The Average Tree Solution for Multi-choice Forest Games

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Abstract In this article we study cooperative multi-choice games with limited cooperation possibilities, represented by an undirected forest on the player set. Players in the game can cooperate if they are connected in the forest. We introduce a new (single-valued) solution concept which is a generalization of the average tree solution defined and characterized by Herings et al. [2008] for TU-games played on a forest. Our solution is characterized by component efficiency, component fairness and independence on the greatest activity level. It belongs to the precore of a restricted multi-choice game whenever the underlying multi-choice game is superadditive and isotone. We also link our solution with the hierarchical outcomes (Demange, 2004) of some particular TU-games played on trees. Finally, we propose two possible economic applications of our average tree solution.

Keywords Average tree solution · Communication graph · (pre-)Core · Hierarchical outcomes · Multi-choice games.

1 Introduction

In the classical cooperative game theory, it is assumed that each player has only two options concerning cooperation, being either active or inactive, and that any coalition of players may form. In order to model in a more accurate way real situations, several extensions of the classical cooperative game in characteristic function form (TU-game) have been proposed. We consider two of them in this article.

The first extension is the class of multi-choice games. Whereas in a TU-game each player is either active or inactive, in a multi-choice context each player may have additional participation opportunities in a finite set of activity levels. Several solutions for multi-choice games have been defined: Hsiao and Raghavan [1993] generalized the Shapley value from TU-games to multi-choice games in which all players have the same number of activity levels; van den Nouweland et al. [1995] and Grabisch and Xie [2007] generalized the core, the dominance core and the Weber set from TU-games to multi-choice games. Other Shapley-like values and their axiomatic

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characterizations can been found in Derks and Peters [1993], Klijn, Slikker and Zarzuelo [1999], Calvo and Santos [2000], Hwang and Liao [2008] and Grabisch and Lange [2007].

The second extension is the class of TU-games with restricted cooperation. In many situations the collection of possible coalitions is restricted by some social, hierarchical, economical, communicational or even technical structure. Examples are games with coalition structures where players meet together in coalitions and form a priori unions (a partition of the player set) or a coalition configuration. Another instance of TU-games with restricted cooperation are graph games in which players can cooperate only if they can communicate through a path in a communication graph. In games with permission structures, the relational structure is not a communication graph but an authority. Here the idea is that the players are part of a hierarchical organization. This implies that they need permission from certain other players before they are allowed to cooperate. For each class of these TU-games with restricted cooperation, the Shapley value and the core have been generalized. We refer the reader to the Aumann-Drèze value [1974] and the Owen value [1977] for games with a priori unions, to Albizuri, Aurrekoetxea and Zarzuelo [2006] for an extension of the Owen value for games with coalition configurations, to the Myerson value [1977] for graph games and to the permission value, due to Gilles et al. [1992], for games with permission structures.

Recently, Albizuri [2009] combined multi-choice games and games with coalition structures and proposed a solution that extends the Owen value [1977]. In this article, we pursue this line of research by combining multi-choice games and graph games. We obtain a solution that extends the average tree solution introduced by Herings et al. [2008] for forest games. Axiomatic characterizations of the average tree solution for forest games can be found in Herings et al. [2008], van den Brink [2009], Béal et al. [2010], Mishra and Talman [2010]. In order to obtain an average tree solution for multi-choice forest games we proceed as follows. First, we introduce three properties for a solution on this class of games: component efficiency, component fairness, and independence on the greatest effort. The first two properties generalize the properties introduced by Herings et al. [2008] from forest games to multi-choice forest games. Independence on the greatest effort states that the payoff variation of a player who does not choose his greatest activity level does not change if we increase the set of his or her activity levels. Combining component efficiency and component fairness, we obtain a set of solutions which distribute to each player choosing his greatest activity level the sum of two parts. One part distributes the worth of the component equally among its members when each one chooses his or her greatest activity level. The second part is determined link by link and can be viewed as a compensation scheme between the two players incident to a link.

Adding independence on the greatest effort to component efficiency and component fairness, we obtain a unique solution on the class of multi-choice forest games. Then, we give a closed expression for this solution. In order to do so, we introduce a restricted multi-choice game and hierarchical outcomes that generalize those introduced by Demange [2004] on spanning trees for tree-games. It turns out that the average of these extended hierarchical outcomes over the set of all possible spanning trees, called the average tree solution, satisfies component efficiency, component fairness and independence on the greatest effort. We also provide a representation of this average tree solution for multi-choice games in terms of an average of specific hierarchical outcomes for tree-games constructed from the underlying multi-choice forest game. That is, the average tree solution for the class of multi-choice forest games can be computed through hierarchical outcomes of specific tree-games.

Our last result establishes that if the underlying multi-choice game is superadditive and isotone, then our average tree solution belongs to the precore of the restricted game. We give a simple condition under which it belongs to the core of the restricted game. Finally our solution is applied to measure a kind of average marginal productivity of each unit of input in a model of production and to distribute the cartel's profit generated by each unit in the price of each firm in a cooperative version of Hotelling's oligopoly on a tree with fixed locations.

This article is organized as follows: Section 2 is a preliminary section containing concepts from multi-choice games and graph games, including the definition of the restricted game. In Section 3, we define the three properties for (single-valued) solutions on the class of multi-choice forest games and we initiate the axiomatic study. In Section 4, we give the expression of the unique solution satisfying these three properties. In Section 5, we show the average tree solution belongs to the pre-core of the restricted game provided that the underlying multi-choice games is superadditive and isotone. In Section 6, we discuss the above-mentioned applications.

2 Preliminaries

Notations

We denote by $|A|$ the cardinality of a finite set A. Given $x \in \mathbb{R}^n_+$, $||x||$ denotes the Minkowski norm of order 1, i.e. $||x|| = \sum_{i=1}^n x_i$.

Multi-choice games

Let $N = \{1, 2, \ldots, n\}$ be a fixed and finite set of players of size $n \in \mathbb{N}$. In a multi-choice (cooperative) game, each player $i \in N$ has a finite number of activity levels at which he or she can choose to play. The worth that the set of players N can obtain depends on the activity of the cooperative players. We set $A_i = \{0, 1, \ldots, m_i\}$ as the finite set of activity levels of player $i \in N$, where action 0 means that player i does not participate. Let A_S be the product set $\prod_{i \in S} A_i$, $S \subseteq N$, $S \neq \emptyset$. Elements a of A_N are called **coalitions**. We will use sometimes the notation $(a_S, a_{N\setminus S}) \in A_S \times A_{N\setminus S}$ instead of $a \in A_N$ in order to insist on the activity levels played by the sets of players S and $N\setminus S$. Denote by $\mathcal{S}(a)$ the **support** of coalition $a \in A_N$, i.e. $\mathcal{S}(a) = \{i \in N : a_i > 0\}$. The set of coalitions A_N endowed with the usual binary relation $\geq$ on $\mathbb{R}^n$ induced a (complete) lattice with greatest element $m = (m_1, \ldots, m_n)$ and least element $0_N = (0, 0, \ldots, 0)$. For any two coalitions a and b of A_N, $a \vee b$ and $a \wedge b$ denote their least upper bound and their greatest lower bound respectively. A **characteristic function** is a real-valued function $v : A_N \longrightarrow \mathbb{R}$ which assigns to every coalition $a = (a_1, \ldots, a_n) \in A_N$ the worth that the players can obtain when each player $i \in N$ plays at activity level $a_i \in A_i$. By convention, the worth $v(0_N)$ of the null coalition 0_N is set to zero. A **multi-choice game** on N is given by a pair (A_N, v) where A_N is the set of coalitions and v the characteristic function. In case $A_i = \{0, 1\}$ for each $i \in N$, $(A_N, \geq)$ is isomorphic to $(2^N, \supseteq)$, where 2^N stands for the set consisting of all subsets of N, so that (A_N, v) is a TU-game. Given a multi-choice game (A_N, v) and a coalition $b \in A_N$, $b \neq 0_N$, we write (A^b_N, v) the **multi-choice subgame** obtained from (A_N, v) by restricting v to the set $A^b_N = \{a \in A_N : a_i \leq b_i, i \in N\}$, a principal of A_N.

A multi-choice game (A_N, v) is **isotone** if $a \geq b$ implies $v(a) \geq v(b)$. It is **superadditive** if for each pair of coalitions a and b of A_N such that $a \wedge b = 0_N$, it holds that $v(a \vee b) \geq v(a) + v(b)$. It is **supermodular** if for each pair of coalitions a and b of A_N, it holds that $v(a \vee b) + v(a \wedge b) \geq v(a) + v(b)$.

Communication graphs

In this article, we study multi-choice games with limited cooperation possibilities, represented by an acyclic communication graph. A **communication graph** on the player set N is a pair (N, L) where the player set N represents the nodes of the graph and L is a subset of $N \times N$. Each pair $\{i, j\} \in L$ represents a communication link between player i and player j. In order to save notations, we write ij instead of $\{i, j\}$ to refer to a link. For each $i \in N$, the set L_i denotes the set of neighbors of i in (N, L), i.e the set of $j \in N$ such that $ij \in L$. A finite sequence of distinct players $(i_1, i_2, \ldots, i_k)$ is a **path** in (N, L) if $i_q i_{q+1} \in L$ for each $q \in \{1, \ldots, k-1\}$. A **cycle** in (N, L) is a finite sequence of players $(i_1, i_2, \ldots i_{k+1})$ such that $i_1 = i_{k+1}$, all players $i_1, i_2, \ldots, i_k$ are different elements of N and $i_q i_{q+1} \in L$ for each $q \in \{1, \ldots, k\}$. In case $k = 1$, the cycle (i_1, i_2) is called a loop. A communication graph (N, L) is **acyclic** if it does not contain any cycle. For each subset of players $C \subseteq N$, the subset of links $L(C) \subseteq L$ represents the set of communication links between players in C, and the pair $(C, L(C))$ is the **subgraph** of (N, L) induced by C. A subgraph $(C, L(C))$ of a graph (N, L) is **connected** if there exists at least one path between any pair of distinct players in C. The subgraph $(C, L(C))$ is maximally connected if it connected and for each $i \in N \backslash C$ the subgraph $(C \cup \{i\}, L(C \cup \{i\}))$ of (N, L) is not connected. If $(C, L(C))$ is a maximal connected subgraph of (N, L), C is a **component** of (N, L). Denote by N/L the set of components of (N, L) and by $C/L(C)$ the set of components of the subgraph $(C, L(C))$ of (N, L) induced by $C \subseteq N$. Note that if (N, L) is acyclic, then there is exactly one path between any pair of distinct players who belong to the same component. An acyclic graph is called a **forest**. If the forest has exactly one component, it is called a **tree**. Therefore, each component of a forest induces a tree. It follows from these remarks that the class of acyclic graphs on N is closed under link deletion, i.e. if (N, L) is a forest or a tree, then the graph $(N, L \backslash ij)$ for $ij \in L$ is a forest on N.

An **orientation** of a graph (N, L) is a directed graph obtained from (N, L) by replacing each link ij by either the directed link (i, j) or the directed link (j, i). Given a forest (N, L), a **rooted tree** t_r^C on the subgraph $(C, L(C))$ is an orientation that arises from a component $C \in N/L$ by selecting player $r \in C$, called the **root**, and directing all links of $L(C)$ away from the root r. Because r belongs to exactly one component of (N, L), we will use the notation t_r instead of t_r^C when no confusion arises. Each agent $r \in N$ is the root of exactly one rooted tree t_r. Note also that for any rooted tree t_r on (N, L), any agent $k \in C \backslash \{r\}$, there is exactly one directed link (j, k); agent j is the unique **predecessor** of k and k is a **successor** of j in t_r. Denote by $s_r(j)$ the possibly empty set of successors of player j in t_r. A player k is a **subordinate** of j in t_r if there is a directed path from j to k, i.e. if there is a sequence of distinct agents $(i_1, i_2, \ldots, i_p)$ such that $i_1 = j$, $i_p = k$ and for each $q \in \{1, 2, \ldots, p-1\}$, $i_{q+1} \in s_r(i_q)$. The set $S_r(j)$ denotes the union of the set of all subordinates of j in t_r and $\{j\}$. So, we have $s_r(j) \subseteq S_r(j) \backslash \{j\}$. A rooted tree reflects the idea that two players incident to a communication link do not have equal access or control to that link.

Multi-choice forest games

The combination of a multi-choice game and a communication graph on N results in a multi-choice graph game given by a triple (A_N, v, L), where A_N is the set of coalitions, v the characteristic function and L the set of communication links of the graph (N, L). Let us denote by $\mathcal{C}_N$ the set of all **multi-choice forest games** (A_N, v, L) on N and by $\mathcal{C}_N^* \subseteq \mathcal{C}_N$ the set of **multi-choice tree games** (A_N, v, L) on N. If (A_N, v) is a TU-game, then we say the combination of a TU-game and a communication graph results in a **graph game**.

In a multi-choice graph game (A_N, v, L), the members of the support of coalition $a \in A_N$ can cooperate and earn their worth $v(a)$ if they can communicate through the links of the communication graph (N, L), i.e if $(\mathcal{S}(a), L(\mathcal{S}(a))$ is a connected subgraph of (N, L). If $\mathcal{S}(a)$ does not induce a connected subgraph, we follow Myerson [1977] and assume that players in $\mathcal{S}(a)$ can only realize the sum of the worths of the components on the subgraph $(\mathcal{S}(a), L(\mathcal{S}(a)))$. This yields a **restricted multi-choice game** (A_N, v^L) define as:

$$\forall a \in A_N, \quad v^L(a) = \sum_{C \in \mathcal{S}(a)/L(\mathcal{S}(a))} v(a_C, 0_{N\setminus C}).$$

In case $A_N = \{0,1\}^n$, the restricted multi-choice game (A_N, v^L) coincides with the restricted game introduced by Myerson [1977] for graph games.

Solution concepts for multi-choice forest games

In multi-choice (forest) games, a payoff vector is defined for any player participating at any level. Given $(A_N, v, L) \in \mathcal{C}_N$, we introduce the set V consisting of all pairs (i, k) where $i \in N$ and $k \in A_i$. A **payoff vector** x is an element of $\mathbb{R}^V$. Each coordinate $x_{i,k} \in \mathbb{R}$ of x represents the payoff variation to player i corresponding to an increase of activity from level $k-1$ to k, and $x_{i,0} = 0$ for each $i \in N$. Therefore, if a player i works at level $k \in A_i$, then he or she obtains, according to x, the amount

$$X_{i,k} = \sum_{q=0}^{k} x_{i,q}.$$

A single-valued solution on $\mathcal{C}_N$, also called an **allocation rule**, is a function $f : \mathcal{C}_N \longrightarrow \mathbb{R}^V$ which assigns to each multi-choice forest game $(A_N, v, L) \in \mathcal{C}_N$ a payoff vector $f(A_N, v, L) \in \mathbb{R}^V$. As above, $F_{i,k}(A_N, v, L)$ denotes the accumulated payoffs of player i when he plays $k \in A_i$ in (A_N, v, L). For convenience, $F^S(A_N, v, L)$ denotes the sum of accumulated payoffs received by the players in $S \subseteq N$ when each one chooses its greatest activity level $m_i \in A_i$, $i \in S$, i.e.

$$F^S(A_N, v, L) = \sum_{i \in S} F_{i,m_i}(A_N, v, L).$$

In a multi-choice forest games $(A_N, v, L) \in \mathcal{C}_N$, a payoff vector $x \in \mathbb{R}^V$ is **efficient** if for it holds that

$$\sum_{i \in N} X_{i,m_i} = v^L(m).$$

A payoff vector $x \in \mathbb{R}^V$ is **level-increased rational** if for all $i \in N$ and all $k \in A_i$, it holds that

$$x_{i,k} \geq v^L(ke^i) - v^L((k-1)e^i),$$

where e^i is the vector in $\mathbb{R}^n$ such that $e^i_j = 0$ if $j \in N\setminus\{i\}$, and $e^i_j = 1$ if $j = i$. This means that $x_{i,k}$ is at least the increase in worth that player i can obtain when he or she is the only one to participate and changes his or her activity from level $k-1$ to k. A payoff vector is an **imputation** if it is efficient and level-increased rational. A payoff vector is x **acceptable** if for each coalition $a \in A_N$, it holds that

$$\sum_{i \in N} X_{i,a_i} \geq v^L(a).$$

The **precore** of $(A_N, v, L) \in \mathcal{C}_N$, denoted by $\mathcal{PK}(A_N, v, L)$, is the set of efficient and acceptable payoff vectors. Note that $\mathcal{PK}(A_N, v, L)$ is a convex set. The **core** of $(A_N, v, L) \in \mathcal{C}_N$, denoted by $\mathcal{K}(A_N, v, L) \subseteq \mathcal{PK}(A_N, v, L)$, is the set of acceptable imputations.

3 Properties for allocation rules

In this section we introduce three properties for allocation rules on $\mathcal{C}_N$. The first one is a generalization of the property of component efficiency from graph games to the class of multi-choice forest games. Component efficiency for graph games is itself a generalization of the property of efficiency from TU-games to graph games. It has been introduced by Myerson [1977] in order to characterize, together with a property of fairness, the so-called Myerson value.

Component efficiency. An allocation rule f on $\mathcal{C}_N$ is component efficient if for each $(A_N, v, L) \in \mathcal{C}_N$ and for each component C of (N, L) it holds that

$$F^C(A_N, v, L) = v(m_C, 0_{N \setminus C}).$$

This property means that the worth of coalition $(m_C, 0_{N \setminus C}) \in A_C \times A_{N \setminus C}$, where only the players of the component C participate and play their greatest activity level, is totally redistributed among themselves. In case (N, L) is a tree, i.e. the only component is N, this property is similar to the property of efficiency introduced by Hsiao and Rhagavan [1993] and others to characterize several extensions of the Shapley value [1953] from TU-games to multi-choice games. In such a special case, component efficiency becomes $F^N(A_N, v, L) = v(m)$.

The second property is a generalization of the property of component fairness from forest games to multi-choice forest games. Component fairness has been introduced by Herings et al. [2008] for the class of forest games. It says that deleting a link between two players yields for both resulting components the same average change in payoff, where the average is taken over the players in the component. To extend this property to the class of multi-choice games, we need the following definition. Given a forest (N, L) and a component C of (N, L), the subgraph $(C, L(C))$ is a tree. Therefore, deleting a link $ij \in L(C)$ generates a bipartition $\{K_{ij}, K_{ji}\}$ of C. The subset K_{ij} of C is the element of the bipartition containing player i and K_{ji} is the element of the bipartition containing player j. It follows that K_{ij} and K_{ji} constitute the two new components of the forest $(N, L \setminus ij)$. We say that the subsets K_{ij} and K_{ji} are the two **cones** of the forest induced by the link $ij \in L(C)$. It follows that each component of size $|C|$ possesses $2|C|-2$ cones.

Component fairness. An allocation rule f on $\mathcal{C}_N$ satisfies component fairness if for each $(A_N, v, L) \in \mathcal{C}_N$ and each link $ij \in L$, it holds that

$$\frac{F^{K_{ij}}(A_N, v, L) - F^{K_{ij}}(A_N, v, L \setminus ij)}{|K_{ij}|} = \frac{F^{K_{ji}}(A_N, v, L) - F^{K_{ji}}(A_N, v, L \setminus ij)}{|K_{ji}|}.$$

Component fairness for multi-choice forest games indicates that deleting a link between two players yields for both resulting components K_{ij} and K_{ji} of the forest $(N, L\backslash ij)$ the same average change in accumulated payoffs when each player chooses its greatest activity level, where the average is taken over the players in the component. On the class of forest games, Herings, et al. [2008] show that the combination of component efficiency and component fairness generates a unique allocation rules. This uniqueness result is no longer true on the class of multi-choice forest games since these two properties only concern accumulated payoffs. We thus introduce a new property, called independence on the greatest activity level, which states that the increase in payoff to player i corresponding to a change of activity from level $k-1$ to k, where $k \neq \{0, m_i\}$, does not depend on his greatest activity level m_i.

Independence on the greatest activity level. An allocation rule f on $\mathcal{C}_N$ satisfies independence on the greatest activity level if for each $(A_N, v, L) \in \mathcal{C}_N$ such that there is $i \in N$ with $m_i \geq 2$, it holds that for each $k \in \{1, \dots, m_i - 1\}$,

$$f_{i,k}(A_N, v, L) = f_{i,k}\big(A_N^{(m_i-1, m_{-i})}, v, L\big).$$

Remark: we could instead assume independence on the greater activity levels without modifying the results obtained in this article.

We now describe the solutions satisfying different combinations of the above list of properties. The first proposition states that if an allocation rule on the set of multi-choice forest games satisfies component efficiency and component fairness, then the accumulated payoffs of the members of a cone is the sum of two parts. One part is the worth of the coalition formed by the players of this cone when they play their greatest activity level and the second part is a share of a surplus generated by the deletion of the link when players of the component choose their greatest activity level. This share is precisely the relative size of this cone. Reciprocally, if the accumulated payoffs of the member of each cone is allocated as above, then the allocation rule satisfies component efficiency and component fairness.

Proposition 3.1 *An allocation rule f on $\mathcal{C}_N$ satisfies component efficiency and component fairness if and only if for each $(A_N, v, L) \in \mathcal{C}_N$, each component $C \in N/L$, each link $ij \in L(C)$, the sum $F^{K_{ij}}(A_N, v, L)$ of accumulated payoffs in K_{ij} is equal to*

$$v(m_{K_{ij}}, 0_{N\backslash K_{ij}}) + \frac{|K_{ij}|}{|C|}\big(v(m_C, 0_{N\backslash C}) - v(m_{K_{ij}}, 0_{N\backslash K_{ij}}) - v(m_{K_{ji}}, 0_{N\backslash K_{ji}})\big) \qquad (3.1)$$

Proof. Assume that f satisfies component efficiency and component fairness on $\mathcal{C}_N$. Pick any multi-choice forest game $(A_N, v, L) \in \mathcal{C}_N$, any component C of (N, L), and delete any link $ij \in L(C)$. By definition $|C| = |K_{ij}| + |K_{ji}|$ and by component efficiency of f, we get:

$$F^{K_{ij}}(A_N, v, L\backslash ij) = v(m_{K_{ij}}, 0_{N\backslash K_{ij}}) \text{ and } F^{K_{ji}}(A_N, v, L\backslash ij) = v(m_{K_{ji}}, 0_{N\backslash K_{ji}}) \qquad (3.2)$$

and

$$F^{K_{ij}}(A_N, v, L) + F^{K_{ji}}(A_N, v, L) = v(m_C, 0_{N\backslash C}) \qquad (3.3)$$

Because $|K_{ji}| = |C| - |K_{ij}|$, component fairness can be rewritten as follows:

$$F^{K_{ij}}(A_N, v, L) - F^{K_{ij}}(A_N, v, L\backslash ij) = \frac{|K_{ij}|\big(F^{K_{ji}}(A_N, v, L) - F^{K_{ji}}(A_N, v, L\backslash ij))\big)}{|C| - |K_{ij}|}$$

Using (3.2) and (3.3) we deduce that

$$F^{K_{ij}}(A_N, v, L) - v(m_{K_{ij}}, 0_{N\setminus K_{ij}}) = \frac{|K_{ij}|\big(v(m_C, 0_{N\setminus C}) - F^{K_{ij}}(A_N, v, L) - v(m_{K_{ji}}, 0_{N\setminus K_{ji}})\big)}{|C| - |K_{ij}|},$$

which is equivalent to write that $F^{K_{ij}}(A_N, v, L)$ is equal to (3.1).

Reciprocally, assume that for each $(A_N, v, L) \in \mathcal{C}_N$, each component C, each link $ij \in L(C)$, the accumulated payoffs $F^{K_{ij}}(A_N, v, L)$ in K_{ij} is equal to (3.1). To show: f satisfies component efficiency and component fairness on $\mathcal{C}_N$. Pick any $(A_N, v, L) \in \mathcal{C}_N$, any component C and any link $ij \in L(C)$. By (3.1), we have

$$F^{K_{ij}}(A_N, v, L) + F^{K_{ji}}(A_N, v, L) = F^C(A_N, v, L) = v(m_C, 0_{N\setminus C}),$$

which proves that f satisfies component efficiency on $\mathcal{C}_N$.

Next, by component efficiency and by (3.1), the variation of accumulated payoffs $F^{K_{ij}}(A_N, v, L) - F^{K_{ij}}(A_N, v, L\setminus ij)$ is equal to

$$\frac{|K_{ij}|}{|C|}\big(v(m_C, 0_{N\setminus C}) - v(m_{K_{ij}}, 0_{N\setminus K_{ij}}) - v(m_{K_{ji}}, 0_{N\setminus K_{ji}})\big).$$

In the same way, $F^{K_{ji}}(A_N, v, L) - F^{K_{ji}}(A_N, v, L\setminus ij)$ is equal to

$$\frac{|K_{ji}|}{|C|}\big(v(m_C, 0_{N\setminus C}) - v(m_{K_{ij}}, 0_{N\setminus K_{ij}}) - v(m_{K_{ji}}, 0_{N\setminus K_{ji}})\big).$$

It follows that

$$\frac{F^{K_{ij}}(A_N, v, L) - F^{K_{ij}}(A_N, v, L\setminus ij)}{|K_{ij}|} = \frac{F^{K_{ji}}(A_N, v, L) - F^{K_{ji}}(A_N, v, L\setminus ij)}{|K_{ji}|},$$

which proves that f satisfies component fairness on $\mathcal{C}_N$. □

The following proposition provides a geometric interpretation of the accumulated payoffs received by each player when the allocation rule satisfies component efficiency and component fairness.

Proposition 3.2 *An allocation rule f on $\mathcal{C}_N$ satisfies component efficiency and component fairness if and only if for each $(A_N, v, L) \in \mathcal{C}_N$, each component $C \in N/L$, the accumulated payoffs $F^{\{i\}}(A_N, v, L)$ of each $i \in C$ is equal to*

$$\frac{v(m_C, 0_{N\setminus C})}{C} + \sum_{j\in L_i} \det M_{ij}^{(A_N, v, L)} \tag{3.4}$$

where $\det M_{ij}^{(A_N,v,L)}$ is the determinant of the 2×2 matrix $M_{ij}^{(A_N,v,L)}$ defined as

$$M_{ij}^{(A_N,v,L)} = \begin{pmatrix} v(m_{K_{ij}}, 0_{N\setminus K_{ij}}) & \dfrac{|K_{ij}|}{|C|} \\ \\ v(m_{K_{ji}}, 0_{N\setminus K_{ji}}) & \dfrac{|K_{ji}|}{|C|} \end{pmatrix}.$$

Before proceeding to the proof of Proposition 3.2, it is instructive to consider the following interpretation of the allocation given by (3.4). First, the accumulated payoffs of each player is the sum of two parts. The first part distributes the worth of the coalition $(m_C, 0_{N\setminus C})$ associated with the component C equally among its members and the second part is computed link by link. Secondly, each determinant $\det M_{ij}^{(A_N,v,L)}$, $j \in L_i$, can be viewed as the oriented area of the parallelogram with vertices at $(0,0)$, $(v(m_{K_{ij}}, 0_{N\setminus K_{ij}}), v(m_{K_{ji}}, 0_{N\setminus K_{ji}}))$, $(|K_{ij}|/|C|, |K_{ji}|/|C|)$ and $(v(m_{K_{ij}}, 0_{N\setminus K_{ji}}) + |K_{ij}|/|C|, v(m_{K_{ji}}, 0_{N\setminus K_{ji}}) + |K_{ji}|/|C|)$. The oriented area is the same as the usual area, except that it is negative when the vertices are listed in clockwise order. Note also that the allocation process associated with a link ij has the zero-sum property: agent i receives $\det M_{ij}^{(A_N,v,L)}$ and agent j receives exactly $-\det M_{ij}^{(A_N,v,L)}$. In order to interpret this situation, assume that the connected component C forms and that the members of C choose their greatest activity level. Because $L(C)$ is minimally connected all communication links are necessary to coordinate the actions inside C. Suppose that the link $ij \in L(C)$ is broken. How should i and j be compensated? The determinant $\det M_{ij}^{(A_N,v,L)}$ offers a compensation scheme between i and j whose geometric interpretation in the form of an orientated area is quite natural.

Proof. Pick any allocation rule f that satisfies component efficiency and component fairness on $\mathcal{C}_N$. Pick any $(A_N, v, L) \in \mathcal{C}_N$ and consider any component C of (N, L) and $i \in C$. First note that

$$\bigcup_{j\in L_i} K_{ji} = C\setminus\{i\} \tag{3.5}$$

where the collection of cones $K_{ji}, j \in L_i$ are pairwise disjoints. Therefore, by component efficiency we have:

$$F^{\{i\}}(A_N, v, L) = v(m_C, 0_{N\setminus C}) - \sum_{j\in L_i} F^{K_{ji}}(A_N, v, L) \tag{3.6}$$

By Proposition 3.1, we know that $F^{K_{ji}}(A_N, v, L)$, $j \in L_i$, is equal to

$$v(m_{K_{ji}}, 0_{N\setminus K_{ji}}) + \frac{|K_{ji}|}{|C|}\big(v(m_C, 0_{N\setminus C}) - v(m_{K_{ji}}, 0_{N\setminus K_{ji}}) - v(m_{K_{ij}}, 0_{N\setminus K_{ij}})\big)$$

or equivalently

$$\frac{|K_{ji}|}{|C|}v(m_C, 0_{N\setminus C}) - \frac{|K_{ji}|}{|C|}v(m_{K_{ij}}, 0_{N\setminus K_{ij}}) + \frac{|K_{ij}|}{|C|}v(m_{K_{ji}}, 0_{N\setminus K_{ji}}).$$

Putting this expression in (3.6) and taking into account (3.5), we obtain that $F^{\{i\}}(A_N, v, L)$ is equal to

$$\frac{v(m_C, 0_{N\setminus C})}{|C|} + \sum_{j\in L_i}\bigg(\frac{|K_{ji}|}{|C|}v(m_{K_{ij}}, 0_{N\setminus K_{ij}}) - \frac{|K_{ij}|}{|C|}v(m_{K_{ji}}, 0_{N\setminus K_{ji}})\bigg),$$

which is precisely (3.4).

Reciprocally, assume that for each $(A_N, v, L) \in \mathcal{C}_N$ and each $i \in N$, $F^{\{i\}}(A_N, v, L)$ is given by (3.4). Pick any $j \in L_i$ and the induced cone K_{ji}. By (3.4), we have:

$$F^{K_{ji}}(A_N, v, L) = \frac{|K_{ji}|}{|C|}v(m_C, 0_{N\setminus C}) + \sum_{k\in K_{ji}}\sum_{q\in L_k} \det M_{kq}^{(A_N,v,L)} \tag{3.7}$$

For each pair ordered pair $(k,q) \in K_{ji} \times L_k$ such that $\{q,k\} \subseteq K_{ji}$, we have $(q,k) \in K_{ji} \times L_q$ and

$$\det M_{kq}^{(A_N,v,L)} + \det M_{qk}^{(A_N,v,L)} = 0.$$

The only ordered pair for which this relation is not true is the pair $(j,i) \in K_{ji} \times L_j$ since $i \notin K_{ji}$. It follows that (3.7) reduces to

$$F^{K_{ji}}(A_N,v,L) = \frac{|K_{ji}|}{|C|} v(m_C, 0_{N\setminus C}) + \det M_{ji}^{(A_N,v,L)}.$$

The right hand side of this equality can be rewritten as:

$$v(m_{K_{ji}}, 0_{N\setminus K_{ji}}) + \frac{|K_{ji}|}{|C|}\big(v(m_C, 0_{N\setminus C}) - v(m_{K_{ji}}, 0_{N\setminus K_{ji}}) - v(m_{K_{ij}}, 0_{N\setminus K_{ij}})\big) \tag{3.8}$$

Since the equality between $F^{K_{ji}}(A_N,v,L)$ and (3.8) is true for each component C and each $ij \in L(C)$, we conclude that f satisfies component efficiency and component fairness by an application of Proposition 3.1. $\square$

We now state the main result of this section, which indicates that there is a unique allocation rule on the class of multi-choice forest games that satisfies component efficiency, component fairness and independence on the greatest activity level. In order to complete the proof of this result, it is useful to define for each $k \geq n$, the subclass $\mathcal{C}_N^k$ of multi-choice forest games $(A_N,v,L) \in \mathcal{C}_N$ such that $n \leq ||m|| \leq k$. Remark that $\mathcal{C}_N^k \subseteq \mathcal{C}_N^{k+1}$ and that

$$\mathcal{C}_N = \bigcup_{k\geq n} \mathcal{C}_N^k.$$

Notice also that Propositions 3.1 and 3.2 can be applied on each $\mathcal{C}_N^k$.

Proposition 3.3 *On the class of multi-choice forest games $\mathcal{C}_N$, there is a unique allocation rule that satisfies component efficiency, component fairness and independence on the greatest activity level.*

Proof. Choose any allocation rule f on $\mathcal{C}_N$ that satisfies component efficiency, component fairness and independence on the greatest activity level. The proof is by induction on $||m||$.
INITIAL STEP: Pick any $(A_N,v,L) \in \mathcal{C}_N$ such that $||m|| = n$. In such a case, v is the characteristic function of the TU-game (N,v) and (A_N,v,L) is the forest game (N,v,L). On the class of forest games, only component efficiency and component fairness can be used. Note also that component efficiency and component fairness reduce to the corresponding properties introduced in Herings et al. [2008]. By Theorem 3.4 in Herings et al. [2008], there is a unique allocation rule that satisfies component efficiency and component fairness on the class of forest games.
INDUCTION HYPOTHESIS: Assume that there is a unique allocation rule f that satisfies component efficiency, component fairness and independence on the greatest activity level on $\mathcal{C}_N^k$ for some $k > n$.
INDUCTION STEP: Pick any $(A_N,v,L) \in \mathcal{C}_N^{k+1}$ such that $||m|| = k+1$. Because $k+1 > n$, there necessarily exists a player with at least two activity levels. Set $Q^{\{i\}}(A_N,v,L)$ equals to the expression (3.4) found in Proposition 3.2. Next, for each ordered pair $(i,k) \in N \times A_i$, define $f_{i,k}(A_N,v,L)$ as follows:

1. If $m_i = 1$, then $f_{i,1}(A_N, v, L) = Q^{\{i\}}(A_N, v, L)$;

2. If $m_i > 1$, then

   (a) $\forall k \in \{1, 2, \ldots, m_i - 1\}$, $f_{i,k}(A_N, v, L) = f_{i,k}(A_N^{(m_i-1, m_{-i})}, v, L)$;

   (b) $f_{i,m_i}(A_N, v, L) = Q^{\{i\}}(A_N, v, L) - F^{\{i\}}(A_N^{(m_i-1, m_{-i})}, v, L)$.

By construction, for each $i \in N$, we have $F^{\{i\}}(A_N, v, L) = Q^{\{i\}}(A_N, v, L)$. By the induction hypothesis and Proposition 3.2, f satisfies component efficiency and component fairness on $\mathcal{C}_N^{k+1}$. By construction and by the induction hypothesis f satisfies also independence on the greatest activity level. Therefore, we have shown that there is an allocation rule f that satisfies component efficiency, component fairness and independence on the greatest activity level on $\mathcal{C}_N^{k+1}$. Next, assume that there is another allocation rule g that satisfies component efficiency, component fairness and independence on the greatest activity level on the class $\mathcal{C}_N^{k+1}$. Pick any multi-choice forest games $(A_N, v, L) \in \mathcal{C}_N^{k+1}$. By Proposition 3.2, we obtain that for each $i \in N$, $F^{\{i\}}(A_N, v, L) = G^{\{i\}}(A_N, v, L)$. By the induction hypothesis and independence on the greatest activity level, for each $(i, k) \in N \times A_i$ such that $1 \leq k < m_i$ we have

$$g_{i,k}(A_N, v, L) = g_{i,k}(A_N^{(m_i-1, m_{-i})}, v, L) = f_{i,k}(A_N^{(m_i-1, m_{-i})}, v, L) = f_{i,k}(A_N, v, L).$$

Because $F^{\{i\}}(A_N, v, L) = G^{\{i\}}(A_N, v, L)$ for each $i \in N$, we necessarily have $g_{i,m_i}(A_N, v, L) = f_{i,m_i}(A_N, v, L)$ for each $i \in N$. Therefore, $g(A_N, v, L) = f(A_N, v, L)$, which proves that f is uniquely determined on $\mathcal{C}_N^{k+1}$. $\square$

4 The average tree solution

Having proved that component efficiency, component fairness and independence on the greatest activity level uniquely determine an allocation rule on $\mathcal{C}_N$, we are now going to give a closed expression for this solution.

Consider a multi-choice forest game $(A_N, v, L) \in \mathcal{C}_N$, and a root tree t_r on the subgraph $(C, L(C))$ induced by the component $C \in N/L$. For each $(i, k) \in V$ define the coalition $a^{r,(i,k)} \in A_N$ with respect to the root r as follows:

$$a_j^{r,(i,k)} = \begin{cases} k & \text{if } j = i, \\ m_j & \text{if } j \in S_r(i) \backslash \{i\}, \\ 0 & \text{otherwise.} \end{cases}$$

The **hierarchical outcome** $h^r(A_N, v, L) \in \mathbb{R}^V$ with respect to $r \in C$ is the payoff vector defined as:

$$h_{i,k}^r(A_N, v, L) = \begin{cases} v^L\big(a^{r,(i,k)}\big) - v^L\big(a^{r,(i,k-1)}\big) & \text{if } i \in C,\ k \neq 0, \\ 0 & \text{otherwise.} \end{cases}$$

The payoff $h_{i,k}^r(A_N, v, L)$ to player $i \in C$ choosing the activity level $k \neq 0$ is a marginal contribution vector in the sense it is equal to the worth with respect to v^L of the coalition whose support contains the player i choosing the activity level k and all the subordinates of player i in t_r when each one chooses his greatest activity level minus the worth of the same coalition except that player i chooses the activity level $k-1$ instead of k. In case $A_N = \{0,1\}^n$, i.e. (A_N, v, L) is a forest game, $h^r(A_N, v, L)$ coincides with the hierarchical outcome defined by Demange [2004]. See also Khmelnitskaya [2010] and van den Brink [2009] for axiomatic characterizations of the hierarchical outcome in forest games. Here is a difference with the class of forest games. If $k \neq 1$, the support of $a^{r,(i,k-1)}$ is equal to the support of $a^{r,(i,k)}$ and so the coalition $a^{r,(i,k)}$ does not split into several components. If $k = 1$, then i no longer belongs to the support of the coalition $a^{r,(i,0)}$. This support is constituted by the groups of players $S_r(j)$, $j \in s_r(i)$, and so

$$v^L\big(a^{r,(i,0)}\big) = \sum_{j \in s^r(i)} v\big(a^{r,(i,0)}_{S^r(j)}, 0_{N \setminus S^r(j)}\big).$$

Herings, et al. [2008] extend Demange's hierarchical outcome on forest games by considering the average of the hierarchical outcomes, where the average is taken over the set of rooted trees that we can create on each component of the forest. The authors show that the average of the hierarchical outcomes, also called the average tree solution, is the unique solution that satisfies component efficiency and component fairness. We propose an extension of this definition from forest game to multi-choice forest games.

On the class of multi-choice forest games $\mathcal{C}_N$, the **average tree solution** AT assigns to any $(A_N, v, L) \in \mathcal{C}_N$ the payoff vector $\mathsf{AT}(A_N, v, L) \in \mathbb{R}^V$ defined as follows: for each $C \in N/L$, each $(i,k) \in C \times A_i$,

$$\mathsf{AT}_{i,k}(A_N, v, L) = \frac{1}{|C|} \sum_{r \in C} h^r_{i,k}(A_N, v, L),$$

i.e. player i choosing the activity level k receives the average of his marginal contributions over the $|C|$ rooted trees t_r in $(C, L(C))$.

Proposition 4.1 *On the class of multi-choice forest games $\mathcal{C}_N$, the average tree solution satisfies component efficiency, component fairness and independence on the greatest activity level.*

Proof. Pick any $(A_N, v, L) \in \mathcal{C}_N$ any component $C \in N/L$ and any root $r \in C$. We have

$$\begin{aligned}
H^{r,C}(A_N, v, L) &= \sum_{i \in C} H^r_{i,m_i}(A_N, v, L) \\
&= \sum_{i \in C} \sum_{k \in A_i} \Big(v^L\big(a^{r,(i,k)}\big) - v^L\big(a^{r,(i,k-1)}\big) \Big) \\
&= \sum_{i \in C} \Big(v^L\big(a^{r,(i,m_i)}\big) - v^L\big(a^{r,(i,0)}\big) \Big) \\
&= \sum_{i \in C} \Big(v\big(a^{r,(i,m_i)}\big) - \sum_{j \in s^r(i)} v\big(a^{r,(i,0)}_{S_r(j)}, 0_{N \setminus S_r(j)}\big) \Big) \\
&= \sum_{i \in C} \Big(v\big(m_{S_r(i)}, 0_{N \setminus S_r(i)}\big) - \sum_{j \in s^r(i)} v\big(m_{S_r(j)}, 0_{N \setminus S_r(j)}\big) \Big) \\
&= \sum_{i \in C} \Big(v^L\big(m_{S_r(i)}, 0_{N \setminus S_r(i)}\big) - v^L\big(m_{S_r(i) \setminus \{i\}}, 0_{N \setminus (S_r(i) \setminus \{i\})}\big) \Big).
\end{aligned}$$

Using an induction argument on the number of elements of $S_r(j)$, one can easily check that for each $j \in N$,

$$\begin{aligned}\sum_{i \in S_r(j)} \Big(v^L\big(m_{S_r(i)}, 0_{N \setminus S_r(i)}\big) - v^L\big(m_{S_r(i)\setminus\{i\}}, 0_{N \setminus (S_r(i)\setminus\{i\})}\big) \Big) &= v^L\big(m_{S_r(j)}, 0_{N \setminus S_r(j)}\big) \\ &= v\big(m_{S_r(j)}, 0_{N \setminus S_r(j)}\big).\end{aligned}$$

Because $C = S_r(r)$, we get

$$H^{r,C}(A_N, v, L) = v\big(m_C, 0_{N \setminus C}\big).$$

It follows that

$$\begin{aligned}\mathsf{AT}^C(A_N, v, L) &= \sum_{i \in C} \sum_{k \in A_i} \mathsf{AT}_{i,k}(A_N, v, L) \\ &= \frac{1}{|C|} \sum_{r \in C} H^{r,C}(A_N, v, L) \\ &= v\big(m_C, 0_{N \setminus C}\big),\end{aligned}$$

which proves that the average tree solution satisfies component efficiency.

To show component fairness, pick any link $ij \in C$ and consider a root $r \in K_{ji}$. By the above argument, we have:

$$H^{r,K_{ij}}(A_N, v, L) = v(m_{K_{ij}}, 0_{N \setminus K_{ji}}).$$

Because $H^r(A_N, v, L) = v\big(m_C, 0_{N \setminus C}\big)$, we also have:

$$H^{r,K_{ji}}(A_N, v, L) = v\big(m_C, 0_{N \setminus C}\big) - H^{r,K_{ij}}(A_N, v, L).$$

Taking the average over all possible roots, we get:

$$\mathsf{AT}^{K_{ij}}(A_N, v, L) = \frac{|K_{ji}| v(m_{K_{ij}}, 0_{N \setminus K_{ji}}) + |K_{ij}|\big(v\big(m_C, 0_{N \setminus C}\big) - v(m_{K_{ji}}, 0_{N \setminus K_{ji}})\big)}{|C|}.$$

On the other hand, by component efficiency of AT, it follows that

$$\mathsf{AT}^{K_{ij}}(A_N, v, L \setminus ij) = v\big(m_{K_{ij}}, 0_{N \setminus K_{ij}}\big)$$

and so

$$|K_{ji}| \Big(\mathsf{AT}^{K_{ij}}(A_N, v, L) - \mathsf{AT}^{K_{ij}}(A_N, v, L \setminus ij) \Big)$$

is equal to

$$|K_{ji}||K_{ij}| \frac{\big(v\big(m_C, 0_{N \setminus C}\big) - v(m_{K_{ji}}, 0_{N \setminus K_{ji}}) - v(m_{K_{ij}}, 0_{N \setminus K_{ij}}\big)}{|C|} \tag{4.1}$$

In the same way, it is easy to see that

$$|K_{ij}| \Big(\mathsf{AT}^{K_{ji}}(A_N, v, L) - \mathsf{AT}^{K_{ji}}(A_N, v, L \setminus ij) \Big)$$

is equal to (4.1), which proves that AT satisfies component fairness. By definition of the hierarchical outcome, it is immediate to verify that AT satisfies independence on the greatest activity level. $\square$

Combining Proposition 3.3 and Proposition 4.1, we derive the following characterization.

Proposition 4.2 *On the class of multi-choice forest games $\mathcal{C}_N$, the average tree solution is the unique allocation rule which satisfies component efficiency, component fairness and independence on the greatest activity level.*

5 Core stability

For the sake of presentation, we assume in this section that the communication graph is a tree. The discussion below can be adapted straightforwardly to the case where the communication graph is a forest. We first show that the average tree solution of a multi-choice tree game is stable in the sense that it belongs to its precore provided that the characteristic function is isotone and superadditive. We thus have the following result.

Proposition 5.1 *Suppose that $(A_N, v, L) \in \mathcal{C}^*_N$ is isotone and superadditive. Then, AT$(A_N, v, L) \in \mathcal{PK}(A_N, v, L)$. If furthermore for each root $r \in N$, each player $i \in N$ and each activity level $k \in A_i$, it holds that*

$$v^L\big(a^{r,(i,k)}\big) - v^L\big(a^{r,(i,k-1)}\big) \geq v^L(ke^i) - v^L((k-1)e^i) \tag{5.1}$$

then AT$(A_N, v, L) \in \mathcal{K}(A_N, v, L)$.

Proof. First pick any $(A_N, v, L) \in \mathcal{C}^*_N$ such that v is isotone and superadditive and (N, L) is a tree. Because the precore is a convex set, it suffices to verify that any hierarchical vector $h^r(A_N, v, L)$, $r \in N$, belongs to $\mathcal{PK}(A_N, v, L)$ in order to show that AT$(A_N, v, L) \in \mathcal{PK}(A_N, v, L)$. In order to do this, consider the set V of all ordered pairs (i, k) where $i \in N$ and $k \in A_i$, and for each nonempty $T \in 2^V$ define for each $i \in N$ the set T_i as the section of T at i, i.e.

$$T_i = \big\{k \in A_i : (i, k) \in T\big\}.$$

Next, define $b^T_i \in A_i$ as follows. If $T_i = \emptyset$ set $b^T_i = 0$; if $T_i \neq \emptyset$ set b^T_i as the largest integer k in T_i such that for each $l \in \{1, \ldots, k\}$, $(i, l) \in T_i$. Construct the TU-game $w : 2^V \longrightarrow \mathbb{R}$ such that for each nonempty coalition $T \in 2^V$, $w(T) = v(b^T)$. Note that the map $T \longmapsto b^T$ is onto, i.e. for each $a \in A_N$, there is $T \in 2^V$ such that $a = b^T$. Pick any pair of disjoint coalitions S and T in 2^V. Obviously, $b^T \wedge b^S = 0_N$ so that

$$v(b^T \vee b^S) \geq v(b^T) + v(b^S) = w(T) + w(S) \tag{5.2}$$

by superadditivity of v and definition of w. On the other hand, we have $b^{T\cup S} \geq b^T$ and $b^{T\cup S} \geq b^S$ so that $b^{T\cup S} \geq b^T \vee b^S$. Because v is isotone, we get

$$w(T \cup S) = v(b^{T\cup S}) \geq v(b^T \vee b^S) \tag{5.3}$$

The combination of (5.2) and (5.3) shows that (V, w) is a superadditive TU-game. Next, pick any root r in (N, L) and construct the forest game (V, w, L^r) where L^r is defined as follows: for each $i \in N$, create the oriented links $(i, k)(i, k-1)$ for $k \in A_i \backslash \{0\}$. Furthermore, for each $i \in N$ and each $j \in s_r(i)$, create the oriented link $(i, 0)(j, m_j)$. Now let us focus on the rooted spanning tree of (V, L^r) obtained when the root is (r, m_r). See Example 5.2 below for an illustration. The hierarchical vector $h^{(r,m_r)}(V,w,L^r)$ of (V,w,L^r) assigns to each player $(i,k)\in V$ the payoff

$$h^{(r,m_r)}_{(i,k)}(V,w,L^r)=w^{L^r}\big(S_{(r,m_r)}(i,k)\big)-w^{L^r}\big(S_{(r,m_r)}(i,k)\backslash\{(i,k)\}\big).$$

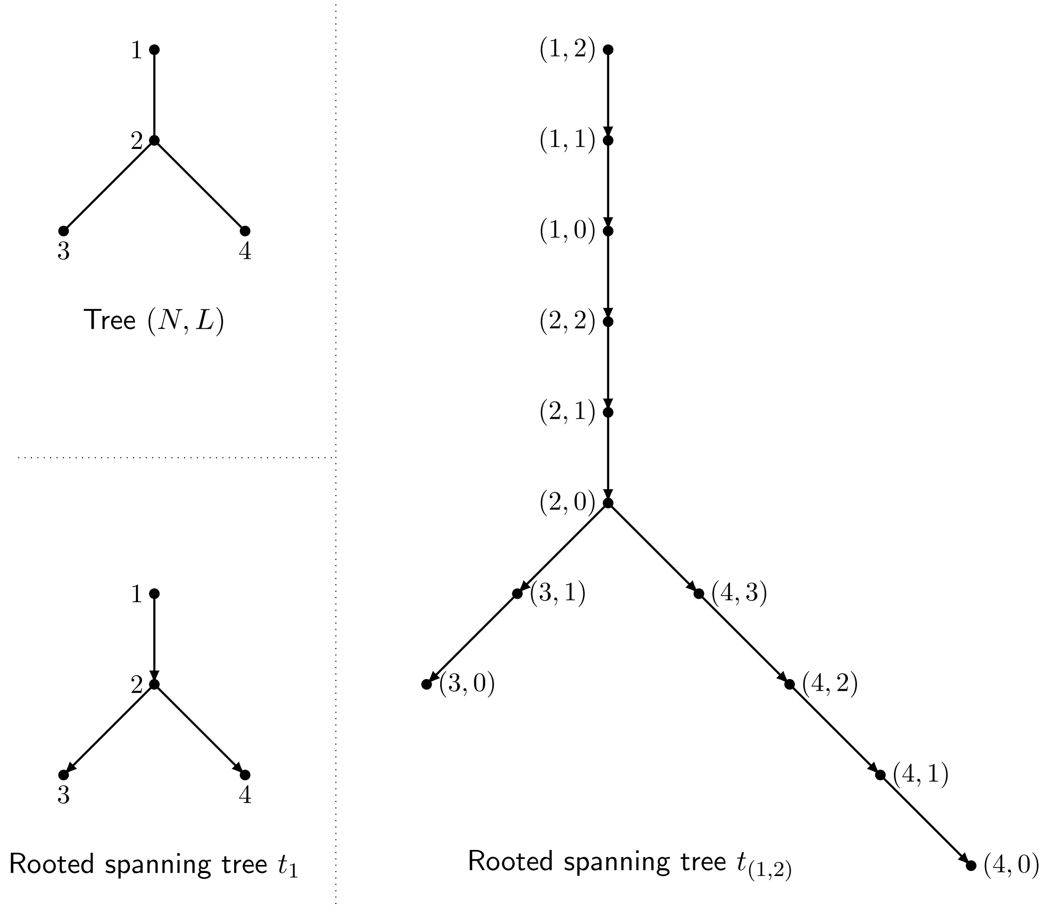
Demange [2004] has shown that this hierarchical vector $h^{(r,m_r)}(V,w,L^r)$ is an extreme point of the core of the restricted TU-game (V,w^{L^r}) when the underlying TU-game (V,w) is superadditive, i.e.

$$\sum_{(i,k)\in V} h^{(r,m_r)}_{(i,k)}(V,w,L^r)=w^{L^r}(V)\text{ and }\forall T\in 2^V,\ \sum_{(i,k)\in T} h^{(r,m_r)}_{(i,k)}(V,w,L^r)\geq w^{L^r}(T).$$

By construction $h^{(r,m_r)}(V,w,L^r)$ coincides with $h^r(A_N,v,L)$. To see this, pick any $(i,k)\in V$ and consider the connected coalition $S_{(r,m_r)}(i,k)\in 2^V$ formed by the subordinates of player (i,k) in the rooted spanning tree $t_{(r,m_r)}$ of (V,L^r) and by player (i,k). The profile $b^{S_{(r,m_r)}(i,k)}$ coincides with the profile $a^{r,(i,k)}$ in (A_N,v,L) when the spanning tree of (N,L) is rooted at r. Therefore, $w(S_{(r,m_r)}(i,k))=v(a^{r,(i,k)})$ and by construction of (V,L^r) and $t_{(r,m_r)}$, we get $w^{L^r}(b^{S_{(r,m_r)}(i,k)})=v^L(a^{r,(i,k)})$. This shows that $h^{(r,m_r)}(V,w,L^r)=h^r(A_N,v,L)$ and we conclude that $h^r(A_N,v,L)$ belongs to the precore of (A_N,v,L) since the map $T\longmapsto b^N$ is onto. If, furthermore, condition (5.1) is satisfied, then the hierarchical vector $h^r(A_N,v,L)$ is an imputation and so it belongs to the core of (A_N,v,L). □

Example 5.2

Let $N=\{1,2,3,4\}$ be the player set and let (N,L) be the tree given by $L=\{12,23,24\}$. The sets of activity levels of players are the following: $A_1=A_2=\{0,1,2\}$, $A_3=\{0,1\}$ and $A_4=\{0,1,2,3\}$. Consider player 1 as the root, which induces the rooted spanning tree t_1. The extended tree of (N,L^r) and the rooted spanning tree $t_{(1,2)}$ is depicted below.



Tree (N, L)

Rooted spanning tree t_1

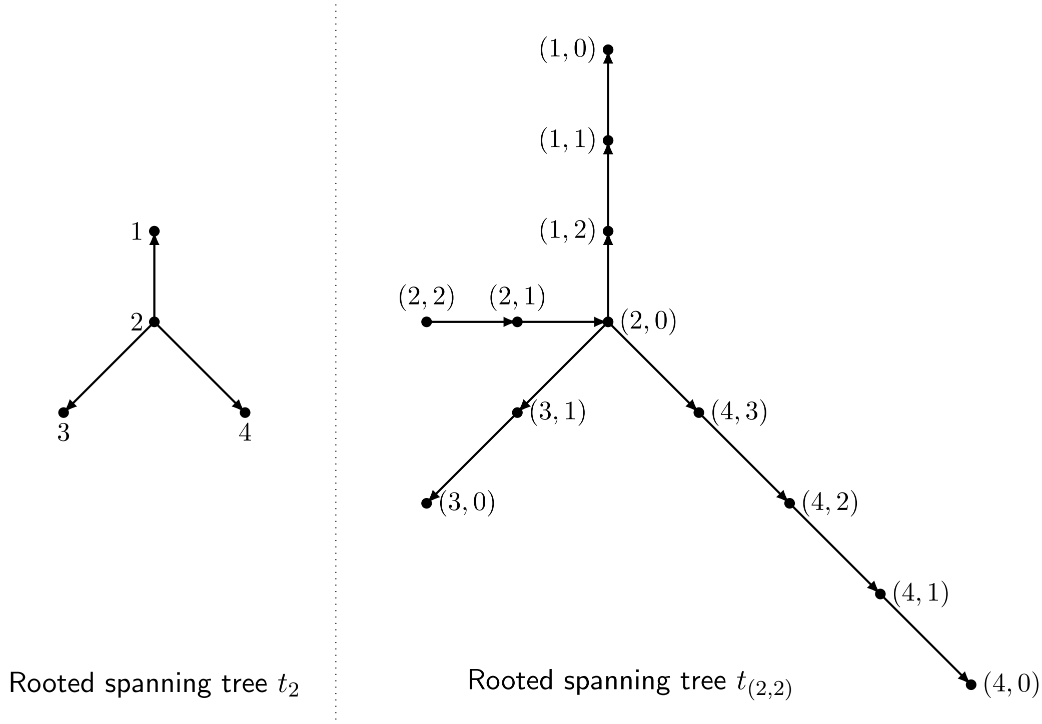
Rooted spanning tree $t_{(1,2)}$

The proof of Proposition 5.1 reveals that the average tree solution for multi-choice tree games (A_N, v, L) can be expressed as the average of hierarchical vectors of the tree games (V, w, L^r), $r \in N$, where the associated spanning tree of each (V, L^r) is rooted at (r, m_r). Note that we can not assert that the average tree solution for (A_N, v, L) is an average tree solution constructed from a given TU-game (V, w) and a fixed tree. The reason is that for each root r of (N, L), we have to construct a new L^r and so a new root spanning tree $t_{(r,m_r)}$. To illustrate this aspect, consider again Example 5.2 and choose 2 as the root of the tree (N, L). The tree (V, L^2) and the corresponding rooted spanning tree $t_{(2,2)}$ are drawn on the same figure, and we see that (V, L^2) is different from (V, L^1).

Proposition 5.3 *For each multi-choice tree game $(N, A_N, L) \in \mathcal{C}_N^*$, the average tree solution assigns to each pair $(i, k) \in N \times A_i$ the payoff*

$$AT_{i,k}(N, A_N, L) = \frac{1}{n} \sum_{r \in N} h_{i,k}^{(r,m_r)}(V, w, L^r),$$

where the TU-game (V, w) and the trees (V, L^r), $r \in N$, are defined in the proof of Proposition 5.1.



6 Applications

Many economic situations can be modeled as multi-choice graph games. In this section, we consider two applications. This first one concerns the problem of coordination of different production activities in a industry and the second one is a cooperative model of price formation.

Coordination of economic activities

A finite set of firms $N = \{1, 2, \ldots, n\}$ are engaged in carrying out n distinct inputs. Each firm is specialized in the production of exactly one input and each input is produced by exactly one firm. The combination of all inputs is a necessary and sufficient condition in order to complete production of the output. The production level depends on the quantity of inputs used by the cooperating firms. More precisely, the set of quantities of input available for each firm $i \in N$ is the finite set $A_i = \{0, 1, \ldots, m_i\}$. The characteristic function $v : A_N \longrightarrow \mathbb{R}$ is a Leontief technology given by the supermodular function

$$v(a) = \min_{i \in N} a_i.$$

Supermodularity implies increasing differences on A_N [see Topkis, 1998, Theorem 2.6.1], i.e. for all distinct firms $i \in N$ and $j \in N$, for all $a_i \geq b_i$ and $a_j \geq b_j$, and for all $a_{-ij} \in A_{N \setminus \{i,j\}}$, it holds that

$$v(a_i, a_j, a_{-ij}) - v(b_i, a_j, a_{-ij}) \geq v(a_i, b_j, a_{-ij}) - v(b_i, b_j, a_{-ij}),$$

which means that the additional production resulting from the availability of any additional unit of input of firm i is increasing with the quantity of input of firm j, all other things being equal. This property reflects the complementarities between the different activities of the process of production in the sense that the set of inputs are different phases of this process. The dependencies among this set of inputs are represented by a tree (N, L), where a link represents the "technological externalities" between two inputs that are closely complementary. Technological externalities refer to the benefits of economic interactions which take place through nonmarket mechanisms, such as bilateral communication between firms. The combination of inputs is undertaken sequentially. A rooted spanning tree describes the positions of the inputs in this order, with the convention that the root is the last input undertaken. Denote by $\underline{m}$ the worth $v(m)$ of the grand coalition m. For each pair $(i, k) \in N \times A_i$, we have

$$h^r_{i,k}(A_N, v, L) = \begin{cases} 1 & \text{if } r = i \text{ and } k \leq \underline{m}, \\ 0 & \text{otherwise.} \end{cases}$$

Because each firm is exactly the root of one spanning tree among the set of n spanning trees, the average tree solution assigns to the k-th unit of input i the following payoff:

$$\mathsf{AT}_{i,k}(A_N, v, L) = \begin{cases} \dfrac{1}{n} & \text{if } k \leq \underline{m}, \\ \\ 0 & \text{otherwise.} \end{cases}$$

It follows that when a firm i chooses to put m_i units of inputs in the process of production, the average marginal contribution of the firm to the $\underline{m}$ units of output is

$$\sum_{k=0}^{m_i} \mathsf{AT}_{i,k}(A_N, v, L) = \frac{\underline{m}}{n}.$$

Not all units of inputs are rewarded since adding extra units of input does not always increase the quantity of output. To see this, recall that when the grand coalition form, the quantity produced is $v(m) = \underline{m} = v(\underline{m}, \ldots, \underline{m})$, which means that the process of production consumes at least $\underline{m}$ units of each input. One possible interpretation is that the average tree solution measures the average marginal contribution of each of these first $\underline{m}$ units to the production of $\underline{m}$ units of output. The marginal contribution of each of these units of input is equal to 1 when the input is the last to enter in the process of production, and zero otherwise. Since each input enters exactly once in the last position, its average marginal contribution is $1/n$.

Derks and Peters [1993] propose a multi-choice Shapley value. One can readily check that this Shapley value of the multi-choice game (A_N, v) coincides with the average tree solution of (A_N, v, L) in this cooperative model of production. This implies that the average tree solution does not depend on the communication tree. The reason is that there is no substitutability among inputs. In the next application, we provide a model of spatial price formation in which the location of a firm on the tree is crucial.

Cooperation in a Hotelling model

Consider a city with a population distributed among a network of roads, represented by a tree of order $n \in \mathbb{N}$. Each firm $i \in N = \{1, 2, \ldots, n\}$ is located at a node of the tree, denoted by (N, L). For simplicity, the length of each link is equal to one, and assimilated to the line segment $[0, 1]$, and firms sell a homogeneous product at zero cost. There is a continuum of consumers (of unit mass) distributed on each link with unit density. Each firm $i \in N$ chooses a unique price $p_i \in A_i = \{0, 1, \ldots, m_i\}$ for the roads $ij \in L_i$, where the activity level $p_i = 0$ means that firm i does not participate on its markets. On each road $ij \in L$, each consumer buys exactly one unit of the product from the firm (i or j) that charges the lowest full price, $i.e.$ the price fixed by the firm plus the linear transportation cost to that firm. This model is a natural extension of the Hotelling [1929] model with fixed locations from a linear city to a city shaped as a tree. For each road $ij \in L$, it is known that if the two firms participate, the market share of firm i is

$$\frac{(p_j - p_i + t)}{2t},$$

while firm j's market share is

$$1 - \frac{(p_j - p_i + t)}{2t} = \frac{(p_i - p_j + t)}{2t},$$

where t is the unit transportation cost. We assume $t > \max_{i \in N} m_i$ so as to ensure that the market shares are always positive. If a unique firm participates on the market associated to road $ij \in L$, then its market share is 1. It follows that $R_i(0, p_j) = 0$ and $R_j(0, p_j) = p_j$. The total revenues of firms i and j when they both participate are given by

$$R_i(p_i, p_j) = p_i \frac{(p_j - p_i + t)}{2t} \text{ and } R_j(p_i, p_j) = p_j \frac{(p_i - p_j + t)}{2t}.$$

Define $v : A_N \longrightarrow \mathbb{R}$ as

$$v(p) = \sum_{ij \in L} \bigg(R_i(p_i, p_j) + R_j(p_i, p_j) \bigg).$$

Consider the market associated with any road $ij \in L$. When firm i reduces its price p_i of one unit, we can distinguish three types of variation of the firms' revenues on this market. If $p_i > 1$ and $p_j > 1$, then straightforward calculations show that

$$R_i(p_i, p_j) - R_i(p_i - 1, p_j) = \frac{p_j - 2p_i + 1 + t}{2t}$$

and

$$R_j(p_i, p_j) - R_j(p_i - 1, p_j) = \frac{p_j}{2t}.$$

If $p_i = 1$ and $p_j \geq 1$, then we have

$$R_i(1, p_j) - R_i(0, p_j) = \frac{p_j - 1 + t}{2t} \text{ and } R_j(1, p_j) - R_j(0, p_j) = -p_j \frac{p_j - 1 + t}{2t}$$

Finally, if $p_i \geq 1$ and $p_j = 0$ we get

$$R_i(p_i, 0) - R_i(p_i - 1, 0) = 1 \text{ and } R_j(p_i, 0) - R_j(p_i - 1, 0) = 0.$$

In order to compute the average tree solution, pick any root $r \in N$. For each $i \in N \backslash \{r\}$ and each $p_i \in A_i \backslash \{0, 1\}$, we have

$$h^r_{i,p_i}(A_N, v, L) = 1 + \sum_{j \in s_r(i)} \frac{(2m_j - 2p_i + 1 + t)}{2t},$$

and for each $p_r \in A_r \backslash \{0, 1\}$, we have

$$h^r_{r,p_r}(A_N, v, L) = \sum_{j \in s_r(r)} \frac{(2m_j - 2p_r + 1 + t)}{2t}.$$

In addition, for each $i \in N \backslash \{r\}$, if $p_i = 1$ we have,

$$h^r_{i,1}(A_N, v, L) = 1 + \sum_{j \in s_r(i)} (1 - m_j) \frac{(m_j - 1 + t)}{2t}.$$

and

$$h^r_{r,1}(A_N, v, L) = \sum_{j \in s_r(r)} (1 - m_j) \frac{(m_j - 1 + t)}{2t}.$$

When firm i decides to not participate instead of charging a unit price on the market associated with road ij, the other firm j can behave as a monopoly on this market. The total revenues of both firms on the market will be greater whenever $m_j > 1$ if firm i decides to not participate. The above two expressions highlights this feature. We conclude that the average tree solution assigns to every firm i the following payoff variations. If $p_i > 1$,

$$\begin{aligned} \mathsf{AT}_{i,p_i}(A_N, v, L) &= \frac{1}{n}\left(n - 1 + \sum_{r \in N} \sum_{j \in s_r(i)} \frac{(2m_j - 2p_i + 1 + t)}{2t}\right) \\ &= \frac{n-1}{n} + \sum_{j \in L_i} \frac{|K_{ij}|}{2tn}(2m_j - 2p_i + 1 + t) \\ &= \frac{n-1}{n} + \sum_{j \in L_i} \frac{|K_{ij}|}{n}\left(\frac{m_j - p_i}{t} + \frac{1+t}{2t}\right), \end{aligned}$$

and

$$\begin{aligned} \mathsf{AT}_{i,1}(A_N, v, L) &= \frac{1}{n}\left(n - 1 + \sum_{r \in N} \sum_{j \in s_r(i)} (1 - m_j) \frac{(m_j - 1 + t)}{2t}\right) \\ &= \frac{n-1}{n} + \sum_{j \in L_i} \frac{|K_{ij}|}{n}(1 - m_j) \frac{(m_j - 1 + t)}{2t}. \end{aligned}$$

Each payoff variation $\mathsf{AT}_{i,p_i}(A_N, v, L)$ measures the share of the total profit generated along the roads of the city which can be attributed to the p_i-th unit of the price m_i charged by firm i.

In the absence of transportation cost, the model resumes to a cooperative version of a spatial Bertrand game. On each road ij, the market share of firm i is 1 if $p_i < p_j$, 1/2 if $p_i = p_j$ and 0 if $p_i > p_j$. In such a model, the average tree solution assigns to every firm i choosing price p_i, the payoff variation

$$\mathsf{AT}_{i,p_i}(A_N, v, L) = \sum_{j \in L_i : 1 \leq p_i \leq m_j} \frac{|K_{ij}|}{n}.$$

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