

# Existence of Equilibrium Prices for Discontinuous Excess Demand Correspondences

Vincenzo Scalzo\*

Received 2 February 2011; Accepted 11 April 2011

**Abstract** The aim of the paper is to obtain the existence of equilibrium prices in economies where the excess demand correspondences—differently from the usual condition—are not necessarily upper semicontinuous. So, in our setting, we cannot use the Gale-Debreu-Nikaido Lemma. The existence of equilibrium prices is obtained for discontinuous excess demand correspondences which obey to a condition like of the weak axiom of revealed preferences.

**Keywords** Discontinuous excess demand correspondences, existence of equilibrium prices, weak axiom of revealed preferences

**JEL classification** C62, D51

## 1. Introduction

Consider an economy with a finite number of commodities. The main investigation over such an economy aims to predict, from the properties of the excess demand correspondence, if equilibrium prices exist. The first result in this sense is the well known Gale-Debreu-Nikaido Lemma (GDNL in shorth; see, for example, Border 1985), where the excess demand correspondence is required to be upper semicontinuous. Let us remark that the upper semicontinuity is an extension to set-valued functions of the continuity of (single-valued) functions (see Berge 1959; Border 1985).

Successively, in some papers, where special cases have been studied, the existence of equilibrium prices have been obtained with techniques different from the GDNL: see Greenberg (1977), Barbolla and Corchon (1989), John (1999). More precisely, besides to satisfy the conditions required by the GDNL, additional properties over the (upper semicontinuous) excess demand correspondence have been introduced.

On the other hand, a recent literature deals with situations which lead to excess demand correspondences that are not upper semicontinuous (we call *discontinuous* a set-valued function which is not upper semicontinuous): see Xie (2005), Bich (2005), Kara (2009). So, in these models the existence of equilibrium prices cannot be investigated by using the GDNL.

In the framework in which the excess demand correspondences are discontinuous, we aim to explore new tools which allow to obtain new sufficient conditions for the existence of equilibrium prices. We deal with excess demand correspondences which

\* University of Naples “Federico II”, Department of Mathematics and Statistics, via Cinthia (Monte S. Angelo), 80126 Napoli, Italy. E-mail: [vincenzo.scalzo@unina.it](mailto:vincenzo.scalzo@unina.it), Phone: +39081675007.

satisfy a property, weaker than upper semicontinuity, that we call *upper hemicontinuity over segments*. This property is different from those considered in Xie (2005) and in Bich (2005). Besides, we require that the excess demand correspondence obeys to a property like of the weak axiom of revealed preferences (see Samuelson 1938). A key role in our result has been played by variational inequalities of Minty's type (see Kinderlehrer and Stampacchia 1980; Baiocchi and Capelo 1984; Baiocchi 1997): for a given excess demand correspondence, we consider a variational inequality which admits solutions and, with the aid of a separating hyperplane theorem, we prove that the solutions of such variational inequality are equilibrium prices.

The paper is organized as follows: the setting and the mathematical tools are presented and discussed in Section 2; the result is given in Section 3, together with examples that clarify the role of the assumptions used in Theorem 1; comments on the tools employed in our result and hints for possible generalization of the model are in Section 4.

## 2. The setting

Assume that the number of commodities of the economy is  $\ell$  (a natural number) and let  $\Delta = \{p \in \mathbb{R}_+^\ell : p_1 + \dots + p_\ell = 1\}$  be the set of prices. Moreover, suppose that the agents are price takers. For any vector of prices  $p$ , let  $\zeta(p)$  be the set of bundles which come from the subtraction between the aggregate demand and the supply corresponding to  $p$ : we assume that  $\zeta(p)$  is non-empty for any  $p \in \Delta$ . The *excess demand correspondence* of the economy is the set-valued function  $\zeta : p \in \Delta \Rightarrow \zeta(p) \subseteq \mathbb{R}^\ell$ , see Border (1985), Mas-Collel et al. (1995).

Now, let us recall some definition (see, for example, Berge 1959; Border 1985; Mas-Collel 1995), where  $\mathbb{R}^\ell$  is endowed with the topology induced by the Euclidean norm  $\|\cdot\|$ :

- (i)  $\zeta$  satisfies the *weak Walras' law* if  $p^\top z \leq 0$  for any  $z \in \zeta(p)$  and any  $p \in \Delta$ ;
- (ii) a price  $p^* \in \Delta$  is said to be an *equilibrium* of the economy<sup>1</sup> if  $\zeta(p^*) \cap -\mathbb{R}_+^\ell \neq \emptyset$ ;
- (iii)  $\zeta$  is said to be *upper semicontinuous* if for any  $p \in \Delta$  and for any open set  $O$  which includes  $\zeta(p)$ , there exists an open neighborhood (relative to  $\Delta$ )  $N$  of  $p$  such that  $\zeta(p') \subseteq O$  for all  $p' \in N$ ;
- (iv)  $\zeta$  is said to be *upper hemicontinuous* if for any  $p \in \Delta$ , for any sequences  $(p_n)_n \subset \Delta$  converging to  $p$  and for any sequence  $(z_n)_n$  such that  $z_n \in \zeta(p_n)$  for each  $n$ , there exists a subsequence of  $(z_n)_n$  which converges to an element of  $\zeta(p)$ .

**Remark 1.** Let  $F : X \Rightarrow Y$  be a set-valued function with non-empty values, where  $X$  and  $Y$  are non-empty subsets of  $\mathbb{R}^\ell$ . About the connections between the upper semicontinuity and the upper hemicontinuity, we observe that:

<sup>1</sup> We consider *free-disposal* economies (see Mas-Collel et al. 1995).

- (i) If  $F$  is upper hemicontinuous then  $F$  is compact-valued and upper semicontinuous. In fact, for any  $x \in X$  and for any sequence  $(y_n)_n \subseteq F(x)$ , it is sufficient to take the sequence  $(x_n)_n$  with  $x_n = x$  for any  $n$  in order to prove that  $F(x)$  is compact. Besides, if  $F$  is not upper semicontinuous, for some  $x \in X$ , there exist an open set  $O$  which includes  $F(x)$  and a sequence  $(x_n)_n$  converging to  $x$  such that  $F(x_n) \setminus O \neq \emptyset$  for any  $n$ . Hence, if  $y_n \in F(x_n) \setminus O$  for any  $n$ , by virtue of the upper hemicontinuity of  $F$ , there exists a subsequence of  $(y_n)_n$  which converges to an element of  $F(x) \subseteq O$  and this is in conflict with  $y_n \notin O$  for any  $n$ .
- (ii) If  $F$  is compact-valued and upper semicontinuous and if  $\bigcup\{F(x) : x \in X\}$  is included in a compact subset of  $Y$ , then  $F$  is upper hemicontinuous. In fact, in light of Berge (1959, Théorème 6, p. 117), the graph  $G = \{(x, y) \in X \times Y : y \in F(x)\}$  of  $F$  is a closed set. Now, if  $x_n \rightarrow x$  and if  $y_n \in F(x_n)$  for any  $n$ , since the sequence  $(y_n)_n$  is included in a compact subset of  $Y$ , there exists a subsequence  $(y_{n'})_{n'}$  of  $(y_n)_n$  which converges to an element  $y \in Y$ . Since  $(x_{n'}, y_{n'})$  belongs to the closed set  $G$  for any  $n'$ , we obtain  $y \in F(x)$ .

**Remark 2.** Let  $(A_n)_n$  be a sequence of subsets of a metric space. We recall that—see Kuratowski (1966), *outer limit* of  $(A_n)_n$  in Rockafellar and Wets (1998)— $\text{Limsup} A_n$  is the set of points  $x$  for which there exists a subsequence  $(A_{n'})_{n'}$  of  $(A_n)_n$  and a sequence  $(x_{n'})_{n'}$  such that  $x_{n'} \rightarrow x$  and  $x_{n'} \in A_{n'}$  for any  $n'$ . Let  $F : X \rightrightarrows Y$  be a set-valued function with non-empty values, where  $X$  and  $Y$  are non-empty subsets of  $\mathbb{R}^\ell$ . We say that  $F$  is *upper convergent in the sense of Painleve-Kuratowski*, or also that  $F$  is *outer semicontinuous* (see Kuratowski 1966; Rockafellar and Wets 1998), if  $\text{Limsup} F(x_n) \subseteq F(x)$  for any sequence  $(x_n)_n$  converging to  $x$  and for any  $x \in X$ . Moreover,  $F$  is said to be *subcontinuous* (see Smithson 1975) if, for each  $x \in X$ , given a sequence  $(x_n)_n$  converging to  $x$  and a sequence  $(y_n)_n$  such that  $y_n \in F(x_n)$  for any  $n$ , there exists a subsequence of  $(y_n)_n$  which converges to an element of  $Y$ . Hence, we have an alternative formulation for the upper hemicontinuity:

$F$  is upper hemicontinuous if and only if  $F$  is upper convergent in the sense of Painleve-Kuratowski (outer semicontinuous) and subcontinuous.

The upper hemicontinuity and the upper semicontinuity are the assumptions widely employed in previous existence results of equilibrium prices: see Greenberg (1977), Border (1985), Barbolla and Corchon (1989), Mas-Collel et al. (1995), John (1999). In the following, we call *discontinuous* a set-valued function which is neither upper hemicontinuous nor upper semicontinuous.

In line with a recent literature which deals with discontinuous excess demand correspondences (see Xie 2005; Bich 2005; Kara 2009), our aim is to obtain new sufficient conditions for the existence of equilibrium prices when the excess demand correspondence is discontinuous. We use the following property, that we call *upper hemicontinuity over segments*:<sup>2</sup>

<sup>2</sup> We recall that, given  $x, y \in \mathbb{R}^\ell$ , the sets  $[x, y] = \{(1-t)x + ty : t \in [0, 1]\}$  and  $]x, y[ = \{(1-t)x + ty : t \in ]0, 1[ \}$  are called *segments*.

For any  $p', p'' \in \Delta$ , the set-valued function  $p \in [p', p''] \rightarrow \zeta(p)$  is upper hemicontinuous, where  $[p', p'']$  has the topology induced by the euclidean norm.

Let us note that the class of upper hemicontinuous set-valued functions is strictly included in the class of upper hemicontinuous over segments set-valued functions and an upper hemicontinuous over segments set-valued function is not necessarily upper semicontinuous. In fact, consider the (single-valued) function  $f$  defined on  $\mathbb{R}_+^2$  as below:

$$f(x, y) = \begin{cases} \frac{x^2 y}{x^4 + y^2} & \text{if } (x, y) \neq (0, 0) \\ 0 & \text{if } (x, y) = (0, 0) \end{cases} \quad (1)$$

The function  $f$  is upper hemicontinuous over segments but it is neither upper hemicontinuous nor upper semicontinuous. First, we show that  $f$  is upper hemicontinuous over segments. Obviously, the only point in which the statement has to be checked is  $(0, 0)$ . The property is trivial on the segments in which  $x = 0$  and  $y = 0$ , respectively. Let  $m \neq 0$  and consider the function over the line  $y = mx$ ; one has, for  $x \neq 0$ :

$$\lim_{x \rightarrow 0} f(x, mx) = \lim_{x \rightarrow 0} \frac{mx}{x^2 + m^2} = 0 = f(0, 0).$$

So,  $f$  is upper hemicontinuous over segments. Now, if one considers the sequence  $(x_n, kx_n^2)_n$ , where  $x_n \rightarrow 0$  and  $x_n \neq 0 \neq k$ , since

$$f(x_n, kx_n^2) = \frac{k}{1 + k^2} \neq f(0, 0) \quad \forall n,$$

one gets that  $f$  is not continuous at  $(0, 0)$ , that is:  $f$  is neither upper hemicontinuous nor upper semicontinuous.

Finally, consider an economy with  $\ell = 3$  commodities. Since the set of prices  $\Delta$  can be described by two coordinates  $(x, y)$ , we can suppose the case in which the excess demand is the function  $\zeta(x, y) = (f(x, y), 0, 0)$ , where  $f$  is defined by (1). Now, the function  $\zeta$  is neither upper hemicontinuous nor upper semicontinuous, but upper hemicontinuous over the segments included in  $\Delta$ .

### 3. Equilibrium prices in a discontinuous case

In this section we obtain the existence of equilibrium prices when the excess demand correspondence  $\zeta$  is upper hemicontinuous over segments. Besides, we consider excess demand correspondences satisfying a condition like of the weak axiom of revealed preferences.<sup>3</sup> The condition that we introduce is the following:

$\mathcal{A}$ : for any  $p, p' \in \Delta$ , one has  $p^\top z' + p'^\top z \geq 0$  for any  $z \in \zeta(p)$  and any  $z' \in \zeta(p')$ .

Note that the discontinuous set-valued function defined at the bottom of Section 2 satisfies axiom  $\mathcal{A}$ .

<sup>3</sup> The *weak axiom of revealed preference*, introduced in Samuelson (1938) for excess demand functions, is the following:  $p^\top \zeta(p') \leq 0$  and  $\zeta(p) \neq \zeta(p')$  imply  $p'^\top \zeta(p) > 0$ .

**Remark 3.** Assume that  $\zeta$  is a (single-valued) function. Consider the following stronger version of axiom  $\mathcal{A}$ : for any  $p, p' \in \Delta$ , one has  $p^\top \zeta(p') + p'^\top \zeta(p) > 0$ . It is easy to see that if  $\zeta$  satisfies this stronger version of axiom  $\mathcal{A}$ , then  $\zeta$  obeys to the weak axiom of revealed preferences.

Our existence result is based on *variational inequalities* of Minty's type (*M-VI* in short), see Kinderlehrer and Stampacchia (1980), Baiocchi and Capelo (1984), Baiocchi (1997): let  $\phi$  be a function from  $K \subseteq \mathbb{R}^\ell$  to  $\mathbb{R}^\ell$ ; the *M-VI* related to  $\phi$  is the following problem:

$$\text{Find } x \in K \text{ such that } \phi(y)^\top (y - x) \geq 0 \quad \forall y \in K.$$

We use the following *M-VI* associated to the excess demand correspondence  $\zeta$ :

$$\text{Find } p^* \in \Delta \text{ such that } h(p, p^*) \geq 0 \quad \forall p \in \Delta, \tag{2}$$

where  $h$  is defined on  $\Delta \times \Delta$  as below:

$$h(p, p') = \inf_{z \in \zeta(p)} p'^\top z.$$

More precisely, for an excess demand correspondence  $\zeta$  with non-empty, convex and compact values, we show that each solution of (2) is an equilibrium price of the economy whether  $\zeta$  satisfies the axiom  $\mathcal{A}$  and is upper hemicontinuous over segments. First, we recall the following result (see Baiocchi 1997, Teorema 4.2):

**Lemma 1.** *Let  $K$  be a non-empty, convex and compact subset of a topological vector space and let  $h$  be a real-valued function defined on  $K \times K$ . If  $h(x, \cdot)$  is concave and upper semicontinuous for any  $x \in K$  and if  $h(x, y) + h(y, x) \geq 0$  for any  $x, y \in K$ , then there exists at least a  $x^* \in K$  such that  $h(x, x^*) \geq 0$  for all  $x \in K$ .*

**Proposition 1.** *If  $\zeta$  satisfies the axiom  $\mathcal{A}$ , then (2) has at least a solution.*

**Proof.** For any  $p, p' \in \Delta$ , by axiom  $\mathcal{A}$  one has:

$$p'^\top z + p^\top z' \geq 0 \quad \forall z \in \zeta(p) \text{ and } \forall z' \in \zeta(p'),$$

which leads to:  $h(p, p') + h(p', p) \geq 0$  for each  $p, p' \in \Delta$ . On the other hand, for any  $t \in [0, 1]$  and any  $p, p', p'' \in \Delta$ , set  $p_t = (1 - t)p' + tp''$ , we have:

$$p_t^\top z = (1 - t)p'^\top z + tp''^\top z \quad \text{for any } z \in \zeta(p)$$

and taking the infimum over  $\zeta(p)$ , we have  $h(p, p_t) \geq (1 - t)h(p, p') + th(p, p'')$ , that is:  $h(p, \cdot)$  is concave. Moreover,  $h(p, \cdot)$  is upper semicontinuous for any  $p \in \Delta$  in light of Berge's Maximum Theorem (Berge 1959). Finally, all the assumptions of Lemma 1 are verified and the statement follows.  $\square$

**Proposition 2.** *Assume that  $p^*$  is a solution of (2) and  $\zeta$  satisfies the weak Walras' law. Then the following property holds:*

$$\text{For any } p \in \Delta \text{ and any } p' \in ]p^*, p] \text{ one has } p^\top z' \leq 0 \text{ for all } z' \in \zeta(p'). \tag{3}$$

**Proof.** Let  $p \in \Delta$ ,  $t \in ]0, 1]$  and  $p' = (1-t)p^* + tp$ . For any  $z' \in \zeta(p')$ , we have:

$$0 \geq p'^{\top} z' = (1-t)p^{*\top} z' + tp^{\top} z'$$

and since  $p^*$  is a solution of (2), we get:  $0 \geq tp^{\top} z'$ . So,  $p^{\top} z' \leq 0$  for any  $z' \in \zeta(p')$  and the thesis follows.  $\square$

Hence, we have:

**Theorem 1.** *If  $\zeta$  is upper hemicontinuous over segments with non-empty, convex and compact values, and if  $\zeta$  satisfies the axiom  $\mathcal{A}$  and the weak Walras' law, then there exists at least an equilibrium price.*

**Proof.** In light of Proposition 1, there exists a solution  $p^*$  of (2). Now, we prove that  $p^*$  is an equilibrium price. By contradiction, assume that  $\zeta(p^*) \cap -\mathbb{R}_+^{\ell} = \emptyset$ . Since  $\zeta(p^*)$  is non-empty, convex and compact, in view of the strict separating hyperplane theorem (see, for example, Border 1985) there exist  $q \in \mathbb{R}^{\ell} \setminus \{0\}$  such that, set  $p = q / \|q\|$ :

$$p^{\top} z > p^{\top} z' \quad \text{for all } z \in \zeta(p^*) \text{ and all } z' \in -\mathbb{R}_+^{\ell}. \quad (4)$$

If  $q \notin \mathbb{R}_+^{\ell}$ , one has  $q_i < 0$  for at least a  $i \in \{1, \dots, \ell\}$ . So, given  $z \in \zeta(p^*)$ :

$$q^{\top} z > \lambda q_i \quad \text{for any } \lambda < 0,$$

which leads to the contradiction  $q^{\top} z \geq +\infty$ . Hence  $p$  belongs to  $\Delta$ .

Now, let  $(p_n)_n$  be a sequence included in  $]p^*, p]$  and converging to  $p^*$  and let  $(z_n)_n$  be a sequence such that  $z_n \in \zeta(p_n)$  for each  $n$ . Since  $\zeta$  is upper hemicontinuous over the segment  $[p^*, p]$ , there exists a subsequence  $(z_{n'})_{n'}$  of  $(z_n)_n$  which converges to an element  $z^* \in \zeta(p^*)$ . In light of (3), we have:

$$p^{\top} z_{n'} \leq 0 \quad \forall n';$$

so  $p^{\top} z^* \leq 0$ . Since  $0 \in -\mathbb{R}_+^{\ell}$ , by (4) we get the contradiction  $p^{\top} z^* > 0 \geq p^{\top} z^*$ , which concludes the proof.  $\square$

The next examples clarify the role of the upper hemicontinuity over segments and of the axiom  $\mathcal{A}$  concerning the issue of the existence of equilibrium prices. More precisely, in Example 1, it is given an excess demand function which satisfies the Walras' law and the axiom  $\mathcal{A}$  and for which equilibrium prices do not exist, just because the upper hemicontinuity over segments does not hold. Example 2 shows that, for upper hemicontinuous over segments excess demand correspondences, it is not possible to give sufficient conditions for the existence of equilibrium prices by using only the axiom  $\mathcal{A}$  and leaving out the Walras' law.

**Example 1.** *Let  $\zeta$  be the function defined on  $\Delta \subset \mathbb{R}^2$  as below:*

$$\zeta(p_1, p_2) = \begin{cases} (0, 1) & \text{if } (p_1, p_2) = (1, 0) \\ (1, 0) & \text{if } (p_1, p_2) = (0, 1) \\ (-p_2, p_1) & \text{if } p_1 p_2 \neq 0 \end{cases}$$

*It is easy to see that the Walras' law and the axiom  $\mathcal{A}$  are verified but  $\zeta$  is not upper hemicontinuous over segments. Moreover, no equilibrium prices exist.*

**Example 2.** Consider the following function  $\zeta$  defined on  $\mathbb{R}_+^2$ , where  $f(x, y)$  is given by (1):

$$\zeta(x, y) = \begin{cases} (f(x, y), 1) & \text{if } (x, y) \neq (0, 0) \\ (0, 1) & \text{if } (x, y) = (0, 0) \end{cases}$$

The function  $\zeta$  is upper hemicontinuous over segments and satisfies axiom  $\mathcal{A}$  but the Walras' law fails to be verified. Moreover, there are no equilibrium prices for  $\zeta$ .

#### 4. Concluding remarks

The technique used in Theorem 1 is founded on two mathematical tools: variational inequalities and separating hyperplane theorems. These tools have allowed to relax the continuity assumption employed in the classical results on the existence of equilibrium prices in finite-dimensional spaces. In particular, since the inner product is continuous, axiom  $\mathcal{A}$  guarantees that a variational inequality has solutions and a separating hyperplane theorem and the upper hemicontinuity over segments allow that each of these solutions is an equilibrium price. Here, it is crucial to have a separation between  $\zeta(p)$  and  $-\mathbb{R}_+^\ell$  by strict inequalities, which is possible if the first set is compact and convex (see Border 1985). So, in our model the excess demand correspondence need to have compact and convex values.

In the setting of infinite-dimensional Banach spaces, the compactness in norm is an assumption often too strong. Moreover, in several models in economics, the sets of commodities are included in infinite-dimensional spaces and they are neither compact nor convex: see, for example, Mordukhovich (2006, ch. 8). Hence, a question arises: what about the existence of equilibrium prices in infinite-dimensional spaces for discontinuous excess demand correspondences with non-convex and non-compact values? Answers to this question will be the subject of a further paper.<sup>4</sup>

**Acknowledgment** The author wish to thank the anonymous referees for their comments and suggestions on the previous version of this paper.

#### References

- Baiocchi, C. and Capelo, A. C. (1984). *Variational and Quasivariational Inequalities. Applications to Free-Boundary Problems*. Chichester, J. Wiley & Sons.
- Baiocchi, C. (1997). Una panoramica sulle disequazioni variazionali. *Bollettino U.M.I.*, 7(11-A), 631–648.
- Barbolla, R. and Corchon, L. C. (1989). An Elementary Proof of the Existence of a Competitive Equilibrium in a Special Case. *Quarterly Journal of Economics*, 104, 385–389.
- Berge, C. (1959). *Espaces Topologiques*. Paris, Dunod.

<sup>4</sup> The author is grateful to a referee who suggested the book Mordukhovich (2006) and the investigation of the infinite-dimensional case.

- Bich, P. (2005). On the Existence of Approximate Equilibria in Discontinuous Economies. *Journal of Mathematical Economics*, 41 (4–5), 463–481.
- Border, K. C. (1985). *Fixed Point Theorems with Applications to Economics and Game Theory*. Cambridge, Cambridge University Press.
- Greenberg, J. (1977). An Elementary Proof of the Existence of a Competitive Equilibrium with Weak Gross Substitutes. *Quarterly Journal of Economics*, 91, 513–516.
- John, R. (1999). Abraham Wald's Equilibrium Existence Proof Reconsidered. *Economic Theory*, 13, 417–428.
- Kara, A. (2009). Implications of Multiple Preferences for a Deconstructive Critique and a Reconstructive Revision of Economic Theory. *Journal of Economic and Social Research*, 11(1), 69–78.
- Kinderlehrer, D. and Stampacchia, G. (1980). *An Introduction to Variational Inequalities*. New York, Academic Press.
- Kuratowski, C. (1966). *Topology*. New York, Academic Press.
- Mas-Collel, A., Whinston, M.D. and Green, J.R. (1995). *Microeconomic Theory*. Oxford, Oxford University Press.
- Mordukhovich, B. S. (2006). *Variational Analysis and Generalized Differentiation II: Applications*. Berlin, Springer-Verlag.
- Rockafellar, R. T. and Wets, R. J.-B. (1998). *Variational Analysis*. Berlin, Springer-Verlag.
- Samuelson, P. (1938). A Note on the Pure Theory of Consumers' Behaviour. *Economica*, 5, 61–71.
- Smithson, R. E. (1975). Subcontinuity for Multifunctions. *Pacific Journal of Mathematics*, 61, 282–288.
- Xie, Z. (2005). A New Approach for General Equilibrium with Discontinuous Excess Demand Function. Available at <http://ssrn.com/abstract=721362>.