)$  denote the

marginal distribution of  $P_2$  conditional on the realization of  $P_1$ . Similar derivations as in section 3 leads to:

$$G_{P_1}(b_1) = \phi_{n-1,n}(F1(\beta^{-1}(b_1))) \quad \text{and} \quad (11)$$

$$G_{P_2|P_1}(b_2|b_1) = \begin{cases} \left[ \frac{F1(b_2)}{F1(\beta^{-1}(b_1))} \right]^{n-2} \cdot \left[ \int_{\beta^{-1}(b_1)}^{\bar{x}} F2(b_2|s) \cdot \frac{d[F1(s)]}{1-F1(\beta^{-1}(b_1))} \right] & \text{if } b_2 < \beta^{-1}(b_1) \\ 1 & \text{if } b_2 \geq \beta^{-1}(b_1) \end{cases} \quad (12)$$

for respectively the first and second stages and where  $\beta$  is given by equation (10). Note that  $G_{P_2|P_1}(b_2|b_1)$  has an atom at  $b_2 = \beta^{-1}(b_1)$  which corresponds to the event where the winner of the first stage also wins a unit at the second stage such that his highest opponent that fix the winning price remains the same. If  $\beta$  were known, then  $F1(\cdot)$  would be identified from equation (11) by

$$F1(x) = \phi_{n-1,n}^{-1}(F_{P_1}(\beta(x))).$$

Subsequently, we could also identify  $\int_{\beta^{-1}(b_1)}^{\bar{x}} F2(b_2|s) \cdot d[F1(s)]$  from equation (12) for any  $b_1, b_2$ . The derivation with respect to  $\beta^{-1}(b_1)$  would lead to the identification of  $F2(b_2|\beta^{-1}(b_1))f1(\beta^{-1}(b_1))$  and then to  $F2(\cdot|x_1)$  for any  $x_1 \in [\underline{x}, \bar{x}]$  such that  $f1(x_1) > 0$ . Since  $f(x_1, x_2) = f2(x_2|x_1)f1(x_1)$ ,  $f(\cdot, \cdot)$  would thus be identified and we would be done.

It remains to show that  $\beta$  is actually identified. For any  $b \in [\underline{x}, \beta(\bar{x})]$ ,  $\beta^{-1}(b)$  corresponds to the atom of the distribution  $F_{P_2|P_1}(\cdot|b)$  which has a unique atom as established by the expression (12) since  $F(\cdot, \cdot)$  is atomless.<sup>9</sup>  $\beta^{-1}$  is identified and thus  $\beta$ .<sup>10</sup>**Q.E.D.**

Contrary to B&BP, we consider in proposition 6.2 identification from the distribution of the bids at both stages and not solely from the one at the last stage. Under A1-A6 and A8,  $F(\cdot, \cdot)$  could not be identified from the winning price of the last stage:

<sup>9</sup> $\beta^{-1}(b)$  can be also uniquely characterized as the upper bound of the support of the distribution  $F_{P_2|P_1}(\cdot|b)$ . We put more emphasis on the ‘atom property’ since we conjecture that from a practitioner’s perspective it would help estimation.

<sup>10</sup>If the identities of the winners were observed, then  $\beta$  could be identified in a more direct way. In the events where the winner is the same in both stages, then heuristic M1 guarantees that the highest losing bidder should be the same in both stages: we obtain then that  $P_1 = \beta(P_2)$ . This observation can be of great help to enhance estimation.

any winning price distribution generated from a CDF  $F(.,.)$  satisfying assumption A3b can be alternatively viewed as coming from the model with symmetric draws from a common uni-dimensional distribution  $F(.)$  as under assumption A3.

## 6.2 Risk aversion

B&BP claim to abstract from the details of the equilibrium behavior, in particular by not imposing any risk neutrality assumption. Next proposition 6.3 points out an important issue if one wants to deal with risk aversion: the impossibility to assume a bidding behavior as heuristic M1 that would allow us to fix B&BP's analysis in the same way as we did in the present paper. We consider that bidders are potentially risk-averse with a von Neuman-Morgensten utility function  $U(.)$  satisfying the following assumption.

**A9.**  $U(.)$  is three times continuously differentiable and satisfies  $U'() > 0$ ,  $U''() < 0$  and  $U(0) = 0$ .

Under risk aversion, the generalized version of the first order condition (10), that any equilibrium candidate has to satisfy, is:

$$U(x - \beta(x)) = \int_x^x U(x - y) \frac{d[[F1(y)]^{n-2} F2(y|x)]}{[F1(x)]^{n-2}}. \quad (13)$$

Nevertheless, we face an important caveat in typical cases: the non-existence of a symmetric increasing pure strategy equilibrium function of the high valuation. A similar issue has been raised in two-stage sequential second price auctions with unit demand by McAfee and Vincent (1993). In our generalized affiliated multi-unit demand framework, a similar result holds as stated below while the proof of the argument is exactly the same as in McAfee and Vincent (1993) after noting that the first order condition (13) has a similar form as the one appearing in McAfee and Vincent (1993). The proof is thus straightforward from theirs and thus omitted.

**Proposition 6.3** *Assume A1-A5 and A9 and that valuations are private information. There exists a symmetric increasing pure strategy equilibrium bidding function of the high valuation  $\beta$  for every distribution  $F(.,.)$  if and only if  $U$  displays non-decreasing absolute risk aversion.<sup>11</sup> Moreover, if  $U$  displays decreasing absolute risk*

<sup>11</sup>The necessary part of this assertion holds also if we restrict ourselves to the (limited) multi-unit demand scheme under assumption A3.

aversion, then no symmetric increasing pure strategy equilibrium bidding function of the high valuation exists for any distribution  $F(.,.)$ .

## 7 Conclusion

B&BP claim a very strong identification result for bidders' valuations only from the last stage winning price distribution and without any assumption on the form of the information asymmetry, risk aversion and also whether agents are bidding according to some equilibrium criterium. On the contrary, we show that non-identification occurs very generally and also even if we assume standard informational asymmetry, risk neutrality and that bidders are playing Bayes Nash equilibrium. Then for identification and estimation purposes we have then limited the analysis to equilibria where bidders are bidding according to a strictly increasing function of their high valuation, the so-called equilibria under heuristic M1. We have also extended significantly B&BP's model by considering a richer sampling scheme for the valuations of a given bidder and for which we have shown that an equilibrium under heuristic M1 still exists. While it is an important departure from an underlying symmetry structure that was implicitly imposed in B&BP, our analysis relies on important restrictions: two-stage auctions and symmetric bidders. Outside this scope and as emphasized in the introduction, we know very few of the equilibrium set from a theoretical perspective. E.g. in two-stage auctions with asymmetric bidders, the assumption that bidders are bidding according to heuristic M1 (and thus symmetrically) in the first stage is not consistent with equilibrium behavior and would be thus an ad hoc assumption. On the whole, the general analysis of multi-stage auctions with asymmetric bidders is a challenging one that is left for further research.

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## A Appendix

### Proof of Proposition 3.1

We show that the derivatives of the polynomials  $\Psi_R$  and  $\Psi_{M1}$  are strictly positive on  $(0, 1)$ , which will guarantee that  $\Psi_i^{-1}$  is differentiable on  $(0, 1)$ . For  $\Psi_R$ , this has been already proved by Brendstrup (2007). We now consider heuristic M1 and work first conditional on  $u$  the highest high valuation among all bidders. From equation (2), we have  $g_{n-1,n}(x|u) > 0$  for any  $x$  on the interval  $(\underline{x}, u)$ . Since the density of the variable  $u$  is strictly positive on  $(\underline{x}, \bar{x})$ , we obtain finally after the

integration with respect to  $u$  that  $g_{n-1,n}(x) > 0$  for any  $x$  on the interval  $(x, \bar{x})$ . Since  $g_{n-1,n}(x) = \Psi'_{M1}(F(x)) \cdot f(x)$ , we obtain finally that  $\Psi'_{M1}(x) > 0$  on  $(0, 1)$ .

A straightforward factorization leads to  $\Psi_{M1}[X] - \Psi_R[X] = \frac{2(n-1)(2n-3)}{3} X^{(2n-3)} [1 - X]^2 \cdot [\frac{n-3}{2n-3} + X]$ . Between two roots, a polynomial has a constant sign. The root  $-\frac{n-3}{2n-3} \notin (0, 1)$ . We obtain finally that  $\Psi_{M1}[X] - \Psi_R[X] < 0$  on  $(0, 1)$  for  $n = 2$  while  $\Psi_{M1}[X] - \Psi_R[X] < 0$  on  $(0, 1)$  for  $n \geq 3$ .

### Proof of Proposition 3.4

For a symmetric equilibrium, let  $\beta(x_1, x_2)$  denote the (common) bidding function in the first stage where  $x_1$  and  $x_2$  denote respectively the high and the low valuations of the given bidder ( $x_1 \geq x_2$ ). Heuristic M1 is then equivalent to:  $\beta(x_1, x_2) = \beta(x_1, x_1)$  for any  $x_2 \leq x_1$  and  $x_1 \rightarrow \beta(x_1, x_1)$  being strictly increasing.

Under heuristic M1, the first unit is allocated to the bidder with the highest high valuation. In the second stage, the last item is allocated to the highest valuation among the remaining ones. On the whole the two units are allocated to the two highest valuations such that the final assignment is efficient.

It remains to show that if bidders do not follow heuristic M1 under a symmetric equilibrium, then efficiency fails in some events. First, bidders would not follow heuristic M1 if  $x_1 \rightarrow \beta(x_1, x_1)$  is not strictly increasing. In such a case, efficiency will obviously fail since a bidder may win the first auction while the efficient allocation consists in assigning the two units to one bidder with a strictly lowest valuation that bid either strictly more or with whom he is in tie.<sup>12</sup> Second, consider now the case where  $x_1 \rightarrow \beta(x_1, x_1)$  is strictly increasing but  $\beta(x_1, x_2) \neq \beta(x_1, x_1)$  for some  $x_2 < x_1$ . If  $\beta(x_1, x_2) \geq \beta(u, u)$  for some  $u > x_1$ , then inefficiency will occur in some events and were are done. If  $\beta(x_1, x_2) \leq \beta(u, u)$  for some  $u < x_1$ , then inefficiency will occur in some events if  $n \geq 3$  (consider the event where the agent with the pair of valuations  $(x_1, x_2)$  is the winning bidder while two bidders have the pair of valuations  $(u, u)$  while the remaining bidders have low valuations) and were are done. Consider then the remaining case where  $\beta(u_1, u_2) < \beta(x_1, x_2) < \beta(u'_1, u'_2)$  if  $u_1 < x_1 < u'_1$ . Since  $x_1 \rightarrow \beta(x_1, x_1)$  is strictly increasing, it is thus continuous almost everywhere. At a point  $x_1$  where it is continuous, then  $x_2 \rightarrow \beta(x_1, x_2)$  is constant and the first order condition implies that this constant should be equal to the equilibrium bid function

<sup>12</sup>We implicitly assume that the tie breaking rule does not depend on the valuations of the bidders but solely on their bids.



$\beta(x_1)$  as derived in Katzman (1999) and defined in eq. (3). Since  $u \rightarrow \beta(u)$  is continuous, we obtain finally that  $u \rightarrow \beta(u, u)$  is continuous in this remaining case such that heuristic M1 should hold in equilibrium which ends the proof.

**Remark** If  $n = 2$ , the equilibrium under heuristic M2 allocates also the items efficiently: first, if the bidder with the highest high valuation wins in the first stage then the same argument that has shown efficiency under heuristic M1 still guarantees efficiency; second, if the bidder with the highest high valuation does not win in the first stage, then it will surely win in the second stage while it could not have been strictly more efficient to give him both units since his low valuation has to be smaller than his opponent's low valuation and thus a fortiori than his opponent's high valuation.

### Proof of Proposition 6.1

Consider that all of  $i$ 's opponents are using a common bid function of the high valuation that is denoted  $\beta$ . Consider bidder  $i$  with the realized vector of valuations  $x = (x_1, x_2)$  and let  $V(T; x)$  denote bidder  $i$ 's expected payoff for the game given that he chooses to bid as if his high valuation  $x_1$  were equal to  $T$ . We consider three cases: case 1 where  $T = T_1 \geq x_1$ , case 2 where  $T = T_2 \in [x_2, x_1]$  and case 3 where  $T = T_3 \leq x_2$ .

$$\begin{aligned} V(T_1; x) &= \int_{\underline{x}}^{x_2} (x_1 + x_2 - \beta(x) - x) d[(F_1(x))^{n-1}] \\ &\quad + \int_{x_2}^{T_1} (x_1 - \beta(x)) d[(F_1(x))^{n-1}] \\ &\quad + \int_{T_1}^{\bar{x}} \int_{\underline{x}}^{x_1} (x_1 - s) \frac{d[(F_1(s))^{n-2} F_2(s|x)]}{(F_1(x))^{n-2}} d[(F_1(x))^{n-1}] \end{aligned}$$

The first term is the contribution to bidder  $i$ 's expected payoff of the case where the highest high valuation of  $i$ 's opponents is smaller than  $x_2$  such that he obtains both units. The second term corresponds to the case where this highest valuation lies between  $x_2$  and  $T_1$  such that he obtains one unit at the first stage and no unit at the second stage. The third term corresponds to the case where this highest valuation is above  $T_1$  such that he may obtain one unit but only at the second stage.

$$\begin{aligned}
 V(T_2; x) &= \int_{\underline{x}}^{x_2} (x_1 + x_2 - \beta(x) - x) d[(F1(x))^{n-1}] \\
 &+ \int_{x_2}^{T_2} (x_1 - \beta(x)) d[(F1(x))^{n-1}] \\
 &+ \int_{T_2}^{x_1} \int_{\underline{x}}^x (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s|x)]}{(F1(x))^{n-2}} d[(F1(x))^{n-1}] \\
 &+ \int_{x_1}^{\bar{x}} \int_{\underline{x}}^{x_1} (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s|x)]}{(F1(x))^{n-2}} d[(F1(x))^{n-1}]
 \end{aligned}$$

The first term is still the contribution to bidder  $i$ 's expected payoff of the case where the highest high valuation of  $i$ 's opponents is smaller than  $x_2$  such that he obtains both units. The second term corresponds to the case where this highest valuation lies between  $x_2$  and  $T_2$  such that bidder  $i$  obtains one unit at the first stage and no unit at the second stage. The third term corresponds to the case where this highest valuation lies between  $T_2$  and  $x_1$  such that he does not win the first auction but he surely obtains one unit at the second stage. The fourth term corresponds to the case where this highest valuation is above  $x_1$  such that he may obtain one unit but only at the second stage.

Taking the derivative of  $V(T_1; X)$  with respect to  $T_1$  and of  $V(T_2; X)$  with respect to  $T_2$  evaluated at  $T_1 = T_2 = x_1$  results in the necessary first order condition (10) that uniquely characterizes  $\beta(x)$ . Moreover,  $\beta(x)$  is actually strictly increasing in  $x$  since, from equation (10), it can be viewed as the mean of a variable that is distributed according to the CDF  $x \rightarrow \mathbf{1}[y \leq x] \cdot \frac{[F1(y)]^{n-2} F2(y|x)}{[F1(x)]^{n-2}}$ , an expression which is strictly decreasing in  $x$  as a corollary of lemma 6.1.

We then check that the candidate solution satisfies the global incentive compatibility conditions. For 'case 1 deviations', it is sufficient to check that

$$\begin{aligned}
 \frac{\partial V(T_1; x)}{\partial T_1} &= (n-1)[F1(T_1)]^{n-2} f1(T_1) \cdot \left( x_1 - \beta(T_1) - \int_{\underline{x}}^{x_1} (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s|T_1)]}{(F1(T_1))^{n-2}} \right) \\
 &= (n-1)[F1(T_1)]^{n-2} f1(T_1) \cdot \left[ \int_{x_1}^{T_1} (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s|T_1)]}{(F1(T_1))^{n-2}} \right] \\
 &\leq 0.
 \end{aligned}$$

For 'case 2 deviations', we can check that  $\frac{\partial V(T_2; X)}{\partial T_2} = 0$ , i.e., in equilibrium, bidders are indifferent between any bid in the interval  $[x_2, x_1]$ .

Finally we are left with ‘case 3 deviations’ where it is sufficient to check that  $\frac{\partial V(T_3; X)}{\partial T_3} \geq 0$ . The expression of the expected payoff with such deviations is given by

$$\begin{aligned} V(T_3; x) &= \int_{\underline{x}}^{T_3} (x_1 + x_2 - \beta(x) - x) d[(F1(x))^{n-1}] \\ &\quad + \int_{T_3}^{x_1} \int_{\underline{x}}^x (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s|x)]}{(F1(x))^{n-2}} d[(F1(x))^{n-1}] \\ &\quad + \int_{x_1}^{\bar{x}} \int_{\underline{x}}^{x_1} (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s|x)]}{(F1(x))^{n-2}} d[(F1(x))^{n-1}]. \end{aligned}$$

The partial derivative with respect to  $T_3$  is then

$$\begin{aligned} \frac{\partial V(T_3; x)}{\partial T_3} &= (n-1)[F1(T_3)]^{n-2} f1(T_3) \cdot (x_1 + x_2 - \beta(T_3) - T_3 \\ &\quad - \int_{\underline{x}}^{T_3} (x_1 - s) \frac{d[(F1(s))^{n-2} F2(s)]}{(F1(T_3))^{n-2} F2(T_3)}) \\ &= (n-1)[F1(T_3)]^{n-2} f1(T_3) \cdot (x_2 - T_3) \geq 0. \end{aligned}$$