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The Shapley value for airport and irrigation games*

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Abstract

In this paper cost sharing problems are considered. We focus on problems given by rooted trees, we call these problems cost-tree problems, and on the induced transferable utility cooperative games, called irrigation games. A formal notion of irrigation games is introduced, and the characterization of the class of these games is provided. The well-known class of airport games (Littlechild and Thompson, 1977) is a subclass of irrigation games. The Shapley value (Shapley, 1953) is probably the most popular solution concept for transferable utility cooperative games. Dubey (1982) and Moulin and Shenker (1992) show respectively, that Shapley's (Shapley, 1953) and Young (1985)'s axiomatizations of the Shapley value are valid on the class of airport games.

In this paper we show that Dubey (1982)'s and Moulin and Shenker (1992)'s results can be proved by applying Shapley (1953)'s and Young (1985)'s proofs, that is those results are direct consequences of Shapley (1953)'s and Young (1985)'s results. Furthermore, we extend Dubey (1982)'s and Moulin and Shenker (1992)'s results to the class of irrigation games, that is we provide two characterizations of the Shapley value for cost sharing problems given by rooted trees. We also note that for irrigation games the Shapley value is always stable, that is it is always in the core (Gillies, 1959).

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1 Introduction

In this paper we consider cost sharing problems given by rooted trees, called *cost-tree problems*. We assign transferable utility (TU) cooperative games (henceforth games) to these cost sharing problems. The induced games are called *irrigation games*. See the following example for the naming: There is given an irrigation ditch joined to the stream by a head gate and a group of users who use this ditch to irrigate their own farms. The functional and maintenance costs of the ditch are given too, and they have to be payed for by the users. One of the main questions is how to share the costs among the users. The ditch and the related users can be represented by a rooted tree.

The root of the tree is the head gate, each node represents one user, and the edges of the rooted tree represent the sections of the ditch. The users are related to the ditch by these sections. In this setting Littlechild and Owen (1973) show that the contribution vector (solution) recommended by the *sequential equal contribution*, called *sequential equal contributions rule*, where the costs of the sections is shared equally among the farmers who use them and the farmers pay the total cost of the sections they use, coincides with the Shapley value (Shapley, 1953).

For an empirical and axiomatic analysis of the sequential equal contribution see Aadland and Kolpin (1998) (they call it *serial cost-share rule*) who examine a real cost-sharing problem where the irrigation ditch is located in a south-central Montana community. The irrigation game by Aadland and Kolpin (1998) is defined as follows: for each nonempty set of players S the value $v(S) = -c(S)$, where $c = (c_1, \ldots, c_n) \in \mathbb{R}^N$ cost-vector gives the cost of the ditch, c_i gives the cost of section i and $c(S)$ means the minimum cost of servicing all users in S. It is easy to verify that this game is convex. A similar definition is presented in Kayi (2007), using c_i for the whole cost joining user i to the headgate.

When we consider special rooted trees: chains, we get the class of airport problems, so the class of airport games (Littlechild and Thompson, 1977) is the proper subset of the class of irrigation games. Thomson (2007) gives an overview on the results for airport games. Up to now, two axiomatizations of the Shapley value are considered on airport games in the literature: Shapley (1953)'s and Young (1985)'s axiomatizations. Dubey (1982) shows that Shapley (1953)'s characterization is valid on the class of airport games, and Moulin and Shenker (1992) prove that Young (1985)'s axiomatization works on this subclass of games.

In this paper, we introduce the formal concept of irrigation games, and characterize the class of them. We show that the class of irrigation games is a non-convex cone which is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave. Furthermore, as a corollary we show that the class of airport games has the very same characteristics as that of the class of irrigation games.

In addition to the previously listed results of this paper, we extend Dubey (1982)'s and Moulin and Shenker (1992)'s results to the class of irrigation games. Furthermore, we translate the axioms used in the cost sharing literature (see e.g. Thomson (2007)) to the language of transferable utility cooperative games, and provide results that clearly show that Dubey (1982)'s and Moulin and Shenker (1992)'s results can be deduced directly from Shapley (1953)'s and Young (1985)'s results. That is, we present two new variants of Shapley (1953)'s and Young (1985)' results, and we provide Dubey (1982)'s and Moulin and Shenker (1992)'s and our characterizations as direct corollaries of the two new variants.

We also notice that the Shapley value (sequential equal contributions rule) is stable for irrigation games, that is it is always in the core (Gillies, 1959). This result is a simple consequence of the Ichiishi-Shapley theorem (Shapley, 1971; Ichiishi, 1981) and that every irrigation game is concave.

Up to our knowledge this is the first paper in the literature which provides a precise characterization of the class of irrigation games, and extends Shapley (1953)'s original and Young (1985)'s axiomatizations of the Shapley value for this class of games. We conclude that applying the Shapley value to cost-tree problems is theoretically well-founded, therefore, since the Shapley value behaves well from the viewpoint of computational complexity theory (Megiddo, 1978), the Shapley value is a desirable tool for solving cost-tree problems.

Further related literature are as follows: Granot et al. (1996) introduce the notion of standard fixed tree game. Irrigations games and standard fixed tree games are the same, the difference between Granot et al. (1996)'s approach and ours is that Granot et al. (1996) assume a fixed tree, while we allow the trees to vary. Koster et al. (2001) also consider standard fixed tree games, and study the core of these games. Ni and Wang (2007) characterize the rules meeting properties additivity and independence of irrelevant costs, on the class of standard fixed tree games.

Ambec and Ehlers (2006) examine how to share a river efficiently among countries joined to the given river. They note that cooperation exerts positive externalities on the benefit of a coalition and explore how to distribute this benefit among the countries of the coalition. By Ambec and Sprumont (2002) the location of an agent (i.e. country) along the river determines the quantity of water it controls, and thereby the welfare it can secure to itself. They call the appropriated cooperative game *consecutive game*. The authors prove that the game is convex so the Shapley value is in the core (see Shapley (1971) and Ichiishi (1981)).

For an other problem, for the allocation of costs and benefits from regional cooperation, see the *regional games* defined by Dinar and Yaron (1986). Different allocations of cooperative game theory are applied, like the core, the nucleolus, the Shapely value and the generalized Shapley value; and are compared with an allocation based on marginal cost pricing. Dinar et al. (1992) analyze a similar problem in the TU and the NTU settings (in the NTU case the core of the related game is non-convex, so the Nash-Harsányi solution is applied).

In this paper, we consider only Shapley (1953)'s and Young (1985)'s axiomatizations, the validity of further axiomatizations of the Shapley value, see e.g. van den Brink (2001) and Chun (1991) among others, on the classes of airport games and irrigation games, is intended to be the topic of an other paper.

The setup of this paper is as follows: in the next section we introduce the concept of irrigation games and characterize the class of them. In Section 3 we present our main results: we show that Shapley (1953)'s and Young (1985)'s axiomatizations of the Shapley value work on the classes of airport games and irrigation games. In the last section we show that how cost sharing axioms (see Thomson (2007)) correspond to axioms for solutions on transferable utility cooperative games, and reformulate our results in the classical cost sharing setting.

2 Airport and Irrigation Games

standard fixed tree games *Notions, notations*: $\#N$ is for the cardinality of set N, and 2^N denotes the class of all subsets of N. $A \subset B$ means $A \subseteq B$, but $A \neq B$. $A \uplus B$ stands for the union of disjoint sets A and B.

A *graph* is a pair $G = (V, A)$, where the elements of V are called *vertices* or nodes, and A stands for the ordered pairs of vertices, called *arcs* or edges. A *rooted tree* is a graph in which any two vertices are connected by exactly one simple path, and one vertex has been designated the root, in which case the edges have a natural orientation, away from the root. The tree-order is the partial ordering on the vertices of a rooted tree with $i \leq j$, if the unique path from the root to j passes through i. The *chain* is a rooted tree such that any vertices $i, j \in N$, $i \leq j$ or $j \leq i$. That is, a chain is a rooted tree with only one "branch". For any pair $e \in A$, $e = \overline{ij}$ means e is an edge between vertices $i, j \in V$ such that $i \leq j$. For each $i \in V$, let $S_i(G) = \{j \in V : j \geq i\}$, that is, for any $i \in V$, $i \in S_i(G)$. For each $i \in V$ let $P_i(G) = \{j \in V : j \leq i\}$, that is, for any $i \in V$, $i \in P_i(G)$. Moreover, for any $V' \subseteq V$, let $(P_{V'}(G), A_{V'})$ be the sub-rooted-tree of (V, A). where $P_{V'}(G) = \cup_{i \in V'} P_i(G)$ and $A_{V'} = \{\overline{ij} \in A : i, j \in P_{V'}(G)\}$.

Let $c : A \to \mathbb{R}_+$. Then c and (G, c) are called *cost function* and *cost-tree* respectively. An interpretation of cost tree (G, c) might be as follows: there is a given irrigation ditch joined to the stream by a headgate, and the users, the vertices of the graph but the root are those who use this ditch to irrigate their own farms. The functional and maintenance costs of the ditch are given by c, and it is payed for by the users. More generally, the vertices might be departments of a company, persons, etc. and for any $e \in A$, $e = \overline{ij}$, c_e is the cost of joining player j to player i.

Let $N \neq \emptyset$, $\#N < \infty$, and $v : 2^N \to \mathbb{R}$ be a function such that $v(\emptyset) = 0$. Then N, v are called set of players, and *transferable utility cooperative game* (henceforth game) respectively. The class of games with player set N is denoted by $\mathcal{G}^N$. It is worth noticing that $\mathcal{G}^N$ is isomorphic with $\mathbb{R}^{2^{\#N}-1}$, henceforth, we assume there is a fixed isomorphism[1] between the two spaces, and regard $\mathcal{G}^N$ and $\mathbb{R}^{2^{\#N}-1}$ as identical.

Game $v \in \mathcal{G}^N$ is *convex*, if for all $S, T \subseteq N$, $v(S) + v(T) \leq v(S \cup T) + v(S \cap T)$, moreover, it is *concave*, if for all $S, T \subseteq N$, $v(S) + v(T) \geq v(S \cup T) + v(S \cap T)$.

The *dual* of game $v \in \mathcal{G}^N$ is game $\bar{v} \in \mathcal{G}^N$ such that for all $S \subseteq N$, $\bar{v}(S) = v(N) - v(N \setminus S)$. It is well known that the dual of a convex game is a concave game and vice versa.

Let $v \in \mathcal{G}^N$ and $i \in N$, and $v'_i(S) = v(S \cup \{i\}) - v(S)$, where $S \subseteq N$. v'_i is called player i's *marginal contribution function* in game v. Alternatively, $v'_i(S)$ is player i's marginal contribution to coalition S in game v.

Let $v \in \mathcal{G}^N$, players $i, j \in N$ are *equivalent*, $i \sim^v j$, if for all $S \subseteq N$ such that $i, j \notin S$, $v'_i(S) = v'_j(S)$.

Let N and $T \in 2^N \setminus \emptyset$, and for all $S \subseteq N$, let

$$u_T(S) = \begin{cases} 1, & \text{if } T \subseteq S \\ 0 & \text{otherwise} \end{cases} .$$

Then game u_T is called *unanimity game* on coalition T.

In this paper we use the duals of the unanimity games. For any $T \in 2^N \setminus \emptyset$ and for all $S \subseteq N$,

$$\bar{u}_T(S) = \begin{cases} 1, & \text{if } T \cap S \neq \emptyset \\ 0 & \text{otherwise} \end{cases} .$$

[1]The fixed isomorphism is the following: we fix a complete ordering on N, that is $N = \{1, \ldots, \#N\}$. Then for all $v \in \mathcal{G}^N$, let $v = (v(\{1\}), \ldots, v(\{\#N\}), v(\{1,2\}), \ldots, v(\{\#N - 1, \#N\}), \ldots, v(N)) \in \mathbb{R}^{2^{\#N}-1}$.

It is clear that every unanimity game is convex, and the duals of the unanimity games are concave.

We assume that the considered cost-tree problems have fixed, at least two, number of players, that is $\#V \geq 3$ and $\#N \geq 2$. First we introduce the notion of irrigation games. Let (G, c) be a cost-tree, and N be the set of the players (the vertices but the root). Moreover, consider $S \subseteq N$ a non-empty coalition, then the cost of connecting the players of S to the root is given by the cost of the minimal rooted tree which covers coalition S. By this method for each cost-tree we can define a game, called *irrigation game*. Formally,

Definition 1 (Irrigation Game). *For any cost-tree (G, c), let $N = V \setminus \{root\}$ be the player set, and for any coalition S (the empty sum is 0) let*

$$v_{(G,c)}(S) = \sum_{e \in A_S} c(e) \ .$$

Moreover, games like v are called irrigation games, and the class of irrigation games with player set N is denoted by $\mathcal{G}_I^N$. If rooted tree G remains fixed, then the class of induced games is denoted by $\mathcal{G}_G$.

The next example is an illustration of the above definition.

Example 2. Consider the cost-tree in Figure 1, where the rooted tree $G = (V, A)$ is as follows, $V = \{\text{root}, 1, 2, 3\}$, $A = \{\overline{\text{root}1}, \overline{\text{root}2}, \overline{23}\}$, and the cost function c is defined as $c(\overline{\text{root}1}) = 12$, $c(\overline{\text{root}2}) = 5$ and $c(\overline{23}) = 8$.

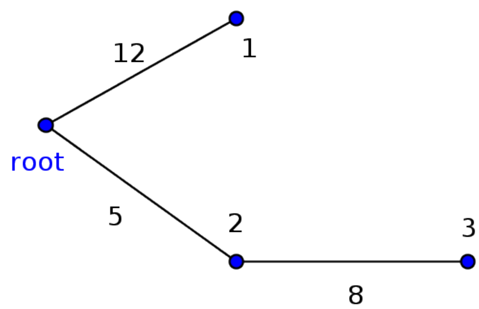

Figure 1: The cost-tree (G, c)

Then irrigation game $v_{(G,c)} = (0, 12, 5, 13, 17, 25, 13, 25)$, that is $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 12$, $v_{(G,c)}(\{2\}) = 5$, $v_{(G,c)}(\{3\}) = 13$, $v_{(G,c)}(\{1, 2\}) = 17$, $v_{(G,c)}(\{1, 3\}) = 25$, $v_{(G,c)}(\{2, 3\}) = 13$ and $v_{(G,c)}(N) = 25$.

The notion of airport games is introduced by Littlechild and Thompson (1977). An airport problem can be illustrated by the following example. There is an airport with one runway, and there are k types of planes. Each type of planes i determines a cost c_i for maintaining the runway. E.g. if i stands for the small planes, then the maintenance cost of a runway for small planes is c_i. If j is the category of big planes, then $c_i < c_j$, since the big planes need longer runway. That is, the player set N is given by a partition: $N = N_1 \uplus \cdots \uplus N_k$, where N_i stands for the planes of category i, and each category i determines a maintenance cost c_i, such that $c_1 < \ldots c_k$. When we consider a coalition of players (planes) S, then the maintenance cost of coalition S is the maximum maintenance cost of the members' maintenance costs of S. That is, the cost of coalition S is the maintenance cost of the biggest plane of coalition S.

We provide two equivalent definitions of airport games; the first one is as follows:

Definition 3 (Definition of Airport Games I)**.** *Let $N = N_1 \uplus \cdots \uplus N_k$ be the player set, and $c \in \mathbb{R}^k_+$, such that $c_1 < \ldots < c_k \in \mathbb{R}_+$ be an airport problem. Then airport game $v_{(N,c)} \in \mathcal{G}^N$ is defined as follows, $v_{(N,c)}(\emptyset) = 0$, and for each non-empty coalition $S \subseteq N$*

$$v_{(N,c)}(S) = \max_{i: N_i \cap S \neq \emptyset} c_i \ .$$

An alternative definition of airport games is the following:

Definition 4 (Definition of Airport Games II)**.** *Let $N = N_1 \uplus \cdots \uplus N_k$ be the player set, and $c = c_1 < \ldots < c_k \in \mathbb{R}_+$ be an airport problem. Let $G = (V, A)$ be a chain such that $V = N \cup \{root\}$ and $A = \{\overline{root1}, \overline{12}, \ldots, \overline{\#N-1\#N}\}$, where $N_1 = \{1, \ldots, \#N_1\}, \ldots, N_k = \{\#N - \#N_k + 1, \ldots, \#N\}$. Moreover, for each $\overline{ij} \in A$, let $c(\overline{ij}) = c_{N(j)} - c_{N(i)}$, where $N(i) = \{N^* \in \{N_1, \cdots, N_k\} : i \in N^*\}$.*

For cost-tree (G, c) airport game $v_{(N,c)} \in \mathcal{G}^N$ is defined as follows, let $N = V \setminus \{root\}$ be the player set, and for any coalition S (the empty sum is 0)

$$v_{(N,c)}(S) = \sum_{e \in A_S} c(e) \ .$$

It is obvious that both definitions above give the same games, and let the class of airport games with player set N be denoted by $\mathcal{G}^N_A$. Furthermore, if chain G remains fixed, then the class of induced games is denoted by $\mathcal{G}_G$. Notice that, the notation $\mathcal{G}_G$ is consistent with the notation introduced in

Definition 1, because if G is a chain, then $\mathcal{G}_G \subseteq \mathcal{G}_A$, in other cases, when G is not a chain, $\mathcal{G}_G \setminus \mathcal{G}_A \neq \emptyset$. Since not every rooted tree is a chain, $\mathcal{G}_A^N \subset \mathcal{G}_I^N$.

Example 5. Consider the airport problem (N, c'), where $N = \{\{1\} \uplus \{2,3\}\}$, and $c'_{N(1)} = 5$ and $c'_{N(2)} = c'_{N(3)} = 8$ $(N(2) = N(3))$. Then consider the cost-tree in Figure 2, where the rooted tree $G = (V, A)$ is as follows, $V = \{\text{root}, 1, 2, 3\}$, $A = \{\overline{\text{root}1}, \overline{12}, \overline{23}\}$, and the cost function c is defined as $c(\overline{\text{root}1}) = 5$, $c(\overline{12}) = 3$ and $c(\overline{23}) = 0$.

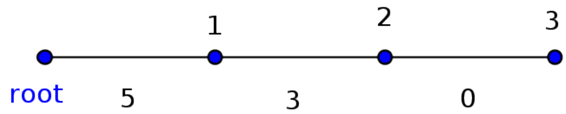

Figure 2: The cost-tree (G, c)

Then the induced airport game is as follows: $v_{(G,c)} = (0, 5, 8, 8, 8, 8, 8, 8)$, that is $v_{(G,c)}(\emptyset) = 0$, $v_{(G,c)}(\{1\}) = 5$, $v_{(G,c)}(\{2\}) = v_{(G,c)}(\{3\}) = v_{(G,c)}(\{1,2\})$ $= v_{(G,c)}(\{1,3\}) = v_{(G,c)}(\{2,3\}) = v_{(G,c)}(N) = 8$.

Next we characterize the classes of airport games and irrigation games. First, we take an obvious observation, for any rooted tree G, $\mathcal{G}_G$ is a cone, that is for any $\alpha \geq 0$, $\alpha \mathcal{G}_G \subseteq \mathcal{G}_G$. Since union of cones is also a cone, both $\mathcal{G}_A^N$ and $\mathcal{G}_I^N$ are cones.

Lemma 6. *For any rooted tree G, $\mathcal{G}_G$ is a cone, therefore, $\mathcal{G}_A^N$ and $\mathcal{G}_I^N$ are cones.*

In the following lemma we show that the dual of any unanimity game is an airport game.

Lemma 7. *For any chain G, $T \subseteq N$ such that $T = S_i(G)$, $i \in N$, $\bar{u}_T \in \mathcal{G}_G$. Therefore, $\{\bar{u}_T\}_{T \in 2^N \setminus \{\emptyset\}} \subset \mathcal{G}_A^N \subset \mathcal{G}_I^N$.*

Proof. For any $i \in N$, $N = (N \setminus S_i(G)) \uplus S_i(G)$, and let $c_1 = 0$ and $c_2 = 1$, that is the cost of the members of coalition $N \setminus S_i(G)$ is 0, and the cost of the members of coalition $S_i(G)$ is 1 (see Definition 3). Then the generated airport game $v_{(G,c)} = \bar{u}_{S_i(G)}$.

On the other hand, it is clear that there is an airport game which is not the dual of any unanimity game. □

It is important to see how the classes of airport games and irrigation games are related to the convex cone spanned by the duals of the unanimity games.

Lemma 8. *For any rooted tree G, $\mathcal{G}_G \subset$ cone $\{\bar{u}_{S_i(G)}\}_{i\in N}$. Therefore, $\mathcal{G}_A \subset \mathcal{G}_I^N \subset$ cone $\{\bar{u}_T\}_{T\in 2^N\setminus\{\emptyset\}}$.*

Proof. First we show that $\mathcal{G}_G \subset$ cone $\{\bar{u}_{S_i(G)}\}_{i\in N}$.

Let $v \in \mathcal{G}_G$ be an irrigation game. Since $G = (V, A)$ is a rooted tree, for each $i \in N$, $\#\{j \in V : \overline{ji} \in A\} = 1$, so we can name the node before player i, let $i_- = \{j \in V : \overline{ji} \in A\}$. Then for any $i \in N$, let $\alpha_{S_i(G)} = c_{\overline{i_- i}}$.

Finally, it is easy to see that $v = \sum_{i\in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$.

Second we show that cone $\{\bar{u}_{S_i(G)}\}_{i\in N} \setminus \mathcal{G}_G \neq \emptyset$. Let $N = \{1, 2\}$, then $\sum_{T\in 2^N\setminus\{\emptyset\}} \bar{u}_T \notin \mathcal{G}_G$, that is game $(1, 1, 3)$ is not an irrigation game. □

The following example is an illustration of the above result.

Example 9. Consider the irrigation game of Example 2. Then $S_1(G) = \{1\}$, $S_2(G) = \{2, 3\}$ and $S_3(G) = \{3\}$. Furthermore, $\alpha_{S_1(G)} = 12$, $\alpha_{S_2(G)} = 5$ and $\alpha_{S_3(G)} = 8$. Finally, $v_{(G,c)} = 12\bar{u}_{\{1\}} + 5\bar{u}_{\{2,3\}} + 8\bar{u}_{\{3\}} = \sum_{i\in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$.

Next we discuss further corollaries of Lemmata 7 and 8. First we show that even if for any rooted tree G, $\mathcal{G}_G$ is a convex set, the classes of airport games and irrigation games are not convex.

Lemma 10. *$\mathcal{G}_A^N$ is not a convex set, moreover $\mathcal{G}_I^N$ is not convex either.*

Proof. Let $N = \{1, 2\}$. From Lemma 6 $\{\bar{u}_T\}_{T\in 2^N\setminus\{\emptyset\}} \subseteq \mathcal{G}_A^N$, however, $\sum_{T\in 2^N\setminus\{\emptyset\}} \frac{1}{3}\bar{u}_T \notin \mathcal{G}_I^N$, that is game $(1/3, 1/3, 1)$ is not an irrigation game. □

The following corollary has a key role in Young (1985)'s axiomatization of the Shapley value on the classes of airport games and irrigation games. It is well-known that the duals of the unanimity games are linearly independent vectors. From Lemma 8, for any rooted tree G and $v \in \mathcal{G}_G$, $v = \sum_{i\in N} \alpha_{S_i(G)} u_{S_i(G)}$, where weights $\alpha_{S_i(G)}$ are well-defined, that is those are uniquely determined. The following lemma says that for any game $v \in \mathcal{G}_G$ if we erase the weight of any basis vector (the duals of the unanimity games), then we get a game belonging to $\mathcal{G}_G$.

Lemma 11. *For any rooted tree G and $v = \sum_{i\in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$, for each $i^* \in N$, $\sum_{i\in N\setminus\{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$. Therefore, for any airport game $v = \sum_{T\in 2^N\setminus\{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N \setminus \{\emptyset\}$, $\sum_{T\in 2^N\setminus\{\emptyset,T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_A^N$, and for any irrigation game $v = \sum_{T\in 2^N\setminus\{\emptyset\}} \alpha_T \bar{u}_T$ and $T^* \in 2^N\setminus\{\emptyset\}$, $\sum_{T\in 2^N\setminus\{\emptyset,T^*\}} \alpha_T \bar{u}_T \in \mathcal{G}_I^N$.*

Proof. Let $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}$ and $i^* \in N$. Then let the cost function c' be defined as follows, for any $e \in A$, (see the proof of Lemma 8)

$$c'_e = \begin{cases} 0, & \text{if } e = \overline{i^*_- i^*} \\ c_e & \text{otherwise} \end{cases} .$$

Then game $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} = v_{(G,c')}$, that is $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$. □

The following example is an illustration of the above result.

Example 12. Consider the irrigation game of Example 2, and take player 2. Then $c'(e) = \begin{cases} 12, & \text{if } e = \overline{\text{root}1} \\ 0, & \text{if } e = \overline{\text{root}2} \\ 8, & \text{if } e = \overline{23} \end{cases}$. Moreover, $\sum_{i \in N \setminus \{i^*\}} \alpha_{S_i(G)} \bar{u}_{S_i(G)} = 12\bar{u}_{\{1\}} + 8\bar{u}_{\{3\}} = v_{(G,c')}$ is an irrigation game.

Finally, an obvious observation:

Lemma 13. *Every irrigation game is concave.*

Proof. The duals of the unanimity games are concave games, hence Lemma 8 implies the statement. □

To sum up our results we conclude as follows:

Corollary 14. *The class of airport games is a union of finitely many convex cones, but it is not convex, and it is a proper subset of the class of irrigation games. The class of irrigation games is also a union of finitely many convex cones, but is not convex either, and it is a proper subset of the finite convex cone spanned by the duals of the unanimity games, therefore every irrigation game is concave, so every airport game is concave too.*

3 Solutions for irrigation games

In this section we propose solutions for irrigation games.

A *solution* on set $A \subseteq \mathcal{G}^N$ ψ is a map-valued mapping $\psi : A \twoheadrightarrow \mathbb{R}^N$, that is a solution assign a set of allocations to each game. In the following, we define two solutions.

Let $v \in \mathcal{G}^N$ and

$$p^i_{Sh}(S) = \begin{cases} \dfrac{\#S!(\#(N \setminus S) - 1)!}{\#N!}, & \text{if } i \notin S \\ 0 & \text{otherwise} \end{cases} .$$

Then $\phi_i(v)$ the Shapley value (Shapley, 1953) of player i in game v is the p^i_{Sh} expected value of v'_i. In other words

$$\phi_i(v) = \sum_{S \subseteq N} v'_i(S)\ p^i_{Sh}(S)\ . \tag{1}$$

Furthermore, let ϕ denote the Shapley solution.

It is obvious from its definition that the Shapley solution is a single valued solution.

Next, we introduce an other solution: the core (Gillies, 1959). Let $v \in \mathcal{G}^N_I$ be an irrigation game. Then the core of irrigation game v is defined as follows

$$\text{core}\ (v) = \left\{ x \in \mathbb{R}^N : \sum_{i \in N} x_i = v(N),\ \text{and for any } S \subseteq N,\ \sum_{i \in S} x_i \leq v(S) \right\}\ .$$

The core consists of the stable allocations of the value of the grand coalition, that is any allocation of the core is such that the allocated cost is the total cost ($\sum_{i \in N} x_i = v(N)$) and no coalition has incentive to deviate from the allocation scheme.

In the following definition we list the axioms we use to characterize a single valued solution.

Definition 15. *A single valued solution ψ on $A \subseteq \mathcal{G}^N$ is / satisfies*

- *Pareto optimal (PO), if for all $v \in A$, $\sum\limits_{i \in N} \psi_i(v) = v(N)$,*
- *null-player property (NP), if for all $v \in A$, $i \in N$, $v'_i = 0$ implies $\psi_i(v) = 0$,*
- *equal treatment property (ETP), if for all $v \in A$, $i, j \in N$, $i \sim^v j$ implies $\psi_i(v) = \psi_j(v)$,*
- *additive (ADD), if for all $v, w \in A$ such that $v + w \in A$, $\psi(v + w) = \psi(v) + \psi(w)$,*
- *marginal (M), if for all $v, w \in A$, $i \in N$, $v'_i = w'_i$ implies $\psi_i(v) = \psi_i(w)$.*

Brief interpretations of the above introduced axioms are as follows: An other, commonly used name of axiom *PO* is *Efficiency*. This axiom requires that the total cost must be shared among the players. Axiom *NP* is about that if a player's marginal contribution is zero, that is she has no influence, effect on the given situation, then her share (her value) must be zero.

On the class of transferable utility games axiom ETP is equivalent with an other well known axiom, with *Symmetry.* These axioms put the requirement that, if two players have the same effects in the given situation, then their evaluations must be equal. Going back to our example, if two users are equivalent in regard to their irrigation costs, then their cost shares must be equal.

A solution meets axiom ADD, if for any two games, adding up the games first then evaluate the players, or evaluate the players first then adding up their evaluations does not matter. Axiom M requires that, if a given player in two games produces the same marginal contributions, then the player must be evaluated equally in the two games.

First we take an obvious observation:

Proposition 16. *Let $A, B \subseteq \mathcal{G}^N$. If a set of axioms S characterizes solution ψ on both classes of games A and B, then set of axioms S characterizes solution ψ on class $A \cup B$.*

In this section we consider two characterizations of the Shapley value on the classes of airport games and irrigation games. The first one is Shapley's original axiomatization (Shapley, 1953).

Theorem 17. *For any rooted tree G, a single valued solution ψ on $\mathcal{G}_G$ is PO, NP, ETP and ADD if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution. Therefore, a single valued solution ψ on the class of airport games is PO, NP, ETP and ADD if and only if $\psi = \phi$, and a single valued solution ψ on the class of irrigation games is PO, NP, ETP and ADD if and only if $\psi = \phi$.*

Proof. if: It is well known that the Shapley solution meets axioms PO, NP, ETP and ADD, see e.g. Peleg and Sudhölter (2003).

only if: From Lemmata 6 and 7 ψ is defined on the cone spanned by $\{\bar{u}_{S_i(G)}\}_{i \in N}$.

Take $i^* \in N$. Then for any $\alpha \geq 0$ and players $i, j \in S_{i^*}(G)$, $i \sim^{\alpha \bar{u}_{S_{i^*}(G)}} j$, and for any player $i \notin S_{i^*}(G)$, $i \in NP(\alpha \bar{u}_{S_{i^*}(G)})$.

Then axiom NP implies that for any player $i \notin S_{i^*}(G)$, $\psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = 0$. Moreover, from axiom ETP for any players $i, j \in S_{i^*}(G)$, $\psi_i(\alpha \bar{u}_{S_{i^*}(G)}) = \psi_j(\alpha \bar{u}_{S_{i^*}(G)})$. Finally, axiom PO implies $\sum_{i \in N} \psi_i(\alpha \bar{u}_{S_{i^*}(G)})$.

Therefore $\psi(\alpha \bar{u}_{S_{i^*}(G)})$ is well-defined (unique), so since the Shapley solution meets axioms PO, NP and ETP, $\psi(\alpha \bar{u}_{S_{i^*}(G)}) = \phi(\alpha \bar{u}_{S_{i^*}(G)})$.

It is also well known that $\{u_T\}_{T \in 2^N \setminus \emptyset}$ is a basis of $\mathcal{G}^N$, and that so is $\{\bar{u}_T\}_{T \in 2^N \setminus \emptyset}$.

Let $v \in \mathcal{G}_G$ be an irrigation game. Then Lemma 8 implies that

$$v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \ ,$$

where for any $i \in N$, $\alpha_{S_i(G)} \geq 0$.

From axiom *ADD* $\psi(v)$ is well-defined (unique), so since the Shapley solution meets axiom *ADD* and for any $i \in N$, $\alpha_{S_i(G)} \geq 0$, $\psi(\alpha_{S_i(G)} \bar{u}_{S_i(G)}) = \phi(\alpha_{S_i(G)} \bar{u}_{S_i(G)})$, $\psi(v) = \phi(v)$.

Finally, we can apply Proposition 16. □

In the proof of Theorem 17 we have applied a modified version of Shapley's original proof. In his proof Shapley uses the unanimity games as the basis of $\mathcal{G}^N$. In the proof above we consider the duals of the unanimity games as a basis and use Proposition 16 and Lemmata 6, 7, 8. It is worth noticing that (we discuss it in the next section) for the airport games Theorem 17 is also proved by Dubey (1982), so in this sense our result is also an alternative proof for Dubey (1982)'s result.

Next, we consider Young's axiomatization of the Shapley value (Young, 1985). This was the first axiomatization of the Shapley value not involving axiom *ADD*.

Theorem 18. *For any rooted tree G, a single valued solution ψ on $\mathcal{G}_G$ is PO, ETP and M if and only if $\psi = \phi$, that is, if and only if it is the Shapley solution. Therefore, a single valued solution ψ on the class of airport games is PO, ETP and M if and only if $\psi = \phi$, and a single valued solution ψ on the class of irrigation games is PO, ETP and M if and only if $\psi = \phi$.*

Proof. if: It is well known that the Shapley solution meets axioms PO, ETP and M, see e.g. Peleg and Sudhölter (2003).

only if: The proof goes, as that Young's proof does, by induction. For any irrigation game $v \in \mathcal{G}_G$, let $\mathcal{B}(v) = \#\{\alpha_{S_i(G)} > 0 : v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)}\}$. It is clear that $\mathcal{B}(\cdot)$ is well-defined.

If $\mathcal{B}(v) = 0$, then axioms PO and ETP imply that $\psi(v) = \phi(v)$.

Assume that for any game $v \in \mathcal{G}_G$ such that $\mathcal{B}(v) \leq n$, $\psi(v) = \phi(v)$. Furthermore, let $v = \sum_{i \in N} \alpha_{S_i(G)} \bar{u}_{S_i(G)} \in \mathcal{G}_G$ be such that $\mathcal{B}(v) = n + 1$.

Let $i^* \in N$ be a player such that there exists $i \in N$ such that $\alpha_{S_i(G)} \neq 0$ and $i^* \notin S_i(G)$. Then Lemmata 8 and 11 imply that $\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \in \mathcal{G}_G$, and

$$\left(\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \right)'_{i^*} = v'_{i^*} \ ,$$

therefore from axiom M

$$\psi_{i^*}(v) = \psi_{i^*} \left(\sum_{j \in N \setminus \{i\}} \alpha_{S_j(G)} \bar{u}_{S_j(G)} \right) \ ,$$

that is $\psi_{i^*}(v)$ is well-defined (uniquely determined).

Assume that $i^*, j^* \in N$ are such that for any $i \in N$ such that $\alpha_{S_i(G)} \neq 0$, $i^*, j^* \in S_i(G)$. Then $i^* \sim^v j^*$, so axiom ETP implies that $\psi_{i^*}(v) = \psi_{j^*}(v)$.

By axiom PO, $\sum_{i \in N} \psi_i(v) = v(N)$. Therefore, $\psi(v)$ is well-defined (u-niquely determined), so since the Shapley solution meets the three considered axioms (PO, ETP an d M), $\psi(v) = \phi(v)$.

Finally, we can apply Proposition 16. □

In the above proof we apply the idea of Young's proof, so we do not need the alternative proofs for Young (1985)'s axiomatization of the Shapley value provided by Moulin (1988) and Pintér (2011). We can do so because Lemma 11 ensures that when we apply the induction step in the only if branch we do not leave the considered classes of games. It is also worth noticing that (we discuss it in the next section) for the airport games Theorem 18 is also proved by Moulin and Shenker (1992), so in this sense our result is also an alternative proof for Moulin and Shenker (1992)'s result.

Finally, Lemma 13 and the well-known results of Shapley (1971) and Ichiishi (1981) imply the following corollary:

Corollary 19. *For any irrigation game v, $\phi(v) \in$ Core (v), that is, the Shapley solution is in the core. Moreover, since every airport game is an irrigation game, for any airport game v, $\phi(v) \in$ Core (v).*

The above corollary shows that on the two considered classes of games the Shapley value is stable, that is it can be considered as a special core concept, a singleton core.

4 Cost sharing results

In this section we reformulate our results in the classical cost sharing setting. To unify the different terminologies appearing in the literature we extensively use Thomson (2007)'s notions. First we introduce the notion of rule. Consider the class of cost-tree allocation problems, that is the set of cost-trees. Then a rule is a mapping which assigns a cost allocation to a cost-tree allocation problem, that is it says the method by which the cost is allocated among the players. Notice that the rule is a single valued mapping. The analogy between solutions and rules is clear, the only important difference is that while the solution is a set-valued mapping, the rule is single-valued.

Next we introduce the rule used in this paper. For any cost tree (G, c) $(G = (V, A))$, the *sequential equal contributions rule*, ξ, is defined as follows, for any player i

$$\xi_i(G, c) = \sum_{j \in P_i(G) \setminus \{\text{root}\}} \frac{c_{\overline{j_- j}}}{\# S_j(G)} \ .$$

In the case of airport games, where the graph G is a chain, the sequential equal contributions rule, ξ, can be given as follows, for any player i

$$\xi_i(G, c) = \frac{c_1}{n} + \cdots + \frac{c_i}{n - i + 1} \ ,$$

where the players are ordered according to the their positions in the chain, that is player i is in the ith position of the chain.

Littlechild and Owen (1973) show that the sequential equal contributions rule and the Shapley value coincide on the class of irrigation games.

Proposition 20 (Littlechild and Owen (1973))**.** *For any cost-tree (G, c), $\xi(G, c) = \phi(v_{(G,c)})$, where $v_{(G,c)}$ is the irrigation game generated by cost-tree (G, c), that is for cost-tree allocation problems the sequential equal contributions rule and the Shapely solution coincide.*

Next we consider certain properties of rules (see Thomson (2007)).

Definition 21. *Let $G = (V, A)$ be a rooted tree. Rule χ defined on the set of cost-trees of G satisfies*

- *non-negativity, if for each cost function c, $\chi(G, c) \geq 0$,*
- *cost boundedness, if for each cost function c, $\chi(G, c) \leq \left(\sum_{e \in A_{P_i(G)}} c_e \right)_{i \in N}$,*
- *efficiency, if for each cost function c, $\sum_{i \in N} \chi_i(G, c) = \sum_{e \in A} c_e$,*
- *equal treatment of equals, if for each cost function c and pair of players $i, j \in N$, $\sum_{e \in A_{P_i(G)}} c_e = \sum_{e \in A_{P_j(G)}} c_e$ implies $\chi_i(G, c) = \chi_j(G, c)$,*

- *conditional cost additivity, if for any pair of cost functions c, c', $\chi(G, c+c') = \xi(G,c) + \chi(G,c')$,*
- *independence of at-least-as-large costs, if for any pair of cost functions c, c' and player $i \in N$ such that for each $j \in P_i(G)$, $\sum\limits_{e \in A_{P_j(G)}} c_e = \sum\limits_{e \in A_{P_j(G)}} c'_e$, $\chi_i(G,c) = \chi_i(G,c')$.*

The interpretations of the above defined rule-properties are as follows (see Thomson (2007)). Non-negativity says that for each problem, the rule should only pick a non-negative cost allocation vector. Cost boundedness is about that the cost allocation vector should be bounded above by the individual costs. Efficiency says that coordinates of the cost allocation vector should add up to the maximal cost. Equal treatment of equals is about that players with equal individual costs should pay equal amounts. Conditional cost additivity says that if two cost-trees are added, then the cost allocation vector chosen for the sum problem should be the sum of the cost allocation vectors chosen for each of them. Finally, independence of at-least-as-large costs is about that what a player pays should not depend on the costs of the segments he does not use.

These properties are similar to the axioms we defined in Definition 15. The following proposition formalizes the similarity.

Proposition 22. *Let G be a rooted tree, χ be defined on cost-trees (G,c), solution ψ be defined on $\mathcal{G}_G$ as $\psi(v_{(G,c)}) = \chi(G,c)$ for any cost function c. Then, if χ satisfies*

- *non-negativity and cost boundedness, then ψ is NP,*
- *efficiency, then ψ is PO,*
- *equal treatment of equals, then ψ is ETP,*
- *conditional cost additivity, then ψ is ADD,*
- *independence of at-least-as-large costs, then ψ is M.*

Proof. Non-negativity and cost boundedness $\Rightarrow NP$: It is obvious that player i is a null-player, only if $\sum_{e \in A_{P_i(G)}} c_e = 0$. Then non-negativity implies that $\chi_i(G,c) \geq 0$, and from cost boundedness, $\chi_i(G,c) \leq 0$, to sum up $\chi_i(G,c) = 0$, so $\psi(v_{(G,c)}) = 0$.

Efficiency $\Rightarrow PO$: From the definition of irrigation games (Definition 1), $\sum_{e \in A} c_e = v_{(G,c)}(N)$, therefore $\sum_{i \in N} \psi_i(v_{(G,c)}) = \sum_{i \in N} \chi_i(G,c) = \sum_{e \in A} c_e = v_{(G,c)}(N)$.

Equal treatment of equals $\Rightarrow ETP$: It is clear that, if $i \sim^{v_{(G,c)}} j$, $i,j \in N$, then $\sum_{e \in A_{P_i(G)}} c_e = \sum_{e \in A_{P_j(G)}} c_e$, so $\chi_i(G,c) = \chi_j(G,c)$. Therefore, if $i \sim^{v_{(G,c)}} j$, $i,j \in N$, then $\psi_i(v_{(G,c)}) = \psi_j(v_{(G,c)})$.

Conditional cost additivity $\Rightarrow ADD$: $\psi(v_{(G,c)} + v_{(G'.c')}) = \psi(v_{(G,c+c')}) = \chi(G, c+c') = \chi((G,c)+(G',c')) = \chi(G,c)+\chi(G',c') = \psi(v_{(G,c)})+\psi(v_{(G',c')})$.

Independence of at-least-as-large costs $\Rightarrow M$: It is easy to check that if for cost-trees (G,c), (G,c') and player $i \in N$, $(v_{(G,c)})'_i = (v_{(G,c')})'_i$, then for each $j \in P_i(G)$, $\sum_{e \in A_{P_j(G)}} c_e = \sum_{e \in A_{P_j(G)}} c'_e$, so $\chi_i(G,c) = \chi_i(G,c')$. To sum up, $(v_{(G,c)})'_i = (v_{(G,c')})'_i$ implies $\psi_i(v_{(G,c)}) = \psi_i(v_{(G,c')})$. □

It is worth noticing that all but the efficiency point are tight, so the cost sharing axioms are stronger than the game theory axioms.

The above results and Theorem 17 implies as a direct corollary, Dubey (1982)'s result.

Theorem 23 (Dubey (1982))**.** *Rule χ on airport problems satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if $\chi = \xi$, that is if and only if χ is the sequential equal contributions rule.*

Proof. If: It is a slight calculation to show that the sequential equal contributions rule satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity (see e.g. Thomson (2007)).

Only if: Proposition 22 implies that we can apply Theorem 17 and get the Shapley solution. Then from Proposition 20 the Shapley solution and the sequential equal contributions rule coincide. □

Similarly to Dubey (1982)'s result we deduce from the results above and Theorem 18 Moulin and Shenker (1992)'s result.

Theorem 24 (Moulin and Shenker (1992))**.** *Rule χ on airport problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if $\chi = \xi$, that is if and only if χ is the sequential equal contributions rule.*

Proof. If: It is a slight calculation to show that the sequential equal contributions rule satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs (see e.g. Thomson (2007)).

Only if: Proposition 22 implies that we can apply Theorem 18 and get the Shapley solution. Then from Proposition 20 the Shapley solution and the sequential equal contributions rule coincide. □

Finally, in the following theorems we extend Dubey (1982)'s and Moulin and Shenker (1992)'s results to any cost-tree allocation problem. The proofs of these results go as that for the two above theorems.

Theorem 25. *Rule χ on cost-tree problems satisfies non-negativity, cost boundedness, efficiency, equal treatment of equals and conditional cost additivity, if and only if $\chi = \xi$, that is if and only if χ is the sequential equal contributions rule.*

Theorem 26. *Rule χ on cost-tree problems satisfies efficiency, equal treatment of equals and independence of at-least-as-large costs, if and only if $\chi = \xi$, that is if and only if χ is the sequential equal contributions rule.*

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