Sequential auction and auction design

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Sequential Auctions and Auction Design*

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Often an auction designer has the option of selling, or purchasing, those lots available in one auction or a sequence of auctions. In addition, bidder opportunities will not be static, in part due to arrival of information, but also because bidders can face deadlines for making decisions. This paper examines the optimal decision about how to divide what is available over time.

* I have benefitted from the helpful comments of writing this paper. The usual disclaimer applies.

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1 Introduction

Auctions for similar or identical items often occur at different times. This poses a decision problem for bidders - how should a bidder adjust its bid in an early auction because of opportunities in subsequent auctions. A bidder's decision problem can be further complicated by the fact that circumstances can change due to new information, changes in circumstances facing the bidders, and due to bidder actions, between auctions. At the same time, the auction manager or originator also has a decision problem of how to allocate the amount to be auctioned over time. Clearly, these are not independent problems. While in any given auction, the auction manager can withdraw some, or all, of what it wants to buy or sell, out of the auction, the bidders might respond strategically, and bid less aggressively. Thus, for the auction manager to reduce auction volume in response to limited competition in one auction may only postpone the inevitable. This paper shows that this logic is incomplete. Indeed, somewhat surprisingly, the auction manager can do better, sometimes much better, by splitting the auction volume over two or more dates *even when bidders respond strategically.*

There are a number of different effects that can explain these results. The primary reason that is identified here is that bidders' rank can change across auctions. More specifically, when the best losing bid in one auction is not guaranteed to be a winner in the next auction, then the opportunity cost for a bidder is *not* expected price in the second auction, but the bidder's expected surplus. When bidders' values do not change from one auction to the next, then Weber's Martingale theorem applies - the expected price in one auction in a sequence of auctions is the realized price in the previous auction. In this paper, bidders' values do change from one auction to the next. A bidder would otherwise win in one auction, but strategically withholds, is not guaranteed a win in the next auction. Thus, a bidder's reservation price in one auction is no longer the expected price for the next auction. This limits the benefits to a bidder of strategic withholding.

Other factors can also limit benefits of strategic withholding. For example, when delay in completing a transaction is costly for one, or both, sides, then auction manager's option to delay can fully or partially offset a bidder's strategic withholding. This also has implications for the definition of market power. These results suggest that the ability of firms to obtain super-normal returns will depend not only on market structure, but market organization, and the ability of market designers to counter firms' potential efforts to exert market power.

These issues arise in sequential auctions in many different sectors. Auctions are often conducted for future, and not immediate, delivery. This analysis was largely motivated by actual experience in the energy sector.[1] Electric utilities, to ensure reliability, must purchase energy generation services prior to realization of demand. Utilities in a number of jurisdictions do so in forward auctions, that can be as much as five years in advance of performance, and as little as six minutes in advance. The decision about how much is purchased in advance can, as explained below, have a large impact on the outcome. Moreover, suppliers bidding into such auctions often have other options, such as selling in adjacent areas, that can alter competition across auctions held at different dates in a manner similar to that modeled in this paper. However, this analysis does not apply only to energy procurement. Another example is the auctions of dairy products.[2] One large dairy producer, Fonterra, conducts bi-monthly auctions, generally for delivery one-to-three months in advance. Fonterra has a fairly inelastic aggregate supply of its products available, and limited ability to defer sales. This analysis is useful to determine how an entity such as an electric utility or a Fonterra should manage its auction volume across possible auction dates.

This analysis also suggests that competition policy guidelines should be based not just

[1]See Loxley and Salant (2004) for detailed discussion of one example.
[2]See `http://www.globaldairytrade.info/`

on market structure, but on market rules - when the market originator can choose to divide auction volume across time, then these decisions, will affect prices and returns. Moreover, when the auction manager in one auction can strategically shift volume over time, then bidder returns are further reduced. In short, auction design and rules can affect relative bargaining power on the two sides of an auction. This can have implications for enforcement as well in cases such as the California energy markets in which prices may have spiked to super-competitive levels in part because of the market rules.

The analysis here applies when bidders have a fixed amount they want to bid for. Bidders in forward auctions may have limited financing, or limited needs. In reverse auctions, bidders will often have limited capacities. These auctions can be for forward delivery, in which the earlier auctions are further in advance of the delivery date than the later auctions. The following analysis applies to at least three factors that can affect the strategic spacing of a given target volume across a number of auctions over time. First, opportunities may change. Actions that might have been feasible at one date, can become infeasible at a later date an vice versa.

Second, bidders can take actions between auctions that can foreclose or open up some options. When firms face scheduling and other time constraints, they will tend to take actions that will affect future options. Firms cannot always delay decisions, or may only be able do so at a cost. Third, information can become more precise over time, reducing bidder uncertainty.

These type of issues arise in many auctions and other types of markets. One specific application of the analysis in this paper is to energy procurement. Many energy resources are purchased in advance of need, and over time. Examples include transmission rights, capacity credits, and default service supply.[3] An energy trader or a load serving entity

[3]BGS, Illinois, RTOs.

may need to assemble combinations of different resources to meet specific obligations, such as serving the electricity demands of a particular set of customers for a specific period of time. These arrangements must be in place usually long before performance, or delivery, is required. Physical resources generally provide different sets of overlapping type of services, such as energy and capacity. Therefore, these type of energy traders can find themselves long or short in specific types of resources as they approach the delivery date.

This paper builds off at least three separate strands of literature. A number of papers have studied both the theory[4] and experience[5] Weber showed that in sequence of auctions of identical objects, in which bidders have independent private values and are risk neutral, and in which bidder valuations do not change across auctions, the expected price in each auction is the realized price of the previous auction. That is, on average, there should be no upwards or downwards trend in prices, or, in more technical terms, the expected price is a martingale.. The intuition is that the marginal bidder in each auction will bid based on what it expects price to be in the next auction.

The empirical results has largely found different price patterns. Ashenfelter observed in sequences of wine auctions, and later in other types of auctions, such as for art, real estate, and cattle, prices tend to decline more often than not. The empirical research that identified the declining price anomaly and afternoon effects have given rise to another strand of research which develops conditions under which prices will decrease, or increase, across a sequence of auctions.

McAfee and Vincent (1993) find that prices will decrease if bidders are risk averse: early bids are equal to expected later prices plus a risk premium. Their explanation relies

[4]See Weber (1983) and Milgrom and Weber (1982).

[5]Ashenfelter (1989) examined sequential auctions of identical cases of wine and port. Subsequently Ashenfelter and Graddy (2002) studied art auctions, Ashenfelter and Genesove (1992) looked at real estate and Engelbrecht-Wiggans and Kahn (1992) looked at cattle auctions.

on the assumption of non-decreasing absolute risk aversion. Bernhardt and Scoones look at two sequential auctions with stochastically equivalent values. They find that even though bidders are risk-neutral, prices fall. Intuitively, bidders with higher valuations in the first auction discount their bids (due to the option value of participating in later auctions) less than those with lower valuations, and it is this first group that determines the price in the first auction, leading prices to fall in subsequent auctions.[6] This paper extends this analysis in a number of ways. I find with a variable number of auctions, bidders and objects per auction, prices can rise or fall. Common to all these models is that one item is auctioned at a time.

Another line of research has recognized that bidder ability to benefit from strategic withholding in multi-unit auctions can be attenuated when the auction manager can adjust the auction volume.[7] . For example, in a uniform-price auction, bidders compete by simultaneously submitting their demand schedules for the divisible good on offer. The seller compares aggregate demand with the auction supply and computes the clearing price, which is then paid by all bidders. Uniform-price auctions of a divisible good in fixed supply may lead to bidders strategically submitting high inframarginal bids, resulting in a lower clearing price. When a bidder can be pivotal, the auction manager or auction designer can offset the strategic withholding incentives of such a bidder by making the auction supply or demand variable. The basic idea is that a bidder's market power is reduced when the supply or demand schedule it faces is more elastic. This assumes that any quantities not purchased or sold in the given auction will never be re-auctioned. This paper relaxes this assumption.[8]

[6] See also Engelbrecht-Wiggans (1993) and Engelbrecht-Wiggans (1994)

[7] See McAdams (2006), Back and Zender (2001), LiCalzi and Pavan (2005), Lengwiler (1999) and Engelbrecht-Wiggans and Kahn (1998).

[8] A more recent paper by Horner and Samuelson (2010) look at pricing over time. Here, the amount for sale or purchase at any point in time is a variable, and the auctions occur at fixed discrete dates rather than continuously over time.

Section 2 explains why bidder valuations can change from one auction to the next. This paper's focus is on changes that result from changes in values or opportunity costs over time, but also allows bidders to learn between auctions, or as a result of information revealed during the early auctions. The following section presents a set of simple two period procurement auction example. This section calculates equilibrium prices and outcomes when the auction manager purchases all units at one time, and compares those ex ante costs with two other cases: (1) the auction manager determines ex ante how it will split the procurement across the two periods and (2) the auction manager sets reservation price for one or two each in the first auction. This following section derives equilibrium bidding rules for N bidders and K identical units being purchased at at most two separate auction dates. When decisions must be made in advance about the allocation of the K units over the two periods, it is shown that the optimal decision is to divide the K units across the two auctions. However, the auction manager can lower expected procurement costs by setting unit specific reservation prices. These results are derived assuming costs at the two auction dates are independently drawn from stochastically equivalent distributions. Section 5 considers arbitrary number of discrete auction dates, and provides an example of a three period auction with a finite number of units. Section 6 concludes with some discussion of other factors that can result in increasing or decreasing prices.

2 How Bidder Values Change Over Time

This section describes ways in which bidder values can change over time. This section is intended to motivate the examples and the assumptions of the more general analysis that follows. I also assume throughout this paper that the auction manager does NOT have the discretion to withhold volume; so demand is totally inelastic if it this is a reverse auction, and supply is totally inelastic for forward auctions.[9] The only discretion given to the auction manager is to delay transactions until the next auction. And in the final auction, the auction manager does not have any further discretion to delay its purchases. Most of the analysis assumes that the auctions are for future delivery, and not current consumption or use. Even where the auction is for current consumption, a seller may be able to delay delivery, for example when it has storage facilities, and a buyer may be able to defer consumption. [10]

One of the motivations for this analysis is experience in energy procurement auctions. In those auctions, a utility will be purchasing energy services, capacity and related products in advance of the required performance. Often the energy product is purchased months in advance, and the contract duration can span years.[11] The auction manager faces decisions about how much to purchase in advance, and when.

Between auction dates, bidder costs can change due to a variety of factors. First, if the auctions are for future delivery, and the later auctions are closer to the delivery date, then bidders may learn more about likely values or costs. This can reduce uncertainy; it can also cause the distribution of values to become more or less disperse. For instance, in energy procurement, some project developers anticipating site approval or environmental approvals, can encounter unforeseen obstacles. At the same time, other bidders might find that approvals are easier to obtain than first anticipated. Whatever the reason that costs or values can change over time, a bidder's optimal decision in one auction should include the expected surplus that can be derived from participating in the next auction, and not just

[9]See McAdams (2006) for a discussion of strategic adjustment of the auction manager's demand or supply schedule.

[10]The Fonterra dairy auctions mentioned above include auctions every two weeks for the sale of whole milk powder, skim milk powder and milk solids. Most of the products can be stored, as least for a limited amount of time. Fonterra has some storage capacity, and can defer delivery at one date if demand is low. See for example
`http://www.fonterra.com/wps/wcm/connect/fonterracom/fonterra.com/our+business/news/media+releases/market+pricing+signs+positive+in+globaldairytrade+event.`

[11]See Loxley and Salant (2004) for a description of one such auction.

the expected value conditional on having the most optimistic signal in the first.

More specifically, consider a reverese auction in which a bidder has a signal of its performance costs, and let e denote its expected cost conditional on its signal and its having a winning bid in the first auction. Then that bidder should not want to accept a price as low as e if its expected surplus from withholding its supply in the first auction and participating in the second is positive. If bidders' expected costs do not change, or if all shocks are common to all bidders, then the low cost bidder's expected surplus in the first auction can be derived from the expected surplus that the winning bidder expects to derive in the last, or n^{th}, auction, and the prices will be a Martingale. However, if the marginal bidder in the $(k-1)^{st}$ is no longer always the winning bidder k^{th} auction in equilibrium, then the equilibrium auction prices need no longer be a Martingale.

Second, bidders can engage in other transactions. A bidder with power to sell at one date, can enter into a contract between auction dates. This will take the bidder out of the second auction, or increase its opportunity costs. Conversely a bidder might find additional resources, or long term contracts might expire, allowing a bidder that had little ability to compete in one auction to participate more aggressively at a later date. Similarly, bidders might encounter unanticipated changes, both for good and bad, in their financial position across auction dates.

Third, bidders may base offers not only on their own resources, but also on complementary offers from third parties.[12] Or bidders in an auction may have buyers, who are not directly participating, lined up to purchase all or part of what is won in the auction. The third party contracts available prior to one auction may not always be available when the next auction takes place. Indeed there could be other auctions. However, the fact that

[12]In energy procurement auction, especially where the winners have to provide an array of services, an individual bidder's offer may be contingent on an array of third party offers for some component parts of what is being bid in the auction. See Loxley and Salant (2004) for a discussion.

there may be competing auctions does not necessarily mean that prices should differ across the auctions.[13] This paper focuses on the case in which bidders view the products being auctioned at different dates as substitutes, and mostly perfect substitutes. When there are sequential auctions of complements, prices are likely to fall across auctions, as is explained below.[14]

Throughout what follows, I assume a finite sequence of procurement auctions. Further, I assume that if the bids are for future delivery, then winning bidders in each auction can achieve costs equal to their cost estimates at the time of the auction (or at the time the auction closes). This imposes no loss of generality as long as bidder expectations are unbiased conditional on the outcome of each auction.

3 Simple Two Auction Examples

This section presents examples in which an auction manager must procure a fixed number, K, lots of an identical product in a sequence of two auctions in which there are N bidders competing.[15] Most of what follows assumes each bidder can only supply at most one unit. Therefore, a winning bidder in the first auction will not be able to participate in the second auction. The basic question addressed in the examples of this section whether the auction manager might want to divide the lots available into two (or more) auctions, and if so, how.

[13] Suppose, for example, there are two auctions, N bidders, each bidder can win at most one lot, and the bidder $j's$ costs in both auctions is a random draw from the uniform distribution on $[0, 1]$. Then, assuming each auction is run as a 2nd price sealed bid, or English auction, the expected price in both auctions will be $\frac{3}{N+1}$.

[14] See, for example Woo et. al. (2004).

[15] This analysis also applies to forward auctions, for selling two objects; the notation is a bit simpler for procurement auctions. This example is a generalization of the two-unit, two-auction example of Bernhardt and Scoones (1994).

3.1 Two Lots - In One Auction or Two?

The simplest situation is one in which there are only two lots available. The question then arises as to whether the auction manager should purchase the two lots in one auction or in two.

The auction manager can either purchase both units at the same date or at two different dates. For each possible auction date, each bidder j has costs (or a signal of what it expects costs will be when it must provide delivery or performance) c_j, $c_j \in [0,1], j = 1, 2, ..., N$. It is assumed that the values are independent draws from an identical distribution for all bidders and all dates. However, it is *not* assumed that a bidder's cost at the first auction date are the same as at the second. This means that the lowest cost losing bidder in the first auction will not necessarily be the low cost bidder in the second auction. So, a bidder's decision about what price to offer in the first auction involves both its prediction for the second auction price as well as that bidder's probability of winning in the second auction. It is assumed that bidders only can supply one unit in total, so that a winner in the first auction does not participate in the second auction. Below, I explain how relaxing this assumption affects the results. However, this assumption limits the possibility of bidder strategically withholding supply in order to obtain a better price.

When $K = 2$, bidder valuations are uniform in each auction, and there are two auctions, following Bernhardt and Scoones, it can be shown that the expected price in the second auction, assuming either a first price or second price sealed bid, is $2/N$. This will provide the winner in the second auction with an expected surplus of $1/N$.[16] This means that the expected price in the first auction will be $\frac{2}{N+1} + \frac{1}{N(N-1)}$. As Bernhardt and Scoones have

[16] If costs were distributed according to the distribution F(c), then the winner in the second auction would receive a price of $F^{-1}(2/N)$ amd would receive an expected surplus of $F^{-1}(2/N) - F^{-1}(1/N)$. This follows as the distribution function associated with any random variable is uniform on $[0, 1]$. Most of what follows will generalize to arbitrary distributions of costs.

shown, the expected price in the first auction is higher than in the second auction whenever $N > 3$. One question not addressed previously is whether expected procurement costs would be larger or smaller if there were one auction or two. It is straightforward to show that the expected price if both units were purchased in one auction would be $\frac{3}{N+1}$. The expected costs in one auction are higher purchasing both units in one auction as compared to purchasing one unit in each of two auctions whenever $N \geq 3$. This also maximizes social welfare. In general, the allocation that minimizes expected procurement costs need not maximize welfare, as the auction manager may want to defer purchases to ensure a lower expected price in the last auction than would minimize total costs of the suppliers. This is discussed in more detail in section 3 below.

This assumption about how costs, or forecasts of costs, evolve over time is very specific. The principles illustrated with the implicit information development assumptions generalize in ways explained in the next section; the particular example is intended to simplify the calculations of expected values.

This example can be generalized in a number of ways. First, I consider the situation where the number of lots in the auction can exceed two. I describe the optimal ex ante (open loop) rule for dividing the lots across auctions. Then, I examine what the optimal reservation price is for each unit. I also derive a rule for the auction manager to use to determine how many units to allocate in the first auction, and what to leave for the next assuming that the auction manager can use closed loop decision rules. Bidders are assumed to be strategic and withhold supply in the first auction.

3.2 How to Divide Auction Quantity Among Two Auctions?

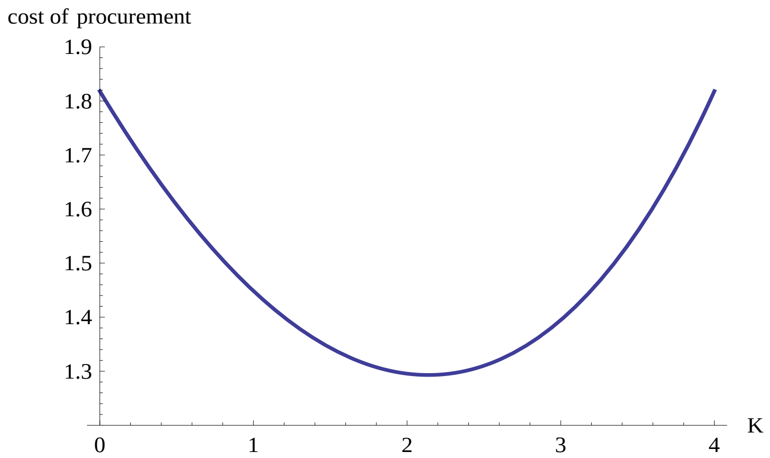
This example generalizes the above by considering the case where $K > 2$. Now, the auction manager has to determine what fraction of the auction volume to purchase in each auction. I assume that it is common knowledge that the auction manager *must* purchase the entire K units in the two auctions. I first consider the case in which the auction manager must decide the number of lots to purchase in each auction in advance of the first auction. I then consider the case in which the auction manager can defer purchases in the first auction based on the offers. The first example is an open loop optimization and the second a closed loop one.

3.2.1 Ex Ante Optimal Division of the Auction Quantity

Assuming X units are purchased in the first auction, and $Y = K - X$ are purchased in the second auction, then the second auction price will be $p_2 = \frac{Y+1}{N-X+1} = \frac{K-X+1}{N-X+1}$. This implies that the average second period winner will have costs of $\frac{Y+1}{2(N-X+1)}$, and will therefore have an average surplus (for winners) of $\frac{Y+1}{2(N-X+1)}$. This implies a first period price of $\frac{X+1}{N+1} + \frac{Y}{(N-X)}\frac{Y+1}{2(N-X+1)} = \frac{X+1}{N+1} + \frac{(K-X)}{(N-X)}\frac{K-X+1}{2(N-X+1)}$- the probability that a first auction loser will be a second auction winner is $\frac{(K-X)}{(N-X)}$. Total costs are then $X\left(\frac{X+1}{N+1} + \frac{K-X+1}{2(N-X+1)}\frac{(K-X)}{(N-X)}\right) + (K - X)\left(\frac{K-X+1}{N-X+1}\right)$. Notice that the first auction expected price is higher, the larger the quantity purchased in the first period and similarly, second auction price is larger the larger is the quantity purchased in the second auction. The optimal ex ante value of $X \in \{0, 1, \ldots, K\}$ will minimize

$$C(X,K) = X\left(\frac{X+1}{N+1} + \frac{K-X+1}{2(N-X)(N-X+1)}\right) + (K-X)\left(\frac{K-X+1}{N-X+1}\right) \quad (1)$$

and, if there is no second auction, then the second term in the right hand side of (**??**) is zero. So, $C(0,K) = C(K,K) = K\left(\frac{K+1}{N+1}\right)$. It will be optimal, ex ante, to divide the auction



quantity between the two dates. The following tables provide some illustrative calculations.

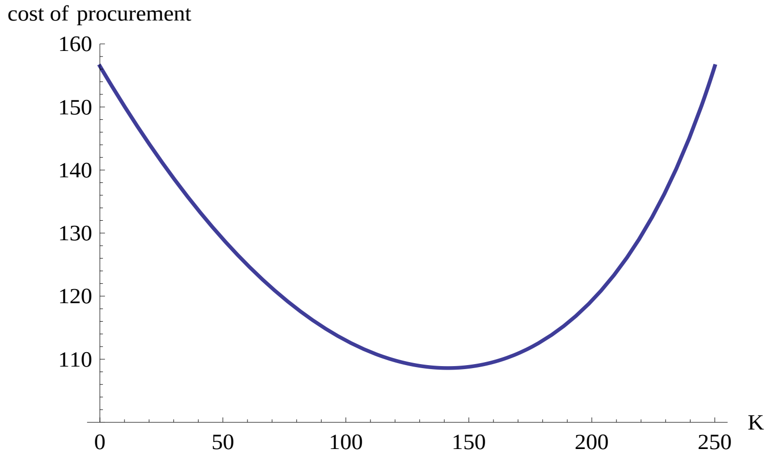
The first example assumes $N = 10$ and $K = 4$

x	y	p_1	p_2	Total Costs
0	4	-	.46	1.82
1	3	.25	.4	1.45
2	2	.31	.33	1.29
3	1	.38	.25	1.39
4	0	.46	-	1.82

The second example considers the case in which N = 400, K = 250

x	y	p_1	p_2	Total Costs
0	250	0.198	0.63	156.48
100	150	0.377	0.501	112.98
140	110	0.442	0.425	108.60
141	109	0.443	0.423	108.598202
142	108	0.445	0.421	108.598178
143	107	0.446	0.419	108.60
145	105	0.449	0.414	108.63
150	100	0.457	0.402	109.79

What these examples illustrate is that there is a benefit in splitting the auction quantity over the two auctions, and some advantage of auctioning more in the first auction than in the second, at least when there is a large quantity to be purchased in aggregate. The reason to divide the auction quantity is that the auction manager gets a larger sample of offers.



A bidder at one date is effectively a different bidder at a different date, in that its costs can be higher or lower. Dividing the volume lowers the expected procurement costs, absent strategic bidding. However, bidders will behave strategically. Bidders in one auction will want a higher profit margin than in a later auction. This strategic withholding effect limits how much the auction manager will want to delay procurement.

3.2.2 Ex Post Division - Optimal Reserve Prices

Here, I supposed that the auction manager can decide, based on the bids, how much to purchase in the first auction. This is not always practical, as to implement such policies will require pauses in the auction while decisions are made, or for formulas to be worked out in advance. At times, it will be possible to work out decision rules in advance.

To see the benefits, I revisit the two unit-two-auction example from above. It is assumed that bidders in the first auction will act strategically, so that a low-cost bidder will decide to withhold supply when that bidder thinks it can expect a better return in the second auction. The fact that in the last auction it is possible to calculate expected costs of one or two units, allows both the auction manager and strategic bidders to explicitly calculate reservation prices. Thus, in the first auction, the auction manager should

1. Never purchase at all when there are no bids below the expected per unit cost of deferring the purchase of both units to a second auction.

2. Purchase only one lot in the first auction, if the purchase price of the first lot is below the expected cost of a second lot in the second auction, and the price of the second lot in the first auction is above its expected cost in the second auction.

3. Purchase *both* lots in the first auction when the price of the second lot in the first auction is below the expected price of a first lot in the second auction.

These three principles pin down an effective demand curve for the auction manager in the first auction. Going back to the above example in which the auction manager must purchase two lots, and bidders have uniform distributions of costs in each auction, it is straightforward to work out the reservation prices. The reservation price for the second lot in the first auction should be $\frac{2}{n}$. The second lowest cost of the n remaining bidders will determine the expected price when only one lot is purchased in the second auction. And the reservation price for the first lot in the first auction is $\frac{3}{n+1}$. In his case, the third lowest cost will determine the expected price when two lots are purchased in the second auction.

As an illustration of the expected savings of the closed loop solution as compared to the open loop one, suppose there are 4 bidders.

First consider the case where the two auctions are conducted with separate reserve prices for each unit in the first auction. In this case, the reserve price for the first unit is 0.6 and 0.5 for the second unit. All four bidders will have costs greater than 0.6 with probability 0.4^4, in which case both units will be purchased in the second period. One unit will be purchased in the first auction when at least one bidder has costs below 0.6 and no more than one bidder has costs below 0.5. This occurs with probability 0.2969. Finally, two units will be purchased in the first auction when two or more bidders have costs no more than 0.5.

This has a probability of 0.6875.

In comparison, when the auction manager decides in advance to purchase one unit in each auction, it would purchase one unit in the first action or pay a higher price in the first auction 50% of the time as compared to the closed loop case. The open loop expected costs are 0.9066 whereas the closed loop expected costs are 0.8566.

3.3 Bidder Strategic Withholding

The above examples assume that each seller has only one unit to sell. When a bidder has a significant share of the auction volume, and can withhold a fraction of it in the first auction, a bidder will have an additional incentive to withhold supply. The simplest example is the case in which the auction manager wants to purchase (or sell) two lots, and one bidder has the capacity to win both. To illustrate the effects of having a "large" bidder, I assume that there are three bidders in total, two of which are small, having the capacity to bid for only one lot each, and the other is large, and can win both. I let $c_1 \leq \tilde{c}_1$ denote the costs of the two lots that the large bidder can provide, and $c_f \leq \tilde{c}_f$ denote the costs of the two small bidders (the subscript f denotes the "field"). As above, each of the $c_j's$ are drawn from a uniform distribution on $[0, 1]$ in each auction. In what follows, I let (c_1, c_2, c_3, c_4) denote the four draws of costs ordered from lowest to highest.

There are essentially six possible orderings:

1. The large bidder has the two lowest cost units. $[(c_1, \tilde{c}_1, c_f, \tilde{c}_f)]$
2. The large bidder has the first and third lowest cost units.$[(c_1, c_f, \tilde{c}_1, \tilde{c}_f)]$
3. The large bidder has the first and fourth lowest cost units.$[(c_1, c_f, \tilde{c}_f, \tilde{c}_1)]$
4. The large bidder has the second and third lowest cost units. $[(c_f, c_1, \tilde{c}_1, \tilde{c}_f)]$

5. The large bidder has the second and fourth lowest cost units.[$(c_f, c_1, \tilde{c}_f, \tilde{c}_1)$], and

6. The large bidder has the two highest cost units.[$(c_f, \tilde{c}_f, c_1, \tilde{c}_1)$]

If there is a single auction, then the large, two-lot, bidder can have an incentive to bid too high, or withhold supply, in cases 1, 2 and 4. It can affect price in all the cases, but would not have an incentive to do so in those other cases - assuming all units are sold at the third lowest price.

In case 1, the large bidder who bids its second lot high, will earn an expected profit of $\frac{4}{5} - c_1$ instead of $\frac{3}{5} - c_1 + \frac{3}{5} - \tilde{c}_1$. On average, the large bidder will be indifferent between over-bidding as compared to sincere bidding, but depending on its realized costs, may choose to over bid its second lot.

In cases 2 and 4, the large bidder's losing bid sets the price, and so it can benefit from over-bidding.

In cases 3 and 5, a bidder from the field sets the price, and so the large bidder has no incentive to overbid.

In case 6, the large bidder does not win, and has no incentive to overbid. On net, the expected per unit cost will be $\frac{3}{5}$ with probability $\frac{2}{3}$, and $\frac{4}{5}$ with probability $\frac{1}{3}$,or a total cost of $\frac{4}{3}$.

Next, consider the case in which there are two auctions, and that one unit is purchased in each auction. Also, assume, as above that each auction is a second-price sealed-bid auction. Again, there are six logically relevant cases in the first period, as described above. The large bidder is the logical winner in cases 1, 2 and 3. In case 1, it clearly can benefit from over-bidding its second lot, and there is no cost in doing so. The question is what is the benefit to the large bidder of overbidding its first lot.

If the large bidder does NOT over bid on the first lot, it has a $\frac{1}{3}$ probability of winning the second auction, and earning an expected surplus of $\frac{1}{4}$ for a total expected surplus, conditional on winning the first auction of $\frac{1}{12}$. If the large bidder loses the first auction, by over-bidding, it then has a $\frac{2}{3}$ probability of winning the second auction, and earning an expected surplus of $\frac{1}{4}$ with probability $\frac{1}{6}$ or $\frac{1}{2}$ with probability $\frac{1}{12}$. Its expected surplus is then again $\frac{1}{12}$. Thus, the large bidder has no expected gain from over-bidding in the first auction, on average, but may want to do so if its low cost lot in the first auction is not very low, that is if $c_1$is large. On the other hand, if c_1 is small, then it will not want to overbid in the first auction. Even if the large bidder over bids its first lot in cases 2 and 3, the expected cost in the first auction will be not more than $\frac{2}{3} \times \frac{2}{5} + \frac{1}{3} \times \frac{3}{5} = \frac{7}{15}$.

If the large bidder is a first auction loser, then it will win with probability $\frac{2}{3}$ and set the price too with probability $\frac{1}{3}$. In this case the expected price in the second auction will be $\frac{3}{4}$. Otherwise, with probability $\frac{2}{3}$, the expected cost in the first auction will be .5. Expected costs in the second auction cannot exceed $\frac{7}{12}$. Total costs in the two auctions cannot exceed $\frac{7}{15} + \frac{7}{12} = 1.05$, which is less than the expected costs when both units are purchased in a single auction. Thus, as this example illustrates, the auction manager will have incentives to split the procurement across dates even where large strategic bidders can also act strategically to shift their supply too. The auction manager can further benefit from setting unit and bidder specific reserve prices.

4 Two Auction Cases

This section considers a more general set of two auction cases. I assume that the auction manager wants to purchase K lots, and can do so at two different, specific dates.

Again there are N bidders, and each bidder j has costs c_{jt} in auction $t = 1, 2$. Here, I assume that, for $t = 1$, these costs are distributed over the unit interval, $[0, 1]$, but no longer assume that the costs are independently drawn at each date. I first consider the case in which costs in period 2 are $c_{j2} = c_{j1} + z$, where z is some random shock, which is the same for all bidders. In this case, the Martingale Theorem still applies.

Proposition 1 *Suppose there are N suppliers, who each can sell at most one unit. Let c_{jt} denote $j's$ cost in period $t = 1, 2$. Suppose that the c_{j1}'s are uniformly distributed over $[0, 1]$, and $c_{j2} = c_{j1} + z$ for each j. Suppose the auctioneer will purchases X units in the first auction, and $Y = K - X$ units in the second auction, where $N > K$. Suppose too that the price in each auction is the highest losing offer. Also suppose that $E[c^{K+1}|c]$ is an increasing function of c, where $E[c^{K+1}|c]$ is the expected value, as of period 1, of the $(K+1)^{st}$ lowest cost in period 2 for a bidder with cost c. Then the first period price is $E[c^{K+1}|c^K]$ and the second period price is $c^{K+1} + z$, where and $c^{K+1} + z$ is the $(K+1)^{st}$ lowest cost in period 2.*

Proof: The proof follows Milgrom (2004). Consider the second period auction. The expected value of the price will be $c^{K+1}+z$ as the $Y = K-X$ lowest cost suppliers remaining in the the second auction will win, and this is the expected value of amount that the lowest cost loser will bid, assuming that the X lowest cost suppliers win in the first auction. If one of these X bidders does not win in the firms auction, but another of the K lowest cost bidders win in the first auction, the expected value of the price in the second auction will still be $c^{K+1}+z$. And, it cannot be an equilibrium for one of the K lowest cost bidders to submit losing bids in both auctions. Now, no bidder in the first auction should accept a price lower than the expected value of the price in the second auction. This means that bidders whose costs are lower than $E[c^{K+1}|c]$ will want to offer this amount in the first auction, and bidders with higher costs will bid their true costs. The assumption that $E[c^{K+1}|c]$ is an increasing function of c, means that the X lowest cost bidders will win the first auction. In the second auction, the next $K - X$ low cost bidders will win, as each bidder will bid its true costs. Note, that these are also the $K - X$ lowest cost losers in the first auction.

■

The above is a straightforward generalization of Weber's Martingale Theorem.

Lemma 1 *Suppose there are N suppliers, who each can sell at most one unit. Let c_{jt} denote $j's$ cost in period $t = 1, 2$.Suppose that the c_{j1}'s are uniformly distibuted over $[0, 1]$, and that $c_{j2} = c_{j1} + z + \varepsilon_j$ for each j, where each ε_j is an independent random draw from a (non-degenerate) distribution with a zero mean. Suppose the auctioneer will purchases X units in the first auction, and $Y = K - X$ units in the second auction, where $N > K$.. Supppose that the price in each auction is the highest losing offer. Also suppose that $E[c^{K+1}|c]$ is an increasing function of c, where $E[c^{K+1}|c]$ is the expected value, as of period 1, of the $(K + 1)^{st}$ lowest cost in period 2 for a bidder with cost c. Then, the expected value of the price in the second auction is $E[c_2^{Y+1}|c = E(c_1^{X+1})]$, and the expected value of the price in the first auction is*

$$E\ [c_1^{X+1}|c = c^{X+1}] + E(\max\{0, c_2^{Y+1} - c - \varepsilon - z\}|c = c^{X+1})]$$

where c_1^{X+1} is the $X + 1^{st}$ lowest cost in period 1, and c_2^{Y+1} is the $(Y + 1)^{st}$ lowest cost in period 2.

Proof: The assumption that $E[c^{K+1}|c]$ is an increasing function of c means that the K lowest cost bidders in period 1 will have the lowest K expected costs for period 2, and the bidder with the $(X + 1)^{st}$ lowest cost in period 1 will have the lowest losing bid in the first auction.

Conditional on $c = c_1^{X+1}$, i.e., on a bidder with costs c having the $(X + 1)^{st}$ lowest cost in period 1, this bidder will want a higher price than c_1^{X+1} in period 1, as its expected payoff in period 2 is $E\Big[\{\max\{0, c^{K+1} - c - \varepsilon - z\} \, |c = c^{X+1}\Big] > 0$. Note, that the expected value of the price in period 1 is $E[c_1^{X+1}] + E[\max\{0, E[c_2^{y+1}] - c - \varepsilon - z|c = c^{X+1}\}] > E[c_2^{Y+1}|c = E(c_1^{X+1})]$, which is the expected price in period 2.

■

Notice that if bidder costs remain the same across auctions, then the first period price is $E[c_2^{K+1}|c_1^{X+1}]$ as in Weber's Martingale theorem.

The above lemma provides conditions that equilibrium prices must satisfy.

Proposition 2 *Suppose the conditions of Lemma 1 are satisfied. Suppose that $X + Y = K$, and $E\big[c_1^{X+1}\big] = E\big[c_2^{Y+1}\big]$, so that as independent auctions, competition would be the same in the two auctions. Then the price will be lower in the second auction.*

Proof: The price in the second auction is $E\big[c_2^{Y+1}\big]$, and in the first auction, the price is

$$
\begin{aligned}
&= E[c_1^{X+1}|c = c^{X+1}] + E\left\{\max\{0, c_2^{y+1} - c - \varepsilon - z\}|c = c^{X+1}\right\} \\
&= E[c_2^{Y}] + E\left\{\max\{0, c_2^{y+1} - c - \varepsilon - z\}|c = c^{X+1}\right\} > E[c_2^{Y+1}]
\end{aligned}
$$

■

Proposition 2 does not impose any conditions other than that the two auctions, if conducted independently, would be equally competitive. Generally, bidders' strategic incentive to delay sales means creates a slight bias in favor of an auction manager purchasing more in the first auction, assuming competition is the same at both dates.[17] Thus, if the auction manager can choose how to divide the amount purchased over time, and must choose this in advance of the first auction, the auction manager will want to purchase less in the second auction, assuming other circumstances are otherwise the same.

[17]In a forward auction, the auction manager would want to sell more in the first auction.

Proposition 3 *Suppose the assumptions of Lemma 1 are satisfied and that $E(z) = 0$. Also, suppose $E[c_i^{(j+1)}]$ are increasing in j, for $1 = 1, 2$ and $j = X, Y$, and that $E[c_1^{K+1}] = E[c_2^{K+1}]$, then minimizing of expected costs will require some lots purchased in both auctions.*

Proof: If all lots are purchased in one auction, then expected costs are $K \times E[c_1^{K+1}] = K \times E[c_2^{K+1}]$. Purchasing one lot in period 1 and $(K-1)$ lots in period 2 reduces expected costs in period 2 to $(K-1) \times E[c_2^K]$ and expected costs in period 1 will be $E[c_1^2] + E\left\{\max\{0, E[c^K|c] - c - \varepsilon - z\right\}$. The assumed monoticity of $E[c_i^{(j+1)}]$ implies that per unit expected costs in both periods 1 and 2 must decrease.

■

Note that differentiation of equation **??** with respect to X at $X = 0$ and $X = K$ can also establish the above result. Essentially, the assumption that increasing volume in an auction leads to a less competitive, or higher, procurement costs, not factoring in strategic incentives of bidders to delay sales, means that there is some cost savings from splitting the procurement across auctions. In the example in the previous section, with uniform cost, stochastically equivalent cost distributions in each period, costs will be minimized when somewhat more is purchased in the first period than in the second. There is also an ex ante bias. If dividing the total amount needed to be auctioned into X in the first auction and Y in the second would tend to minimize total expected costs absent bidder incentives to withhold supply in the first auction, then Proposition 2 implies that on the margin, the amount allocated in the second auction should be kept so low that expected costs in the second are no higher than the first auction. If not, bidders will have incentives to wait. Of course, costs can change, or information can be revealed, that offsets this affect, and would still mean that the optimal ex ante decision could still entail more being purchased at the later date.

The above assumes that the auction manager set X and Y in advance. This is essentially an "open loop" approach. A question arises as to whether the auction manager need, or should commit, to a specific auction volume, or to retain discretion to purchase a larger fraction than initially planned, or than would be purchased in the first auction assuming an open loop optimal allocation. A commitment not to purchase more at a given date than a given amount can those bidders in the first auction to bid more aggressively as the opportunities to sell in the second auction will be limited.[18] However, offsetting this are the benefits that accrue to the auction manager from being able to adjust purchases based on the actual level of competition in the auction. In the simple case, with only two auctions, discretion, or a closed loop solution, will result in lower expected costs.

Proposition 4 *Suppose the assumptions of Lemma 1 are satisfied. Suppose that the auction manager can set unit specific reserve prices in the first auction. Then, the expected costs will be minimized when the unit specific reserve prices p_j^r for unit j in the first auction as follows:*

$$p_j^r = (K - j + 1)P_{j-1,k-j+1} - (K - j)P_{j,K-j}$$

where $P_{j,K-j}$ is the expected price in the second auction when j units are sold in the first auction and $K - j$ units are sold in the second.

Proof: By construction, the terms $E[c_i^{(j+1)}]$ are increasing in j, for $1 = 1, 2$ and $j = X, Y$. Let $X^* + Y^* = K$ minimize ex ante expected costs in a sequence of two auctions. Then a reserve price for the j^{th} in the first auction equal to the difference in the expected costs in the second auction when purchasing one more unit in that auction.

■

[18] For example, Kydland and Prescott (1977) explain the advantages of rules over discretion.

5 Multiple Auctions

When there are three or more auctions, it is still possible to work backwards. A bidder's opportunity cost in one auction is not just the price it can get in the next auction. Rather, it is its expected surplus. When bidders are very similar ex ante, so there is little benefit in winning an early auction, but late in a sequence of auctions costs diverge, so that winners in late auctions can expect high surplus, bidder reservation prices in early auction can be relatively high. The converse is true when, ex ante, bidders have quite different costs, or expectations, that tend to diverge over time. To see this I slightly modify the example of Section 3 to allow for three auctions for one lot each. As above, it is assumed that there N bidders to start, and each bidder can win at most one lot. I suppose that in each auction each bidder's cost is a uniform draw from $[0, 1]$ in all three periods.

In this case, price will be increasing for all $N > \dot{3}$. The following table provides some sample values, where $P(t)$ denotes the price in the t^{th} auction.

N	20	10	5
P(1)	0.105	0.222	0.5
P(2)	0.103	0.214	0.483
P(3)	0.101	0.205	0.446
Total Cost	0.309	0.641	1.429

However, if there is costs diverge quite a bit at the end, this trend will reverse. The following table assumes that costs will spread out in the last period, so that expected surplus for the last period winner is ten times larger other things equal than in the previous example.

N	10
P(1)	0.222
P(2)	0.339
P(3)	1.305
Total Cost	1.866

The incentives illustrated by the above example are relevant when bidders do have different abilities to perform, but may only discover these differences over time. On the other hand, when there are common value components to costs, as well as idiosyncratic ones, bidders will have relatively little incentive to wait, as surplus will be relatively higher in the early auctions.

When there is a sequence of K procurement auctions, there is some tendency for prices to rise over time. To see this consider the following simple situation in which, as above, the auction manager purchases one lot in each of K auction[19]. Then,

Proposition 5 *Suppose, there is a sequence of K auctions $N > K$ bidders, and that the auction manager will purchase one unit in each auction. Also suppose that no bidder can win more than one auction. Suppose that in each auction, each bidder's cost is an independent draw from the same distribution. For N sufficiently larger than K, the expected price will necessarily be an increasing function of the number of auctions remaining, or decreasing from one auction to the next .*

Proof:

Let $c_j(t)$ denote bidder j's costs in auction t and $\Delta = E[c^2(t) - c^1(t)]$ where $c^j(t)$ denotes the j^{th} lowest (realized) cost in auction t. Let ρ denote probability that a bidder has the

[19]Englebrecht-Wiggans (1994) provied a result in which N is large, and $(N - K)$ is fixed in size.

lowest cost in the first auction, and $\rho(t)$ this probability in auction $t = 2, 3, \ldots, K$. Notice that $\rho(t)$ is increasing in t but, for large enough N, it is arbitrarily close to ρ. Notice that bidder j's willingness to sell in auction t is approximately $c_j(t) + \Delta \frac{1}{\rho}(1-\rho)\left[1-(1-\rho)^{k-t+1}\right]$ and that this is decreasing in t.

■

The above Proposition examines the case when the number of bidders becomes large, but the total number of lots remains constant, or small relative to the number of bidders. There are several other ways in there can be a large number of bidders and auctions. One variant is that there can be a large number of auctions, but on average there is a constant ratio of bidders to objects. This would be the case when the buyers and sellers both have other options, and there is continuing exit and entry, or that participation is a variable. Another is where there are a large number of lots, and so, most bidders will eventually become winners.[20]

6 Other Types of Sequential Auctions

This paper provides an analysis of a few types of sequential auctions. The main insight is that expected surplus in later auctions and not just expected price will affect prices in early auctions. This insight applies where bidder opportunities change over time, or where there is learing, and information aggregation within auctions. There are four other factors that can be significant in sequential auctions, and which are not addressed in the above analysis.

First, it is assumed that the auction manager has a fixed, inelastic demand. Clearly, this is unrealistic. In addition, the auction manager will often have a reserve price or a cap on what it can or will pay. As noted in the introduction, a number of previous papers ahve looked at various aspects of sequential auctions in this case.

[20]Engelbrecht-Wiggans (1994) analyzed this case.

Second, bidders, at times, view products purchased or supplied in sequential auctions as complements. For example, it has been argued that bidders in spectrum auctions in the US and Europe faced complementarities.[21] Here I very briefly sketch how complementarities can affect bidding in sequential auctions. Here I consider two lots, and N bidders. Each bidder has a cost of 0 for 0 lots, and a cost of 1 for one lot or two lots. I assume that the auction manager purchases the two lots, one at a time, in consecutive second price sealed-bid auctions. The equilibrium is easily calculated. Bidders will each offer 0 in the first auction and 1 in the second, assuming ties in the second auction also are resolved in favor of the low cost bidder. All bidders get zero profits, and price rises. More generally, complements will cause prices to rise in reverse auction and fall in forward auctions. Arguably, this is the pattern of prices in the European 3G spectrum auctions in which prices fell. Budget constraints can also give rise to declining prices, and can face bidders in auctions.[22] There was also declining participation in the European 3G auctions - there were also fewer bidders in the later auctions, although the some of the losers in the first few auctions *did* often particpate in, and win, later auctions.

European 3G Auctions

Country	Date	Per Capita Revenue	Number of bidders/licenses
UK	March, April 2000	\$4.90	13/5
Germany	July, August 2000	\$4.74	7/ 4 - 6
Netherlands	July 2000	\$2.33	6/5
Italy	October 2000	\$1.84	6/5
Austria	November 2000	\$0.62	6/4-6
Switzerland	December 2000	\$0.13	4/4

Third, bidders may be able to bid into different markets. A budget constrained bidder might be able to purchase a spectrum license in the UK or Germany, but may not be able to acquire both. An energy service provider may be able to sell electricity in Northern Californa

[21] See Klemperer (2004) and Milgrom (2004).

[22] Salant (1997) and Bulow, Levin and Milgrom (2008) have analyzed the impact of budget constraints in spectrum auctions.

or Southern Californa, but will have capacity constraints. This analysis has not explicitly incorporated an additional auction. However, the results should apply to the extent that bidder opportunity costs woudl still vary in the ways described across auctions.

Fourth, bidders may have market power. An auction manager's decisions about how to divide what needs to be auctioned over time may have limited effect when there are bidders who serve a large share of the auction volume. This does mean that the auction manager's discretion will not affect price, just that it has less impact within a given set of constraints about the timing of the auctions and no inability to reduce the aggregate volume auctioned. At an extreme, if the auction manager can set low reserve prices for procurement auctions, and high reserve prices in forward auctions, and very, very gradually adjust the reserve price if there are no bidders, the sequence of auction will have many of the properties of a one-sided bargaining game, and the auction manager will be able to extract much of the bidders' surplus.

The analysis has useful implications for market design. For example, the California energy crisis in the Summer 2000 was often blamed on the lack of long term contracts. This analysis suggests that problem was perhaps more with the excessive reliance on real-time contracts.[23]

[23] See Joskow (2001), and Joskow and Kahn (2001).

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