Pareto Efficiency, Relative Prices, and Solutions to CGE Models

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Abstract

This paper analyzes Walrasian general equilibrium systems and calculates the static and dynamic solutions for competitive market equilibria. The Walrasian framework encompasses the basic multi-sector growth (MSG) models with neoclassical production technologies in N sectors (industries). The endogenous behavior of all relative prices and the sectorial allocation of the two primary factors (labor and capital) are analyzed in detail. The dynamic systems of Walrasian multi-sector economies and the family of solutions (time paths) for steady-state and persistent growth per capita are parametrically characterized. The technology parameters of the capital good industry are decisive for obtaining long-run per capita growth in closed (global) economies. Brief comments on the MSG literature are offered, together with short remarks on studies of industrial (structural) evolution and economic history.

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1. Introduction

Dynamics is concerned with calculating the motions of the widest variety of objects, and with deriving (computing) the implications (effects) of these motions. Its basic principles and logical structure have long been a model for other scientific disciplines. The domain of dynamics in physics (mechanics) have been extended tremendously on both macro- and microscopic scales, Pais (1986). In the discipline of economics, dynamics began in macroeconomics, in particular with business cycle and the basic one- and two-sector growth models. However, standard microeconomic (producer/consumer) theory is more naturally involved with decentralized mechanisms for resource allocation in static multi-sector modelling and general equilibrium dynamics. We may briefly refer to passages from important theoretical and empirical contributions to multi-sectoral model building.

Regarding the character of the works in *Applied General Equilibrium Analysis* (AGE-modelling), we may quote, Shoven & Whalley (1992,pp.1): "The central idea underlying this work is to convert the *Walrasian general equilibrium structure* (formalized in the 1950s by Arrow, Debreu, and others) from an abstract representation of an economy into realistic models of actual economies. Numerical, empirically based general equilibrium can then be used to evaluate concrete policy options by *specifying production and demand parameters* and incorporating data reflective of real economies"- "Most contemporary applied *general models* are *numerical analogs* of traditional *two-sector general equilibrium models* popularized by James Meade, Harry Johnson, Arnold Harberger, and others in the 1950s and 1960s. Earlier analytic work with these models has examined the distortionary effects of taxes, tariffs, and other policies, along with functional incidence questions."

Computation of general equilibria usually involves solving systems of nonlinear equations. Thus, according to Judd (1998, p.147, p.3): "The Arrow-Debreu concept of general equilibrium reduces to finding a *price vector* at which *excess demand* is zero; it is the most famous nonlinear equation problem in economics", and "The *computational general equilibrium* (CGE) literature is the most mature computational literature in economics". Much of the numerical (algorithmic) methodology derives from the work of Scarf et al. (1967,1973). Numerical approaches give approximate solutions; many papers on numerical methods offer little in the way of showing the qualitative dependence of the solutions upon critical parameter values. Obtaining analytical solutions in theorem-proof style is preferable wherever this is possible.

Our *emphasis* in this paper will be to provide a *conceptual framework* that supports *economic intuition* and offer a general and *unified analytical approach* to the mathematical procedures of obtaining the *static, comparative* *static and dynamic general equilibrium solutions of N-sector economies.* We extend the methodology and mathematical-economic analysis of two-sector dynamics, Jensen (2003), to the multi-sectorial dynamics of temporal Pareto-efficient labor and capital accumulation.

An important contributor to multi-sector generalizations of various macro-dynamic models was Jorgenson (1961), who extended the input-output methods of Leontief to *dynamic input - output analysis.* In the *macro-planning* and *development* literature, our topic of multi-sector economies has another important contributor in Leif Johansen (1959, 1974, pp.1): "It is a well known fact that the *various sectors* of an economy do *not expand* in the *same proportion* in a process of economic growth. The flows of investment and new labor are not allocated proportionately to all production sectors. Existing quantities of capital and labor may be *reallocated* during the growth process. Terms of trade between the production sectors may change in a systematic way, and so on. Such considerations illustrate aspects of the economic growth process which we shall attempt to explain and analyse within the framework of a *multi-sectoral growth model* ".

Regarding structural analysis and *sectorial developments*, the three categories of Colin Clark were dealing with the reallocation of the labour force over the groups of industries: Primary industries (Agriculture, Fisheries and Forestry), Secondary industries (Manufacturing, Handicraft, Building and Construction, Mining, Electric power production), Tertiary industries (Wholesale and Retail Commerce, Transport, Financial services and Public administration), and globally observing that, Clark (1951, p.365): " by careful generalisation of available facts to be the most important concomitant of economic progress, namely the movement of working population from agriculture to manufacture and from manufacture to commerce and services." Concomitant changes in the composition of demand, e.g.,the budget share of food (*Engel's law* of 1857) is confirmed by all surveys, Houthakker (1957).

Controversies related to "convergence" and "balanced growth" models of being incompatible with *structural change* and the process of economic development are given renewed attention in both the empirical and theoretical growth literature, Pasinetti (1981), Islam (1995), Echevarria (1997), Laitner (2000), Kongsamut, Rebelo, Xie (2001), Meckl (2002). Patterns of industrial growth will here be obtained and discussed.

As we do consider relationship between *resources, technology* and *economic evolution*, we may finally quote the historian, Abbot P. Usher (1954, p.1): "Economic history is concerned with the description and the analysis of the mutual transformations taking place between human societies and their environment. The *study* of *costs* and *prices* is *important*, and the institutional structure of organized social life demands careful attention, but the *basic problems* of *economic history* lie in the field of the *management* of *resources.* - The *quantitative* analysis of economic activities requires *study* of the processes and accomplishments of the *system* of *production* in *physical units* as well as in *value units.*"

This paper deals with *dynamic foundations* of MSG models for closed (global) economies. We attempt to explain well-known empirical facts by modelling in terms of *basic microeconomic principles.* It is organized such that section 2 presents an analytical framework with concepts, definitions and various micro and macro equivalence relations. Section 3 studies the relationships between *factor prices* and *relative commodity prices.* Section 4 analyses and derives GDP expenditure shares from some specific parameterizations of utility functions. Section 5 uses the proper NIPA version of *Walras's law* to obtain the *Walrasian equilibrium* of the multi-sector economy and derives the timeless (static and comparative static) *competitive general equilibrium solutions* for all the variables as composite functions of the factor endowments. Section 6 analyzes the *dynamic systems* and alternative *evolutions* of the multi-sector *general equilibria*, and section 7 gives various asymptotic sectorial growth rates of a persistent growing multi-sectoral economy. Section 8 offer final comments.

2. Analytical Framework for Multi-Sector (MS) Economies

2.1. The supply side, technology and efficient factor endowments allocation

Consider an economy consisting of N *industries* (sectors), and let sector 1 be a capital good industry. Sector *technologies*, $F_i(L_i, K_i)$, are described by nonnegative smooth concave homogeneous production functions with *constant returns* to scale in labor and capital, $i = 1, \cdots, N$,

$$Y_i = F_i(L_i, K_i) = L_i F_i(1, k_i) \equiv L_i f_i(k_i) \equiv L_i y_i, L_i \neq 0; \tag{1}$$

where the function $f_i(k_i)$, is strictly concave and monotonically increasing in the capital-labor ratio $k_i \in [0, \infty[$, i.e., f_i has the properties

$$\forall k_i > 0\colon \quad f_i'(k_i) = df_i(k_i)/dk_i > 0, \quad f_i''(k_i) = d^2 f_i(k_i)/dk_i^2 < 0 \tag{2}$$

$$\lim_{k_i \to 0} f_i'(k_i) \equiv \bar{\mathfrak{b}}_i \leq \infty, \ \lim_{k_i \to \infty} f_i'(k_i) \equiv \underline{\mathfrak{b}}_i \geq 0, \ f_i'(k_i) \in J_i \equiv \left[\underline{\mathfrak{b}}_i, \bar{\mathfrak{b}}_i\right]. \tag{3}$$

It is worth observing that if F is defined and continous on the axes, we have

$$F_i(0, K_i) = K_i F_i(0, 1), \quad F_i(L_i, 0) = L_i F_i(1, 0) = L_i f_i(0) \tag{4}$$

$$F_i(0, 0) = 0, \quad F_i(0, 1) \geq 0, \quad F_i(1, 0) \equiv f_i(0) \geq 0 \tag{5}$$

Furthermore, we note that

$$\lim_{k_i\to 0} f_i'(k_i) = \frac{\partial F_i}{\partial K_i}(1,0),\ \lim_{k_i\to\infty} f_i'(k_i) = F_i(0,1),\ \frac{\partial F_i}{\partial L_i}(1,0) = F_i(1,0) \tag{6}$$

The *sectorial output elasticities*, $\epsilon_{L_i}, \epsilon_{K_i}$, ϵ_i, with respect to marginal and proportional factor variation are, cf. (1),

$$\epsilon_{L_i} \equiv \frac{\partial \ln Y_i}{\partial \ln L_i} \equiv \frac{\partial Y_i}{\partial L_i}\frac{L_i}{Y_i} = \frac{MP_{L_i}}{AP_{L_i}} = 1 - \frac{k_i f_i'(k_i)}{f_i(k_i)} > 0, \quad k_i \neq 0, \tag{7}$$

$$\epsilon_{K_i} \equiv \frac{\partial \ln Y_i}{\partial \ln K_i} \equiv \frac{\partial Y_i}{\partial K_i}\frac{K_i}{Y_i} = \frac{MP_{K_i}}{AP_{K_i}} = \frac{k_i f_i'(k_i)}{f_i(k_i)} = \frac{\partial \ln y_i}{\partial \ln k_i} > 0, \tag{8}$$

$$\epsilon_i \equiv \epsilon_{L_i} + \epsilon_{K_i} = 1 \tag{9}$$

The *factor endowments,* total labor force (L) and the total capital stock (K), are inelastically supplied and are fully employed (utilized), i.e.,

$$L = \textstyle\sum_{i=1}^N L_i, \quad \sum_{i=1}^N L_i/L \equiv \sum_{i=1}^N \lambda_{L_i} = 1;\ \lambda_{L_i} = l_i \tag{10}$$

$$K = \textstyle\sum_{i=1}^N K_i, \quad \sum_{i=1}^N K_i/K \equiv \sum_{i=1}^N \lambda_{K_i} \equiv 1 \tag{11}$$

$$K/L \equiv k \equiv \textstyle\sum_{i=1}^N l_i k_i \tag{12}$$

where λ_{L_i}, λ_{K_i}, (10-11) are the *factor allocation fractions.*
At any point of the isoquants (1), the *marginal* rates of technical *substitution,* $\omega_i(k_i)$ are, by (2), positive *monotonic* functions,

$$\omega_i = \omega_i(k_i) = \frac{MP_{L_i}}{MP_{K_i}} = \frac{f_i(k_i)}{f_i'(k_i)} - k_i = \frac{\epsilon_{L_i}}{\epsilon_{K_i}} k_i > 0, \quad \forall k_i > 0 \tag{13}$$

and the *substitution elasticities,* σ_i between labor and capital is the proportionate change in the ratio (K_i/L_i) of inputs divided by the proportionate change in the ratio (ω_i) of the marginal products of inputs.

$$\sigma_i \equiv \frac{d\ln k_i}{d\ln \omega_i} \equiv \frac{dk_i}{d\omega_i}\frac{\omega_i}{k_i} = \frac{dk_i}{d\omega_i}\frac{\epsilon_{L_i}}{\epsilon_{K_i}} = \frac{d\ln AP_{L_i}}{d\ln MP_{L_i}} = \frac{d\ln AP_{K_i}}{d\ln MP_{K_i}} > 0 \tag{14}$$

The general relation between the sectorial factor output elasticities, (7-8), and the sectorial substitution elasticities, σ_i, is:

$$\frac{d\ln \epsilon_{K_i}}{d\ln k_i} \equiv \frac{d\epsilon_{K_i}}{dk_i}\frac{k_i}{\epsilon_{K_i}} = \frac{\epsilon_{L_i}(\sigma_i - 1)}{\sigma_i},\ \frac{d\ln \epsilon_{L_i}}{d\ln k_i} \equiv \frac{d\epsilon_{L_i}}{dk_i}\frac{k_i}{\epsilon_{L_i}} = \frac{\epsilon_{K_i}(1-\sigma_i)}{\sigma_i} \tag{15}$$

Free *factor mobility* between the multiple industries and also *efficient* factor *allocation* impose the *common* MRS condition, cf. (13),

$$\omega = \omega_i = \omega_i(k_i) \quad \forall i = 1, \cdots, N \tag{16}$$

For the variables k_i to satisfy (16), it is, beyond (2), further required that the intersection of the sectorial range for $\omega_i(k_i)$ is not empty,

$$\omega_i(k_i) \in \Omega_i = [\underline{\omega}_i, \overline{\omega}_i] \subseteq \mathbf{R}_+, \quad \omega \in \Omega \equiv \cap \Omega_i = [\underline{\omega}, \overline{\omega}] \neq \emptyset, \tag{17}$$

2.2. Efficient factor allocation, costs, and price systems/relative prices

All industries are assumed to operate under *perfect competition* (zero excess profit); absolute (money) input (factor) *prices* (w, r) are the same in both industries; and absolute (money) output (product, commodity) prices (P_i) represent unit cost. Thus, in each sector we have the *competitive producer equilibrium* equations,

$$w = P_i \cdot MP_{L_i}, \quad r = P_i \cdot MP_{K_i}; \quad \omega = w/r, \qquad P_i \neq 0 \tag{18}$$

$$P_i Y_i = wL_i + rK_i, \quad \epsilon_{L_i} = wL_i/P_i Y_i, \quad \epsilon_{K_i} = rK_i/P_i Y_i \tag{19}$$

Hence, (18) gives any *relative commodity price* P_i/P_j as

$$p_{ij} \equiv \frac{P_i}{P_j} = \frac{MP_{K_j}}{MP_{K_i}} = \frac{f_j'(k_j)}{f_i'(k_i)} = \frac{f_j(k_j) - k_j f_j'(k_j)}{f_i(k_i) - k_i f_i'(k_i)} = \frac{MP_{L_j}}{MP_{L_i}} = \frac{y_j \epsilon_{L_j}}{y_i \epsilon_{L_i}} \tag{20}$$

The *connection* between relative *factor* (service) prices and relative *commodity* prices follows from (16, 18, 20),

$$p_{ij}(\omega) = \frac{P_i}{P_j}(\omega) = \frac{MP_{K_j}[k_j(\omega)]}{MP_{K_i}[k_i(\omega)]} = \frac{f_j'[k_j(\omega)]}{f_i'[k_i(\omega)]}, \quad \omega = w/r \tag{21}$$

Average and marginal productivities were objects of pioneering empirical studies by von Thünen. In Table 1, factors are inessential (5), and accordingly the substitution elasticity is larger than one. Hence ϵ_L, (7, 15, 18, 19), is falling with mechanisation. Evidently with $\sigma > 1$, AP_L increases more than MP_L, (14). This table will serve us as a relevant and simple illustration of key sectorial growth numbers.

Table 1. von Thünen´s Productivity Data in Agriculture

K/L	AP_L	AP_K	MP_K	MP_L	ω	$\frac{1}{\omega} = \frac{r}{w}$	$MP_K \cdot k$	ε_L
0	110	–	–	110	–	–	–	1.00
1	150	150	40	110	2.75	0.364	40	0.73
2	186	93	36	114	3.17	0.315	72	0.61
3	218.4	72.8	32.4	121.2	3.74	0.267	97.2	0.55

Source: von Thünen (1850[1930, p.507]); Brems (1986, p.86).

2.3. Macro equivalence relations of the supply-demand side in MS-economies

Gross domestic product (GDP), national income, Y, is the *total* of sectoral producer *revenues* [monetary value of sector outputs, (1)]

$$Y_i = Ly_il_i, \qquad Y \equiv \sum_{i=1}^{N} P_iY_i = L\left[\sum_{i=1}^{N} P_iy_il_i\right] \equiv Ly \tag{22}$$

and is equivalent with (18-19) to the total *factor income,*

$$Y = wL + rK = L(w + rk) = L(\omega + k)P_if_i'[k_i] = Ly \tag{23}$$

Hence the factor income *distribution shares* $\delta_K + \delta_L = 1$, becomes

$$\delta_K \equiv rK/Y = rk/y, \delta_L \equiv wL/Y = w/y; \delta_K = k/(\omega + k), \delta_K/\delta_L = k/\omega \tag{24}$$

The “Final Demand” decomposition of GDP (Y) into the aggregate expenditures on investment (I) and consumption (C) is

$$I = P_1Y_1, \quad C = \sum_{i=2}^{N} P_iY_i, \quad Y = C + I \tag{25}$$

and the *composition* of GDP (22) *expenditure shares,* s_i, is

$$s_i = P_iY_i/Y, \quad \sum_{i=1}^{N} s_i \equiv \sum_{i=1}^{N} P_iY_i/Y = 1 \tag{26}$$

Budget studies and consumption (demand) theory mostly normalizes the budget shares with the total expenditure (“income”) constraint, C, i.e., the expenditure shares, e_i, as

$$e_i = P_iY_i/C, \quad \sum_{i=2}^{N} e_i = 1 \tag{27}$$

Evidently, the connection between the GDP shares (26) and (27) is

$$s_i = e_i(1 - s_1), \quad i = 2, \cdots, N \tag{28}$$

For later purposes, we introduce the notation

$$\underline{s} = \sum_{\sigma_i<1} s_i, \qquad \bar{s} = \sum_{\sigma_i>1} s_i, \quad \bar{s} + \underline{s} = 1. \tag{29}$$

Lemma 1. *The macro factor income shares δ_L, δ_K, (24), are GDP expenditure weighted combinations of sectorial factor (cost) shares, ϵ_{L_i}, ϵ_{K_i},*

$$\delta_L = \sum_{i=1}^{N} s_i\epsilon_{L_i}, \qquad \delta_K = \sum_{i=1}^{N} s_i\epsilon_{K_i}, \quad \delta_K + \delta_L = 1 \tag{30}$$

The factor allocation fractions (10-11) are obtained by

$$L_i/L = \lambda_{L_i} = l_i = s_i\epsilon_{L_i}/\delta_L \qquad K_i/K = \lambda_{K_i} = s_i\epsilon_{K_i}/\delta_K \tag{31}$$

The total factor endowment ratio K/L, (12), satisfy the identity, (24):

$$K/L = k = \frac{\omega\delta_K}{\delta_L} = \omega \sum_{i=1}^{N} s_i\epsilon_{K_i} \Big/ \sum_{i=1}^{N} s_i\epsilon_{L_i} \tag{32}$$

which is a representation of the Walras's law.

Proof. By definition we have,

$$\delta_L = wL/Y = [wL_1 + wL_2 + \cdots wL_N]/Y \tag{33}$$
$$\delta_K = rK/Y = [rK_1 + rK_2 + \cdots rK_N]/Y \tag{34}$$

From (19)and (26), we get

$$wL_i = \epsilon_{L_i} P_i Y_i = s_i \epsilon_{L_i} Y, \qquad rK_i = \epsilon_{K_i} P_i Y_i = s_i \epsilon_{K_i} Y \tag{35}$$

Hence, by (33-34) and (35), we obtain (30). Next, as stated in (31)

$$\lambda_{L_i} = \frac{L_i}{L} = \frac{wL_i}{wL} = \frac{s_i \epsilon_{L_i} Y}{\delta_L Y} = \frac{s_i \epsilon_{L_i}}{\delta_L} \tag{36}$$
$$\lambda_{K_i} = \frac{K_i}{K} = \frac{rK_i}{rK} = \frac{s_i \epsilon_{K_i} Y}{\delta_K Y} = \frac{s_i \epsilon_{K_i}}{\delta_K} \tag{37}$$

3. Relative Prices in MS–Economies with CES Sector Technologies

The general CES forms of $F_i(L_i, K_i)$, (1), $\gamma_i > 0$, $0 < a_i < 1$, $\sigma_i > 0$, are

$$Y_i = F_i(L_i, K_i) = \gamma_i L_i^{1-a_i} K_i^{a_i} = L_i \gamma_i k_i^{a_i} \equiv L_i f_i(k_i) \tag{38}$$
$$Y_i = F_i(L_i, K_i) = \gamma_i \left[(1-a_i) L_i^{\frac{\sigma_i - 1}{\sigma_i}} + a_i K_i^{\frac{\sigma_i - 1}{\sigma_i}} \right]^{\frac{\sigma_i}{\sigma_i - 1}} \tag{39}$$
$$= L_i \gamma_i \left[(1-a_i) + a_i k_i^{(\sigma_i - 1)/\sigma_i} \right]^{\sigma_i/(\sigma_i - 1)} \equiv L_i f_i(k_i) \tag{40}$$
$$f_i'(k_i) = \gamma_i a_i k_i^{a_i - 1}, \quad f_i'(k_i) = \gamma_i a_i \left[a_i + (1-a_i) k_i^{-(\sigma_i - 1)/\sigma_i} \right]^{1/(\sigma_i - 1)} \tag{41}$$

By evaluating (38-41), the limits of $f_i(k_i)$ and $f_i'(k_i)$ become,
$(\forall i : \sigma_i \gtrless 1 \Rightarrow a_i^{\sigma_i/(\sigma_i - 1)} \lessgtr 1)$,

$$\sigma_i < 1\colon \begin{cases} \lim\limits_{k_i \to 0} f_i(k_i) = \lim\limits_{k_i \to 0} \gamma_i a_i^{\frac{\sigma_i}{\sigma_i - 1}} k_i = 0, & \lim\limits_{k_i \to \infty} f_i(k_i) = \gamma_i (1-a_i)^{\frac{\sigma_i}{\sigma_i - 1}} \\ \lim\limits_{k_i \to 0} f_i'(k_i) = \gamma_i a_i^{\frac{\sigma_i}{\sigma_i - 1}}, & \lim\limits_{k_i \to \infty} f_i'(k_i) = 0 \end{cases} \tag{42}$$

$$\sigma_i > 1\colon \begin{cases} \lim\limits_{k_i \to 0} f_i(k_i) = \gamma_i (1-a_i)^{\frac{\sigma_i}{\sigma_i - 1}}, & \lim\limits_{k_i \to \infty} f_i(k_i) = \lim\limits_{k_i \to \infty} \gamma_i a_i^{\frac{\sigma_i}{\sigma_i - 1}} k_i = \infty \\ \lim\limits_{k_i \to 0} f_i'(k_i) = \infty, & \lim\limits_{k_i \to \infty} f_i'(k_i) = \gamma_i a_i^{\frac{\sigma_i}{\sigma_i - 1}} \end{cases} \tag{43}$$

For the CES technologies, the *monotonic* relations between marginal rates of substitution, factor proportions, and output elasticities are, cf. (39-41)

$$\omega_i = \frac{1-a_i}{a_i}\, k_i^{1/\sigma_i}, \qquad k_i = \frac{1}{c_i}\,[\omega_i]^{\sigma_i}, \quad c_i = \left[\frac{1-a_i}{a_i}\right]^{\sigma_i} \quad i = 1, \cdots, N. \tag{44}$$

$$\epsilon_{K_i} = \left[1 + \frac{1-a_i}{a_i} k_i^{\frac{1-\sigma_i}{\sigma_i}}\right]^{-1} = \frac{1}{1 + c_i\omega^{1-\sigma_i}}, \quad \epsilon_{L_i} = \frac{c_i\omega^{1-\sigma_i}}{1 + c_i\omega^{1-\sigma_i}} \tag{45}$$

With multi-sector models and CES technologies, it is apparent from (44) that sectorial *factor* ratio ("intensity") *reversals* can only be avoided if and only if $\sigma_i = \sigma_j$ and $a_i \neq a_j$. Hence, with $\sigma_i \neq \sigma_j$, there will be a *reversal* point, $(k_i, \omega_i) = (\bar{k}, \bar{\omega})$:

$$\bar{k} = \left[\frac{a_i(1-a_j)}{a_j(1-a_i)}\right]^{\frac{\sigma_i\sigma_j}{\sigma_j-\sigma_i}} = \left[\frac{c_j^{\sigma_i}}{c_i^{\sigma_j}}\right]^{\frac{1}{\sigma_j-\sigma_i}}, \quad \bar{\omega} = \left[\frac{c_j}{c_i}\right]^{\frac{1}{\sigma_j-\sigma_i}} \tag{46}$$

3.1. Product price and factor price correspondence with CES technologies

The exact form of the function (21) needs particular attention. With (41) and (44), the *relative commodity* prices (comparative costs) (21) become, with $\sigma_i = 1$, $\sigma_i \neq 1$, and $\sigma_i = \sigma$, respectively,

$$p_{ij}(\omega) = \frac{f_j'[k_j(\omega)]}{f_i'[k_i(\omega)]} = \frac{\gamma_j a_j k_j(\omega)^{a_j-1}}{\gamma_i a_i k_i(\omega)^{a_i-1}} = \frac{\gamma_j a_j^{a_j}(1-a_j)^{1-a_j}}{\gamma_i a_i^{a_i}(1-a_i)^{1-a_i}}\,\omega^{a_j-a_i} \tag{47}$$

$$p_{ij}(\omega) = \frac{f_j'[k_j(\omega)]}{f_i'[k_i(\omega)]} = \frac{\gamma_j a_j\left[a_j + (1-a_j)k_j(\omega)^{-(\sigma_j-i)/\sigma_j}\right]^{1/(\sigma_j-1)}}{\gamma_i a_i\left[a_i + (1-a_i)k_i(\omega)^{-(\sigma_i-1)/\sigma_i}\right]^{1/(\sigma_i-1)}} \tag{48}$$

$$= \frac{\gamma_j a_j^{\sigma_j/(\sigma_j-1)}\left[1 + c_j\omega^{1-\sigma_j}\right]^{1/(\sigma_j-1)}}{\gamma_i a_i^{\sigma_i/(\sigma_i-1)}\left[1 + c_i\omega^{1-\sigma_i}\right]^{1/(\sigma_i-1)}} \tag{49}$$

$$p_{ij}(\omega) = \frac{f_j'[k_j(\omega)]}{f_i'[k_i(\omega)]} = \frac{\gamma_j}{\gamma_i}\left[\left[\frac{a_j}{a_i}\right]^{\sigma}\frac{1 + c_j\omega^{1-\sigma}}{1 + c_i\omega^{1-\sigma}}\right]^{1/(\sigma-1)}_{,} \quad c_i = \left[\frac{1-a_i}{a_i}\right]^{\sigma} \tag{50}$$

The elasticity of (21, 47, 48, 49, 50) is generally, by the composite rule:

$$E\left[p_{ij}, \omega\right] = E\left[MP_{K_j}, k_j\right] E(k_j, \omega) - E\left[MP_{K_i}, k_i\right] E(k_i, \omega) \tag{51}$$

$$= (-\epsilon_{L_j}/\sigma_j)\sigma_j - (-\epsilon_{L_i}/\sigma_i)\sigma_i = \epsilon_{L_i} - \epsilon_{L_j} = \epsilon_{K_j} - \epsilon_{K_i} \tag{52}$$

Evidently, $p_{ij}(\omega)$ is always *inelastic*, as (52) is numerically less than unity; but this is not directly seen from the explicit CES-expressions (48, 49, 50). Hence, with $\sigma_i \neq \sigma_j$, and (46),

$$\bar{p}_{ij} = p_{ij}(\bar{\omega}) = \frac{\gamma_j a_j^{\sigma_j/(\sigma_j-1)} \left[1 + c_j \left[c_j/c_i\right]^{\frac{1-\sigma_j}{\sigma_j-\sigma_i}}\right]^{\frac{1}{\sigma_j-1}}}{\gamma_i a_i^{\sigma_i/\sigma_i-1} \left[1 + c_i \left[c_j/c_i\right]^{\frac{1-\sigma_i}{\sigma_j-\sigma_i}}\right]^{\frac{1}{\sigma_i-1}}} \tag{53}$$

Since the CES marginal rate of substitution ω_i, (44), always has the limit values zero and infinity, we need, for precise geometry and intuition, to know the limits of the relative prices $p_{ij}(\omega)$, (49) for ω going to zero and to infinity. To this end, let

$$p_{ij}^{*} \equiv \frac{\gamma_j}{\gamma_i} \frac{a_j^{\sigma_j/(\sigma_j-1)}}{a_i^{\sigma_i/(\sigma_i-1)}}, \qquad p_{ij}^{**} \equiv \frac{\gamma_j}{\gamma_i} \frac{(1-a_j)^{\sigma_j/(\sigma_j-1)}}{(1-a_i)^{\sigma_i/(\sigma_i-1)}} \tag{54}$$

Proposition 1. *The graphs of the relative prices, $p_{ij}(\omega)$, (49) – the CES factor price-commodity price (FPCP) correspondence – have limits, classified by σ_i, as follows:*

$$\sigma_i < 1, \quad \sigma_j < 1: \qquad \lim_{\omega\to 0} p_{ij} = p_{ij}^{*} \quad \lim_{\omega\to\infty} p_{ij} = p_{ij}^{**} \tag{55}$$

$$\sigma_i > 1, \quad \sigma_j > 1: \qquad \lim_{\omega\to 0} p_{ij} = p_{ij}^{**} \quad \lim_{\omega\to\infty} p_{ij} = p_{ij}^{*} \tag{56}$$

$$\sigma_i > 1, \quad \sigma_j < 1: \qquad \lim_{\omega\to 0} p_{ij} = 0 \quad \lim_{\omega\to\infty} p_{ij} = 0 \tag{57}$$

$$\sigma_i < 1, \quad \sigma_j > 1: \qquad \lim_{\omega\to 0} p_{ij} = \infty \quad \lim_{\omega\to\infty} p_{ij} = \infty \tag{58}$$

The reversal price ratio, $\bar{p}_{ij} \equiv p_{ij}(\bar{\omega})$, (53), is always a maximum (iff $\sigma_i > \sigma_j$) or a minimum (iff $\sigma_j > \sigma_i$), cf. Figure 1. For the substitution elasticities (55)-(57), the range of $p_{ij}(\omega)$, (49) is bounded. With $\gamma_i = \gamma_j$, and both substitutions elasticities, either small (55) or large (56), the range of $p_{ij}(\omega)$ becomes a narrow interval, and there will be only small differences between the values of p_{ij}^{*} and p_{ij}^{**} (54) - especially when a_i, $i = 1, \cdots, N$ have similar size, cf. (49).
Iff $\sigma_i = \sigma_j \neq 1$, the functions, $p_{ij}(\omega)$, (50), are always monotonic, bounded, and increasing between p_{ij}^{*} and p_{ij}^{**}, iff $a_j > a_i$. Only the CD relative prices, $p_{ij}(\omega)$, (47), are both monotonic and unbounded.

Proof. Proof is given in Jensen et al. [2001], cf. Jensen [2003]. □

Insert figure 1 about here

4. Demand, Preferences and Expenditure Composition of GDP

4.1. Consumption and Saving

On the demand side of the economy, the actual division of national income between saving and consumption is our first problem, posing major issues of long theoretical and empirical standing. However, the attention will here only be given - as in NIPA accounting - to the accumulation of new productive capital (tangible assets), excluding intangibles such as all services, education (human capital), and portfolio & wealth evaluations ("capital gains") from entering NIPA saving accounts. Hence only the newly produced final goods of a few manufacturing, constructing and building industries enter as NIPA saving/investment shares of the GDP. Thus, with only one capital good ("equipment") industry, we have

$$Y = C + S, \qquad s = S/Y = I/Y = P_1 Y_1/Y = s_1 \tag{59}$$

Per capita saving/consumption, in any numeraire, are denoted, s_L/P_i, c_L/P_i,

$$s = S/Y = (S/L)/(Y/L) = s_L/y = (s_L/P_i)/(y/P_i), \qquad y - s_L = c_L \tag{60}$$

Optimum paths for savings and consumption by a representative agent may be described in terms of the intertemporal utility functional:

$$U = \int_0^\infty u\left[\frac{c_L}{P_i}(t)\right] e^{-R[u(t)]} dt \tag{61}$$

where $R[u(t)]$ is an *accumulated* rate of time preference by which future utility is discounted – that summarizes the preference structure of the agent regarding the time profile of the continuous utility stream from present and future consumption, see Uzawa (1969, p. 630), (1968, p. 488). According to Uzawa (1969, p. 634), (1973, p. 58), if the intertemporal preference orderings are homothetic, then the optimal per capita saving (s_L/P_i) is separable and linear with respect income, (y/P_i), i.e.,

$$s_L/P_i = s_L(r, w, y) = s_L(r/P_i, w/P_i, y/P_i) \tag{62}$$

$$s_L/P_i = s_L(r/P_i, w/P_i, y/P_i) = s_L^*(r/P_i, w/P_i) \cdot (y/P_i) \tag{63}$$

Thus with (63) and (60), the saving rate s is solely determined by the "real" factor prices, and such that the saving rate increase with the real rental rate and decrease in the real wages, i.e.,

$$s = s_L^*(r/P_i, w/P_i); \qquad ds_L^*/d(r/P_i) > 0, \qquad ds_L^*/d(w/P_i) < 0 \tag{64}$$

A further reduction in the the arguments of saving function (64) may occur in a general equilibrium context with Pareto optimality, cf. (16, 18, 21)

$$s = s_L^*\left[(r/P_i)(\omega), (w/P_i)(\omega)\right] = s_L^{**}(\omega) \tag{65}$$

which may allow for the budget share of saving to be formally handled by the homothetic utility functions applied for genuine consumer goods below.

In the tradition of the Böhm-Bawerk & Fisherian theory of time preference, Koopmans (1960), Tinbergen (1960), Uzawa (1973, p. 59), (1968, p. 494), (1969, p. 630, 637), the optimal saving rate is essentially dependent on (increasing with) the expected real rate of interest, i_r^e, with its level also being equal to the *marginal* rate of time preference, $\rho[u(t)]$, i.e., in short,

$$s = s(i_r^e); \qquad (r/P_1)^e = i_r^e = \rho[u(t)] \tag{66}$$

A serious problem so far is, however, that the saving functions (62-66) have not obtained generally accepted functional forms, or even

less been tractable specified and examined for critical values of some fundamental parameters. Furthermore, the saving/investent share of GDP is not just a matter of optimizing consumer preferences (temporal/intertemporal). It should actually be based on the joint considerartions of the producer-consumer agent.
Therefore, with fair empirical support, we adopt the provisional assumption of analytically treating the saving rate as a parametric constant :

$$s \quad = \quad s_1 \quad = \quad \text{constant (parametric variation)} \tag{67}$$

This assumption does not preclude us from seeing favorable effects of relatively declining capital good prices upon investment (capital stock accumulation), cf. (59). On the observed long-run constancy of the gross private (household and business combined) saving rate, see Denison (1958, p.267), David & Scadding (1974, p.238), Tobin (1980, p.65). The constancy of the total private saving rate (67) may be interpreted, (Tobin,1980), as an extension of the Modigliani-Miller theorem beyond *finance* of *real capital* accumulation.

4.2. CES-class utility preferences and consumer expenditure systems

The purpose of budget (expenditure) systems is to describe mathematically how in a certain period a given money amount of total consumption expenditures is allocated to item-specific expenditure categories. The key elements determining this allocation are, as usual, here assumed to be the consumer preferences, prices of consumer goods & services, and the given level of the total budget (expenditure). Rather than giving the Marshallian demand or Hicks/Slutsky ("compensated") demand functions for various standard preference [direct, $U(.)$, and indirect utility, $V(.)$] functions, we need explicit analytical expressions of the budget shares e_i (27) for subsequently solving the static and dynamic general equilibrium systems.

We shall now make use of a few standard (benchmark) specifications of preference (utility) functions. Among many available references, we may just refer to, Deaton & Muelbauer (1992), Silberberg & Suen (2001, p. 359), Chung (1994), Wold (1952, p.111). Introducing the notation of expenditure ("income") and price elasticities as, $\partial \ln Y_i/\partial \ln C \equiv E(Y_i, C)$, $\partial \ln Y_i/\partial \ln P_j \equiv E(Y_i, P_j)$, then demand (expenditure) systems must satisfy four basic, well-known, Frisch (1959, p.180), restrictions:

$$\text{Homogeneity:} \quad \textstyle\sum_{j=2}^{N} E(Y_i, P_j) \quad + \quad E(Y_i, C) \ = \ 0 \tag{68}$$

$$\text{Engel Aggregation :} \quad \textstyle\sum_{j=2}^{N} e_j \cdot E(Y_j, C) \ = \ 1 \tag{69}$$

$$\text{Cournot Aggregation :} \quad \textstyle\sum_{j=2}^{N} e_j \cdot E(Y_j, P_i) \ = \ -\, e_i < 0 \tag{70}$$

$$\text{Symmetry:} \quad E(Y_i, P_j)/e_j + E(Y_i, C) \ = \ E(Y_j, P_i)/e_i + E(Y_j, C) \tag{71}$$

The CES form of the direct/indirect utility functions is the only functional form with the property of self-duality, i.e., the dual can be expressed exactly with same parameters; cf., Samuelson (1965), Houthakker (1965). We have:

$$\textbf{CES:} \quad U(Y_2, \cdots, Y_N) = \gamma_u \left[\textstyle\sum_{i=2}^{N} \alpha_i Y_i^{\frac{\sigma_u - 1}{\sigma_u}} \right]^{\frac{\sigma_u}{\sigma_u - 1}}, \quad \textstyle\sum_{i=2}^{N} \alpha_i = 1 \tag{72}$$

$$V(P_2, \cdot, P_N, C) = \frac{1}{\gamma_u} \left[\sum_{j=2}^{N} \alpha_j^{\sigma_u} \left[\frac{C}{P_j} \right]^{\sigma_u - 1} \right]^{\frac{1}{\sigma_u - 1}} = \frac{1}{\gamma_u} \left[\sum_{j=2}^{N} \alpha_j^{\sigma_u} p_{ij}^{\sigma_u - 1} \right]^{\frac{1}{\sigma_u - 1}} \frac{C}{P_i} \tag{73}$$

$$e_i = \frac{P_i Y_i}{C} = \frac{\alpha_i^{\sigma_u} P_i^{1-\sigma_u}}{\sum_{j=2}^{N} \alpha_j^{\sigma_u} P_j^{1-\sigma_u}} = \left[\textstyle\sum_{j=2}^{N} \left[\frac{\alpha_j}{\alpha_i} \right]^{\sigma_u} p_{ij}^{\sigma_u - 1} \right]^{-1} \tag{74}$$

$$e_i = e_i\left(p_{i2}, \cdots, p_{iN}\right) = \left[1 + \sum_{j=2, j \neq i}^{N} \left[\alpha_j/\alpha_i\right]^{\sigma_u} p_{ij}^{\sigma_u - 1} \right]^{-1} i = 2, \cdots, N \tag{75}$$

Linking the relative *consumer prices* of e_i, (75), to *cost prices,* (21), and (48), we get

$$e_i(\omega) = \left[1 + \sum_{j=2, j \neq i}^{N} \left[\alpha_j/\alpha_i\right]^{\sigma_u} p_{ij}(\omega)^{\sigma_u - 1} \right]^{-1} \quad i = 2, \cdots, N \tag{76}$$

Lemma 2. *The limiting consumer expenditure shares – with CES technologies (39–41) and CES utility functions (72) – are given by*

$$\sigma_u \neq 1 : \quad \forall i \ \sigma_i < 1 : \qquad \lim_{\omega \to 0} e_i(\omega) = e_i^{*} \ \lim_{\omega \to \infty} e_i(\omega) = e_i^{**} \tag{77}$$

$$\sigma_u \neq 1 : \quad \forall i \ \sigma_i > 1 : \qquad \lim_{\omega \to 0} e_i(\omega) = e_i^{**} \ \lim_{\omega \to \infty} e_i(\omega) = e_i^{*} \tag{78}$$

where

$$e_i^* = \left[1 + \sum_{j=2, j\neq i}^{N} [\alpha_j/\alpha_i]^{\sigma_u} p_{ij}^{*\,\sigma_u - 1}\right]^{-1} \quad i = 2, \cdots, N; \; \textstyle\sum_{i=2}^{N} e_i^* = 1 \quad (79)$$

$$e_i^{**} = \left[1 + \sum_{j=2, j\neq i}^{N} [\alpha_j/\alpha_i]^{\sigma_u} p_{ij}^{**\sigma_u - 1}\right]^{-1} \quad i = 2, \cdots, N; \; \textstyle\sum_{i=2}^{N} e_i^{**} = 1 \quad (80)$$

and p_{ij}^*, p_{ij}^{**} *are given by (54), cf. Proposition 1.*
When some of the industries have CES substitution elasticities larger and smaller than one, then the limiting GDP expenditure shares become

$$\sigma_u < 1: \qquad \sigma_i > 1 > \sigma_j: \quad \lim_{\omega\to 0,\infty} e_i(\omega) = 0 \quad \lim_{\omega\to 0,\infty} e_j(\omega) = \bar{e}_j^* \quad (81)$$

$$\sigma_u > 1: \qquad \sigma_i > 1 > \sigma_j: \quad \lim_{\omega\to 0,\infty} e_i(\omega) = \underline{e}_i^{**} \quad \lim_{\omega\to 0,\infty} e_j(\omega) = 0 \quad (82)$$

where

$$\bar{e}_j^* = \left[1 + \sum_{j\neq i, \sigma_i > 1} [\alpha_j/\alpha_i]^{\sigma_u} p_{ij}^{*\,\sigma_u - 1}\right]^{-1} \quad j = 2, \cdot\cdot, N; \; \bar{s} = \sum_{j=2}^{N} \bar{e}_j^* \leq 1 \quad (83)$$

$$\underline{e}_i^{**} = \left[1 + \sum_{j\neq i, \sigma_j < 1} [\alpha_j/\alpha_i]^{\sigma_u} p_{ij}^{**\sigma_u - 1}\right]^{-1} \quad i = 2, \cdot\cdot, N; \; \underline{s} = \sum_{i=2}^{N} \underline{e}_i^{**} \leq 1 \quad (84)$$

$$\bar{s} + \underline{s} = 1 \quad (85)$$

Proof. The limiting shares (77-80) follow from (76) and (55-56). The limiting shares (81-84) follow from (76) and (57-58). □

4.3. Homothetic and Non-homothetic Consumption Preferences

The main deficiency of many conventional and in fact any homothetic direct utility function is that all the consumer goods have an "income (C)" elasticity of one. Hence some indirect utility function attracts attention. But any demand (expenditure) functions with constant price and income elasticities cannot satisfy (69), except for a very narrow range of the price and income variables involved, Wold (1952, p.106). A flexible expenditure system satisfying (69) was first proposed by Leser (1941), and it was related to the indirect addilog by Houthakker (1960). Its empirical application, e.g., Somermeyer (1972), Jensen (1980), works reasonably well; its Engel curve patterns allows negative, zero, and positive income elasticities restricted above by around two, but collectively (69) is satisfied.

Indirect Addilog (additive logarithmic) **Utility:** $\sum_{i=2}^{N} \alpha_i = 1,\ 0 < \beta_i < 1$

$$V(P_2, \cdots, P_N, C) = \gamma_v \sum_{i=2}^{N} \alpha_i \left(C/P_i\right)^{\beta_i} = \gamma_v \sum_{j=2}^{N} \alpha_j p_{ij}^{\beta_j} \left(C/P_i\right)^{\beta_j} \tag{86}$$

$$e_i = \frac{\alpha_i \beta_i \left[\dfrac{C}{P_i}\right]^{\beta_i}}{\displaystyle\sum_{j=2}^{N} \alpha_j \beta_j \left[\dfrac{C}{P_j}\right]^{\beta_j}} = \left[\sum_{j=2}^{N} \frac{\alpha_j \beta_j}{\alpha_i \beta_i} \left[\frac{P_i^{\beta_i}}{P_j^{\beta_j}}\right] C^{\beta_j - \beta_i}\right]^{-1} = \left[\sum_{j=2}^{N} \frac{\alpha_j \beta_j}{\alpha_i \beta_i} p_{ij}^{\beta_j} \left[\frac{C}{P_i}\right]^{\beta_j - \beta_i}\right]^{-1} \tag{87}$$

$$\forall j \quad \beta_j = \beta: \quad e_i(\omega) = \left[1 + \sum_{j=2, j \neq i}^{N} \left[\alpha_j / \alpha_i\right] p_{ij}(\omega)^{\beta}\right]^{-1} \tag{88}$$

$$e_i(\omega, L, K) = \left[1 + \sum_{j=2, j \neq i}^{N} \left[\alpha_j \beta_j / (\alpha_i \beta_i)\right] p_{ij}(\omega)^{\beta_j} \left[\frac{C}{P_i}(\omega, L, K)\right]^{\beta_j - \beta_i}\right]^{-1} \tag{89}$$

The budget system (88) is with $\beta = \sigma_u - 1 > 0$ seen to be a subsystem of the homothetic CES system (76) already characterized in Lemma 2.

With $\beta_i \neq \beta_j$ the non-homothetic Indirect Addilog Utility function has budget shares (89) with terms $C/P_i = (1-s)Y/P_i$, cf. (59), where Y/P_i is well-defined in terms of ω, L, K, see (23). A very particular parameter constellation implies the limiting consumer expenditure shares , cf. (57):

$$\forall j \neq i : \sigma_i > 1 > \sigma_j \quad \Rightarrow \lim_{\omega \to \infty} e_i = 1 \tag{90}$$

The *translog* indirect utility function, Christensen, Jorgenson, and Lau (1971, 1975) is one of the most widely used flexible functional forms in empirical demand analysis. It can be a second-order local approximation to an arbitrary indirect utility function. Its budget shares is here restated in convenient form for our purposes.

Translog (transcendental log.) **Indirect Utility:** $\sum_{j=2}^{N} \alpha_j = 1, \beta_{ij} = \beta_{ji}$

$$\ln V(P_2, \cdots, P_N, C) = \ln \alpha_0 + \sum_{j=2}^{N} \alpha_j \ln \frac{C}{P_j} + \frac{1}{2} \sum_{i=2}^{N} \sum_{j=2}^{N} \beta_{ij} \ln \frac{C}{P_i} \ln \frac{C}{P_j} \tag{91}$$

$$e_i = \frac{P_i Y_i}{C} = \left[\alpha_i + \sum_{j=2}^{N} \beta_{ij} \ln \frac{C}{P_j}\right] \Bigg/ \left[1 + \sum_{k=2}^{N} \sum_{j=2}^{N} \beta_{kj} \ln \frac{C}{P_j}\right] \tag{92}$$

$$e_i = \frac{\alpha_i + \sum_{j=2}^{N} \beta_{ij} \ln(C/P_j)}{\alpha_i + \sum_{j=2}^{N} \beta_{ij} \ln(C/P_j) + \sum_{i \neq k=2}^{N} \left[\alpha_k + \sum_{j=2}^{N} \beta_{kj} \ln(C/P_j)\right]} \tag{93}$$

Rewriting $\ln\left(C/P_j\right) = \ln\left[C/P_i \cdot P_i/P_j\right] = \ln\left(C/P_i \cdot p_{ij}\right) = \ln\left(C/P_i\right) + \ln p_{ij}$, we get

$$e_i(\omega, L, K) = \left[1 + \frac{\sum_{i \neq k=2}^{N} \left[\alpha_k + \sum_{j=2}^{N} \beta_{kj} \ln p_{ij} + \ln(C/P_i) \sum_{j=2}^{N} \beta_{kj} \right]}{\alpha_i + \ln(C/P_i) \sum_{j=2}^{N} \beta_{ij}} \right]^{-1} \quad (94)$$

With the homothetic translog with restrictions

$$\textstyle\sum_{j=2}^{N} \beta_{kj} = 0, \quad k = 2, \cdots, N \quad (95)$$

the budget shares e_i become, (94)

$$e_i(\omega) = \alpha_i \; [1 + \textstyle\sum_{j=2}^{N} \beta_{ij} \ln p_{ij}(\omega) \;]^{-1} \quad i = 2, \cdots, N \quad (96)$$

In the non-homothetic case, the terms, C/P_i, (94) are - as mentioned above for the non-homothetic Addilog - well-defined in the arguments of e_i, (94).

5. Walrasian General Equilibrium of Multi-Sector Economies

The demand side of the multisector economy is expressed by the respective GDP expenditure shares, s_i, derived in section 4. The supply side of the economy – operating under constant returns to scale and with full (10-11) and Pareto-efficient factor utilization, (16) – is always summarized by sectorial factor allocation fractions λ_{L_i}, λ_{K_i}, which in turn are determined by s_i and the sectorial cost shares ϵ_{L_i}, ϵ_{K_i}, (31).

Theorem 1. *The Walrasian equilibrium (competitive general equilibrium, CGE) states, – with market clearing prices on the commodity/factor markets and Pareto efficient endowments allocation – are with homothetic preferences given by,* $\forall k \in R_+, \forall \omega \in \Omega$, *(32), (26), (28), (76), (88), (96),*

$$k = \frac{\omega \delta_K(\omega)}{\delta_L(\omega)} = \omega \sum_{i=1}^{N} s_i(\omega) \epsilon_{K_i}(\omega) \Big/ \sum_{i=1}^{N} s_i(\omega) \epsilon_{L_i}(\omega) \; = \Psi(\omega) \quad (97)$$

Corollary 1.1. *With CES sector technologies and any homothetic utility function, the Walrasian equilibrium* $k = \Psi(\omega)$ *becomes by (97) and (45):*

$$k = \frac{\omega \sum_{i=1}^{N} s_i(\omega)(1 + c_i \omega^{1-\sigma_i})^{-1}}{1 - \sum_{i=1}^{N} s_i(\omega)(1 + c_i \omega^{1-\sigma_i})^{-1}} = \Psi(\omega) \quad (98)$$

Theorem 2. *For sector technologies with constant returns to scale and non-homothetic utility function, and homogeneous production function of degree one, the Walrasian equilibrium becomes by (32),(26, (28), (89) and (94):*

$$K/L = \frac{\omega \delta_K(\omega, L, K)}{\delta_L(\omega, L, K)} = \frac{\omega \sum_{i=1}^{N} s_i(\omega, L, K) \epsilon_{K_i}(\omega)}{\sum_{i=1}^{N} s_i(\omega, L, K) \epsilon_{L_i}(\omega)} = \Upsilon(\omega, L, K) \quad (99)$$

Corollary 2.1. *With CES sector technologies, Walrasian equilibrium, (99), becomes with (45):*

$$K/L = \frac{\omega \sum_{i=1}^{N} s_i(\omega, L, K)(1 + c_i\omega^{1-\sigma_i})^{-1}}{1 - \sum_{i=1}^{N} s_i(\omega, L, K)(1 + c_i\omega^{1-\sigma_i})^{-1}} = \Upsilon(\omega, L, K) \tag{100}$$

Locus expressions (99,100) give ω implicitly as a graph of a function of L,K

$$\omega = \Lambda(L, K) \tag{101}$$

Proof. The theorem and corollaries are obtained by turning Walras' law (identity), (32), into the respective Walrasian equilibrium condition, (97), (99). Rather than relying on fix-point methods for searching (iterating) the equilibrium prices (vector) of numerous supply and demand equations of goods and factor markets, our general equilibrium solution procedure is formulated in variables having simple economic and observable NIPA counterparts. Since relative commodity prices are endogenous variables, they are by construction properly eliminated from our "structural" general equilibrium equations. Hence we end up with a "reduced" form of just one equation (explicit or implicit) between the remaining endogenous relative factor prices of general equilibrium and the exogenous (given) factor endowments. □

The *competitive general equilibrium* functions, $k = \Psi(\omega)$, $\omega = \Lambda(L, K)$, are crucial for inquiring into the statics, comparative statics, and dynamics of multi-sector economies, and they will be called a *factor endowment-factor price* (FEFP) *correspondence*. Having obtained ω from (97), we can go back through (21), (49), (76) (45), (30), (31), (22) to get the associated general equilibrium values of all other endogenous variables (sector outputs, allocation fractions of inputs, income shares, relative commodity prices).

Regarding the *shape* of the graph of Ψ, (98), it is evident that, if *all* substitution elasticities are *larger* than one, $\sigma_i > 1,\ i = 2, \cdots, N$, then the numerator (denominator) expression in (98) will increase (decrease), cf. ϵ_{K_i}, ϵ_{L_i} (15), which always ensures that the Walrasian locus $\Psi(\omega)$, (98), is *monotonically* increasing. When all σ_i are *less* than one, $\sigma_i < 1$, only a detailed examination will reveal the *global* and *local* shape of the graph.

Proposition 2. *The graph of the Walrasian equilibrium, $\Psi(\omega)$,(98),is within Fig.1, located between the extreme monotonic CES ω_i-curves,(44), and for any value of the GDP shares, s_i, and for any size of the sectorial substitution elasticities, $\sigma_i, i = 1, \cdots, N$, the function, $\Psi(\omega)$, has the limit properties:*

$$\lim_{\omega\to 0} \Psi(\omega) = 0,\ \lim_{\omega\to\infty} \Psi(\omega) = \infty; \tag{102}$$

$$\lim_{\omega\to 0} \Psi(\omega)/\Psi'(\omega) = 0,\ \lim_{\omega\to\infty} \Psi(\omega)/\Psi'(\omega) = \infty \tag{103}$$

With (98), the elasticities, $E(k,\omega)$, of $\Psi(\omega)$ have finite limits as follows:

$$\forall i: \ \sigma_i < 1: \qquad \lim_{\omega\to 0} E(k,\omega) = \lim_{\omega\to\infty} E(k,\omega) = \max_i \sigma_i \quad (104)$$

$$\forall i: \ \sigma_i > 1: \qquad \lim_{\omega\to 0} E(k,\omega) = \lim_{\omega\to\infty} E(k,\omega) = \min_i \sigma_i \quad (105)$$

$$\exists i,j: \ \sigma_i > 1 > \sigma_j: \qquad \lim_{\omega\to 0} E(k,\omega) = \lim_{\omega\to\infty} E(k,\omega) = 1 \quad (106)$$

Proof. The limits (102-103) are seen immediately from (98). The limits (104-106) follow from the formula, cf., (97), (98)

$$E(k,\omega) \ = \Psi'(\omega)\omega/\Psi(\omega) = 1 + E(\delta_K,\omega) - E(1-\delta_K,\omega) \quad (107)$$

$$E(\delta_{K,}\omega) \ = \omega\delta'_K(\omega)/\delta_K \quad E(1-\delta_K,\omega) = -\omega\delta'_K(\omega)/(1-\delta_K) \quad (108)$$

The numerators of (108), $\pm\,\omega\delta'_K(\omega)$ go to zero for both $\omega \to 0$ and $\omega \to \infty$, as is seen by simple calculations. The denominators, δ_K, $1-\delta_K$ go to $\bar{s}$ or $\underline{s}$ for $\omega \to 0$ and $\omega \to \infty$ in accordance with (143-146). Hence in case of $0 < \bar{s} < 1$ cf., (143), (145), last terms of (107) go to zero - giving (106). In case of $\bar{s} = 0, 1$ one term goes to zero, while the other goes to the dominating power, in some cases proved by use of l´Hospital. This proves (104), (105). In the case of *variable* $s_i(\omega)$ we just have to correct the proof for the constant case by adding the limits of $s'_i(\omega)\omega/s_i(\omega)$, which, however, are both zero, since the budget shares $s_i(\omega)$ have specific limits s_i^* and s_i^{**} for the respective utility functions, cf. (77-84), (88, 96). □

6. Dynamic Systems and Evolution of Multi-Sector Economies

In the multi-sectoral planning literature, we often find that e.g., Johansen (1974, p.22): "The *growth process* is generated by the following factors, all of which are considered to be *exogenously determined* : a) Total investment, b) Growth in population; working and total, c) Growth in productivity; shifts in production functions over time, d) Changes in exogenous demand; mainly government and net foreign demand." This broad view of the exogeneity concept refers to exogeneity assumptions of rather diverse nature. Let us consider them in reverse order. Some "applied/planning" growth modeling may include a public sector with government expenditure and taxation assumptions/specifications, but this sector can properly be excluded with other theoretical/analytical purposes. Similarly, some growth models may include international trade and international factor mobility. Certainly, open and in particular small open economies operate differently from closed economies, but for obtaining some actual insights about economic evolution, a relevant closed (global) economy growth model may suffice. Next, "shifts" in

production (or utility) functions refer to "parametric" changes. Since most economic parameters are not "natural constants" and occasionally undergo critical changes, an important object of growth models is to qualitatively understand and identify the crucial parameters involved and their critical numerical values. Regarding labour endowments (population), no attention has so far been directed to its exogenously given size. As a state variable in a dynamic model, it may still be treated, without violating the general equilibrium model above, as evolving exogenously with specified parameters. But time paths of total investments and capital endowments cannot be extrapolated exogenously without violating the general equilibrium solutions of Theorem 1-2. A coherent general equilibrium evolution for the capital endowments in continous time can only be derived by integrating the output paths from sector 1. Thus the *macrodynamic* role of the *capital* good (*machinery*) in the multi-industry-economies now need to be carefully examined.
The equations of *factor accumulation* multi-sector growth models – with two primary factors and flexible constant return to scale sector technologies – are formally given, by (δ is the depreciation rate of capital),

$$dL/dt \equiv \dot{L} = nL, \tag{109}$$
$$dK/dt \equiv \dot{K} = Y_1 - \delta K = Ly_1 l_1 - \delta K = L\left\{f_1(k_1)l_1 - \delta k\right\}. \tag{110}$$

6.1. General equilibrium dynamics with homothetic preferences

In the general equilibrium models of multi-sector economies, k_1 and l_1 are, through ω (16), *uniquely* determined by the *factor* endowments *ratio* k, cf. (45). Hence, the accumulation equations (109-110) become genuine *autonomous* (time invariant) *differential equations* in the state variables L and K and represent a standard *homogeneous dynamic system,*

$$\dot{L} = Ln \equiv L\mathfrak{f}(k), \tag{111}$$
$$\dot{K} = L\left\{f_1(k_1[\Psi^{-1}(k)])l_1[\Psi^{-1}(k)] - \delta k\right\} \equiv L\mathfrak{g}(k),\ L \neq 0. \tag{112}$$

As $\mathfrak{g}(k)$, (112), are intricate functions of k, we rewrite $\mathfrak{g}(k)$ in alternative forms by (110), (23-24),

$$\dot{K} = s_1 Y/P_1 - \delta K = Ls_1(\omega + k)f_1'(k_1) - \delta K \tag{113}$$
$$= Lk\{\frac{s_1 f_1'(k_1)}{\delta_K} - \delta\} = L\mathfrak{g}(k) \tag{114}$$

where to succinctly express and decompose the governing functions of capital accumulation (113), the bounded variable $\delta_K(k)$ is mainly a formal auxiliary term helpful in evaluating concrete cases.

From the *governing* functions $\mathfrak{g}(k)$, $\mathfrak{f}(k)$, (111–114), the *director* function, $h(k)$, that controls $dk/dt \equiv \dot{k}$ becomes, $h(k) \equiv \mathfrak{g}(k) - k\mathfrak{f}(k)$, i.e.,

$$\dot{k} = h(k) = k\left[\frac{s_1 f_1'\left[k_1[\omega(k)]\right]}{\delta_K[\omega(k)]} - (n+\delta)\right]; \quad \omega(k) = \Psi^{-1}(k) \tag{115}$$

The dynamic system (115) in k is difficult to evaluate *quantitatively* and are generally intractable; e.g., if $\sigma_i \neq \sigma_j$, then, $k = \Psi(\omega)$, (97) cannot be inverted (although Ψ^{-1} exists) in closed form. But $k = \Psi(\omega)$, (97), is *continuously differentiable* functions of ω, and dynamics in k can, whenever convenient, be converted into *dual autonomous dynamics* in ω,

$$\dot{\omega} = \frac{\dot{k}}{dk/d\omega} = \frac{h(k)}{dk/d\omega} = \frac{h(\Psi[\omega])}{\Psi'(\omega)} \equiv \hbar(\omega) \tag{116}$$

Hence we get, cf. (115),

$$\hbar(\omega) = \frac{\Psi(\omega)}{\Psi'(\omega)}\left[\frac{s_1 f_1'[k_1(\omega)]}{\delta_K(\omega)} - (n+\delta)\right] \tag{117}$$

$$= \frac{\omega}{E(k,\omega)}\left[\frac{s_1 f_1'[k_1(\omega)]}{\delta_K(\omega)} - (n+\delta)\right] \tag{118}$$

With CES technologies, we have, from (117), (30), (28), (45), (67)

$$\hbar(\omega) = \frac{\Psi(\omega)}{\Psi'(\omega)}\left[\frac{s_1\gamma_1 a_1^{\frac{\sigma_1}{\sigma_1-1}}\left[1 + c_1\omega^{1-\sigma_1}\right]^{1/(\sigma_1-1)}}{\sum_{i=1}^{N} s_i(\omega)\left[1 + c_i\omega^{1-\sigma_i}\right]^{-1}} - (n+\delta)\right] \tag{119}$$

with $\Psi(\omega)$ given in (98).

Existence and uniqueness of steady states or persistent growth

The complete set (family) of $k(t)$ solutions to the dynamic systems (115) is *qualitatively* described and classified in:

Theorem 3. *The multi-sector growth models (109) have no positive, stationary $k(t)$-solution [$k(t) = 0$ is attractor], iff*

$$\forall k > 0 \;: \quad \bar{\mathfrak{b}}_1 < ([n+\delta]/s_1)\delta_K(k) \tag{120}$$

and have at least one steady state [ray path in (L,K)-space], iff, cf. (3)

$$\exists k > 0 : \;\; ([n+\delta]/s_1)\delta_K(k) \in J_1 \tag{121}$$

The stationary capital-labor ratios $\forall t : k(t)\kappa$ *are obtained by*

$$f_1'[k_1(\kappa)] = ([n+\delta]/s_1)\delta_K(\kappa) \tag{122}$$

With existence (121), a sufficient condition for a unique root of $h(k)$ *is*

$$\forall k > 0 : E(h(k)/k, k) < 0 \Leftrightarrow \forall \omega \in \Omega : E(k, \omega) \geq 1, \tag{123}$$

The time paths of the growth model solutions, $k(t)$*, display persistent growth –* $\lim_{t\to\infty} k(t) = \infty$ *– if and only if*

$$\forall k > 0 \ : \quad \underline{\mathfrak{b}}_1 > ([n+\delta]/s_1)\,\delta_K(k) \tag{124}$$

Proof. The family of solutions to (115) depends entirely on the *shape* of the *director* function, $h(k)$, and the *number* of roots of $h(k)$. The existence of nonzero roots requires that $([n+\delta]/s_1)\delta_K(k)$ belongs to the range of f_1' as stated in (121). If no root exists, we have either the case (120) with origo as attractor, or the case (124) with persistent growth.

If it exists, a unique attractor in the interval stated in (121) always occurs with a global negative sign of the elasticity, $E(h(k)/k, k) < 0$, that can be derived from (115); the necessary and sufficient condition for such a negative sign is shown explicitly in (123); cf. (123, 105 106) and Jensen [1994, p. 138, p. 129], [2003, p. 73]. □

6.2. Dynamics of MSG models with CES sector technologies

The *qualitative* properties of the family of *Walrasian general equilibrium solutions* $k(t)$ in multi-sector growth models with CES sector technologies are summarized in:

Theorem 4. *For the multi-sector growth models (109-112) with CES sector technologies, the sufficient conditions for the existence of at least one positive steady-state solution are [no positive, attractive, steady state solution* κ *(122) exists with the RHS inequalities of (125) reversed] :*

$$\forall i : \ \sigma_i < 1 \ \ \bar{\mathfrak{b}}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta)/s_1 \tag{125}$$

$$\sigma_1 < 1 \ \ \bar{\mathfrak{b}}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta)\underline{s}/s_1 \tag{126}$$

With $\sigma_1 \leq 1$ *(sufficient condition), persistent growth of* $k(t)$ *is impossible. With* $\sigma_1 > 1$*, necessary and sufficient conditions for* $\lim_{t\to\infty} k(t) = \infty$ *are:*

$$\forall i : \ \sigma_i > 1 \ \ \underline{\mathfrak{b}}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta)/s_1 \tag{127}$$

$$\sigma_1 > 1 \ \ \underline{\mathfrak{b}}_1 = \gamma_1 a_1^{\frac{\sigma_1}{\sigma_1 - 1}} > (n+\delta)\overline{s}/s_1 \tag{128}$$

except that (128) is occasionally not sufficient for small initial values.

Proof. The proof proceeds with the dual version, $\hbar(\omega)$. The term $\Psi(\omega)/\Psi'(\omega)$ has no influence on these limit analyses, cf. Proposition 2.

Ad $\sigma_1 < 1$: The large fraction in the bracket (119) goes to zero for $\omega \to \infty$; hence, there are no permanent increasing solutions of $\omega(t)$. If (125-126) are satisfied, then the large fraction passes at least once monotonically through the constant when ω goes from zero to infinity. The difference s in the constant comes from the denominator taking values 1 or $\underline{s}$ for $\omega \to 0$, depending on the size of σ_i. If the inequalities are reversed, then $\hbar(\omega)$ is negative. The role of $\underline{s}$ in(124) follows from (119), (29), (85).

Ad $\sigma_1 > 1$: The large fraction in the bracket (119) goes to infinity for $\omega \to 0$. If and only if (127-128) are satisfied, then the large fraction eventually remains above the constant when ω goes from zero to infinity. The difference s_1 in the constant comes from the denominator going towards the values 1 or $\overline{s}$ for $\omega \to \infty$, depending on the size of σ_i. Hence $\hbar(\omega)$ is positive for large values of k. If the inequalities are reversed, then $\hbar(\omega)$ eventually becomes negative. The necessary conditions (127) are also sufficient, as $k(k)$ is, with respectively $\sigma_i > 1$ and $\sigma_1 > 1$, monotonically decreasing, but remain above RHS values in (127). The role of $\overline{s}$ in (128) follows from (29), (119), (84) and (85). □

Theorem 4 shows explicitly that the *global existence issues* of any *steady state* or *persistent* growth depend on the *size* of the key parameters: σ_i, a_1, γ_1, s_1, n, δ. While the *accumulation* parameters (s_1,$\overline{s}$,n,δ) play some roles, the *fundamental* role of the *technology* parameters in the *capital* good sector (σ_1, γ_1, a_1) – for deciding the types of the long-run *evolution* in multi-sectoral general equilibrium growth models – *complies* with *observation* and economic *intuition*. Evidently, the strategic importance ascribed to capital good industries by economic historians and the general public, cf. Rosenberg (1963), evidently makes good sense, at least for closed economies.

The most important parameter in Theorem 4 is the *substitution elasticity* in the capital good sector, σ_1 . It must be larger than one for persistent growth. But the "*total productivity*" parameter γ_1 in the capital good sector matters in all the stated conditions (125-128), and they can all be violated by giving γ_1 any value between 0 and ∞. A larger TFP parameter of the capital good sector γ_1 may give a "big push", cf. Murphy et al. [1989], Parente & Prescott [1999], Prescott [1998]. We cannot here enter a discussion about the dispersion of σ_1 and γ_1, cf. Easterly & Fischer (1995), Prescott [1998]. But if we restrict $\gamma_1 = 1$ and if $\sigma_1 \simeq 2$, then (127) will usually be satisfied for other relevant parameters, in particular with high saving rates. The key role of the *technology* in the *capital good* industry had escaped the "mainstream" literature on the multi-sector growth models, cf. Jensen (2003, p.75).

As to *empirical evidence*, the theoretical general equilibrium predictions of Theorem 4 tally with some observations and studies of long-run growth conducted by De Long & Summers, (1991), Rebolo (1991), and Jones [1994]. In particular, high rates of *equipment* investment ("mechanization") are prime determinants for national growth performance (per capita growth).

Furthermore, the making of various equipments become eventually highly mechanized by making various engines (steam, combustion, electric) " *cheap* as well as *good,*" cf. Mokyr [1990, p. 87]. This supports factor substitution and mechanization subsequently in the consumer good industries. In this way, the *capital good* (multi-purpose machinery/equipment) is a "*Lever of Riches*" (productivity and per capita growth) in *several sectors* with the capital good industry itself and its technology parameters being naturally of primary importance for sustaining the economic growth process - as properly mathematically demonstrated within a Walrasian general equilibrium framework.

On the advances in technology and economic evolution, Usher(1954, p.9) writes: " It is important *not* to presume a continuous *development* of *technology* at a *constant rate,* but it is important, also, to recognize that the process of social *evolution* consists in part in the *cumulative* development of science and technology. We *need* both a *general understanding* of the process or processes, and, when records make it possible, a *documented account* of the history of *particular periods* and *particular achievements"*– (p. 380):

"The *technique* of *interchangeable -part manufacture* was thus established in general outline before the invention of the sewing machine or *harvesting machinery.* The new technique was a fundamental condition of the great achievements realized by inventors and manufacturers in those fields. It made it possible to place the sewing machine in the home and it generalized the use of harvesting machinery of *McCormick* and *Deere* with astonishing rapidity. American *engineering and manufacturing firms* took the lead in this general development, achieving distinctive results over an important field that was steadily enlarged decade after decade. The group of *machine tools* became more and more *automatic,* and it became possible to build highly specialized *machinery* for *manufacturing firms.*

Great refinements of execution were achieved with the *simplest labor* of attendance. These highly developed machine tools are the most distinguished "iron men" of the modern *industrial world,* for they make possible that *substitution* of *machinery* for *labor* that is so happily described as effecting a "transfer of skill"." Can historical stages (parametric changes) of increasing substitution elasticities in several consumer and especially capital goods be better and more eloquently described ? Multi-sectoral dynamics offer the same economic message about key parameters behind industrial evolution.

Such qualitative insight gained by Theorem 4 and the discussion above about the dynamic role of critical parameter values is not confined to economies of homothetic consumer preferences. Evidently, with non-homothetic preferences, the factor accumulation equations (109, 110, 113, 114) still apply, cf.

$$\dot{L} = nL, \quad \dot{K} = Y_1 - \delta K = K\{\frac{s_1 f_1'(k_1[\omega(\Lambda[L,K])])}{\delta_K(L,K)} - \delta\} \tag{129}$$

Although the equations, (129), cannot be reduced to a single equation in the capital-labor ratio, k, and certainly neither in the wage-rental ratio ω, (116-119), the accumulation equations, (129), still represent a well-defined dynamic system in the state variable, L, K. Without explicit solutions, however, the logarithmic time derivative of $k(t)$ is easily obtained as, cf. (115)

$$\frac{d \ln k}{dt} = \hat{k} = \frac{\dot{k}}{k} = \frac{s_1 f_1'(k_1[\omega])}{\delta_K} - (n+\delta) \tag{130}$$

The question of steady state or persistent growth similarly depends cf., (124), on the condition

$$\forall\, L, K > 0 : \qquad \underline{\mathfrak{b}}_1 > ([n+\delta]/s_1)\, \delta_K(L,K) \tag{131}$$

Analogously to (128), $\bar{s}$ is here replaced by

$$\bar{\bar{s}} = \textstyle\sum_{i=2,\sigma_i>1}^{N} \bar{\bar{e}}_i^*(\omega, L, K) \leq 1 \qquad \delta_K(L,K) \to \bar{\bar{s}} \tag{132}$$

Hence, there are certainly quantatively great differences for economies with non-homothetic preferences (income elasticities different from one, Engel´s law), but the critical role of the capital goods sector and its substitution elasticity carry over from Theorem 4.

7. Persistent economic growth and asymptotic growth rates

To complement the persistent growth solutions of the *state* variable $k(t)$ or $\omega(t)$ with *disaggregate* information about the general equilibrium *evolution* for *sectorial* and other endogenous per capita variables, we characterize the respective time paths by their asymptotic growth rates [$\hat{\omega}(t) \equiv \dot{\omega}/\omega(t)$, etc.]:

Theorem 5. *With (127-128), the long-run growth rates of $k(t)$ and $\omega(t)$ in Walrasian multi-sector growth models (109) with CES technologies are:*

$$\forall i: \ \sigma_i > 1: \ \lim_{t\to\infty} \hat{\omega} = \frac{s_1 \underline{\mathfrak{b}}_1 - (n+\delta)}{\min\{\sigma_i\}}; \ \lim_{t\to\infty} \hat{k} = s_1 \underline{\mathfrak{b}}_1 - (n+\delta) \tag{133}$$

$$\sigma_1 > 1, \quad \exists i: \ \sigma_i < 1 : \lim_{t\to\infty} \hat{\omega} = \lim_{t\to\infty} \hat{k} = \frac{s_1}{\bar{s}} \underline{\mathfrak{b}}_1 - (n+\delta) \tag{134}$$

With (133), the long-run sectorial and per capita growth rates are

$$\lim_{t\to\infty} \hat{k}_i = \lim_{t\to\infty} \hat{y}_i = \lim_{t\to\infty} (w\hat{/}P_i) = \lim_{t\to\infty} (y\hat{/}P_i) \tag{135}$$

where

$$\forall i:\ \sigma_i > 1 : \lim_{t\to\infty} \hat{k}_i = \frac{\sigma_i}{\min\{\sigma_j\}}[s_1\underline{\mathfrak{b}}_1 - (n+\delta)] \tag{136}$$

With (134), some long-run sectorial and per capita growth rates are

$$\sigma_1 > 1,\ \sigma_i > 1:\ \exists j : \sigma_j < 1 : \lim_{t\to\infty} \hat{k}_i = \sigma_i \left(\frac{s_i}{\bar{s}}\underline{\mathfrak{b}}_1 - (n+\delta)\right) \tag{137}$$

If only the capital good sector has a high substitution elasticity, then we have, $\sigma_1 > 1,\ \forall i : \sigma_i < 1:$

$$\hat{k} \to \underline{\mathfrak{b}}_1 - (n+\delta) \quad \hat{k}_1 \to \sigma_1 (\underline{\mathfrak{b}}_1 - (n+\delta)) \quad \hat{k}_i \to \sigma_i (\underline{\mathfrak{b}}_1 - (n+\delta)) \tag{138}$$

$$\lim_{t\to\infty} \hat{k}_1 = \lim_{t\to\infty} \hat{y}_1 = \lim_{t\to\infty} (w\hat{/}P_1) = \lim_{t\to\infty} (y\hat{/}P_1) \tag{139}$$

whereas the output of all other sectors will ultimately stagnate.

$$\lim_{t\to\infty} \hat{y}_i = 0; \quad \lim_{t\to\infty} y_i = \gamma_i(1-a_i)^{\frac{\sigma_i}{\sigma_i - 1}} \tag{140}$$

Proof. Theorem 5 follows immediately from Theorem 3: (127), combined with (117),(105), and next using,(44-45),(43),(22) and (16). Thus by (117):

$$\lim_{t\to\infty} \hat{\omega} = [\lim_{\omega\to\infty} E(k,\omega)]^{-1}[s_1\underline{\mathfrak{b}}_1 - (n+\delta)] \tag{141}$$

The FEFP correspondence $k = \Psi(\omega)$ next gives

$$\hat{k} = E(k,\omega)\hat{\omega}; \quad \hat{k}_i = \sigma_i\hat{\omega} \quad \hat{y}_i = \epsilon_{k_i}\hat{k}_i \tag{142}$$

holds generally with CES. These relations and limits establish the relevant asymptotic growth rates in Theorem 5. □

As Theorem 5 supplements Theorem 3 and Proposition 2, only a few remarks is needed. The asymptotic growth rates of $\hat{k}$, (133,134) correspond, respectively, to those implied by (127,128). Evidently, with more industries to be highly mechanized $(\sigma i > 1)$ and hence larger $\bar{s}$, the slower will be the overall accumulation rate $\hat{k}$. The same applies to the sectorial $\hat{K}_i$, (136-137), but for $\hat{K}_i$ hat also matters the elasticity of Ψ and its own σ_i, cf.(142). If *maximum* growth of per *capita consumption* is the goal, then the *ranking* with all $\sigma_j > \sigma_1 > 1$, will be preferred – which contributes to mechanizing and maintaining the growth rate of the consumer goods and thereby increases the welfare per capita in any numeraire (sectorial good),(135). The other extreme is capital accumulation for its own sake (134-140). Capital accumulation with increasing wage-rental rate makes it impossible to avoid increasing sectorial capital intensities anywhere, even though diminishing returns with $\sigma_i < 1$ eventually terminates increases in labor productivity. Thus, even with homothetic preferences and only price elasticities involved on demand side, a diverse pattern of industrial growth may emerge with different CES technologies on the supply side of th multi-sector economy.

8. Final Comments

On capital goods (machinery), the opinions of Ricardo (1965, p.263-69) were: "Ever since I first turned my attention to questions of political economy, I have been of opinion that such an application of machinery to any branch of production as should have the effect of saving labour was a general good, accompanied only with that portion of inconvenience which in most cases attends the removal of capital and labour from one employment to another.

These were my opinions, and they continue unaltered, as far as regards the landlord and the capitalist; but I am convinced that the substitution of machinery for human labour is often very injurious to the interests of the class of laborers.

The statements which I have made will not, I hope, lead to the inference that machinery should not be encouraged. To elucidate the principle, I have been supposing that improved machinery is suddenly discovered and extensively used; but the truth is that these discoveries are gradual, and rather operate in determining the employment of the capital which is saved and accumulated than in diverting capital from its actual employment".

His contemporary von Thünen [1850 (1930, p.499)] had the opinion : "Während man in Europa den gedrückten Zustand der arbeitenden Klasse so haüfig der zunehmenden Anwendung von Maschinen zuschreibt, wird in dem gesellschaftlichen Zustand, den wir hier vor Augen haben, die Lage der Arbeiter immer blühender and glänzender, je ausgedehnter beim Anwachsen des Kapitals die Anwendung von Maschinen wird".

Both issues - factor reallocations and capital accumulation combined with the GDP growth per capita of multi-sector economies - are still with us and will continue to be so, in Europe and globally. In such historical and future human circumstances, it should help - our spirit, knowledge, daily problems and nerves, as in natural sciences – being able to fundamentally understand the logic and able to formally describe the economic laws of motion (change). Mathematical models of general equilibrium dynamics for growing economies serve such purposes, as attempted in this paper. The professional economic extensions of such work are legio.

Appendix A. Asymptotic Walrasian GDP and Factor Allocations

The *comparative static analysis* of exogenous factor endowment, (L, K), variations for *Walrasian equilibria* with CES sector technologies is helpful for the economic understanding of the sectorial allocation implications of critical parameter values. As benchmarks ,the asymptotic factor allocations of multi–sector general equilibrium economies with various GDP income shares, δ_K and sectorial factor allocations, λ_{K_i}, λ_{L_i}, are calculated in Lemma A:

Lemma A. *For the CES multi-sector competitive general equilibrium economy, the limits of the factor allocation fractions and factor income shares, – with a demand side specification with constant* e_i,$(\sigma_u = 1)$*, (72), (75), – are:*

$$\begin{array}{lllllll} \sigma_i & & k \to 0 & & & k \to \infty & \\ 1 > \sigma_i \neq \sigma_{max}: & l_i \to 0 & \lambda_{K_i} \to \frac{s_i}{\underline{s}} & \delta_K \to \underline{s} & l_i \to \frac{s_i}{\underline{s}} & \lambda_{K_i} \to 0 & \delta_K \to \overline{s} \end{array} \quad (143)$$

$$1 > \sigma_i = \sigma_{max}: \quad l_i \to 1 \quad \lambda_{K_i} \to s_i \quad \delta_K \to 1 \qquad l_i \to s_i \quad \lambda_{K_i} \to 1 \quad \delta_K \to 0 \quad (144)$$

$$1 < \sigma_i \neq \sigma_{min}: \quad l_i \to \frac{s_i}{\overline{s}} \quad \lambda_{K_i} \to 0 \quad \delta_K \to \underline{s} \qquad l_i \to 0 \quad \lambda_{K_i} \to \frac{s_i}{\overline{s}} \quad \delta_K \to \overline{s} \quad (145)$$

$$1 < \sigma_i = \sigma_{min}: \quad l_i \to s_i \quad \lambda_{K_i} \to 1 \quad \delta_K \to 0 \qquad l_i \to 1 \quad \lambda_{K_i} \to s_i \quad \delta_K \to 1 \quad (146)$$

where $\underline{s}$ *and* $\overline{s}$ *are given by (29).*
If the sector technologies have the same $\sigma_i = \sigma, \forall i$*, then the limits of (143-146) become,* $[\bar{l}_i \equiv s_i c_i / \sum s_2 c_2,\ \overline{\lambda}_{K_i} \equiv s_i / c_i / \sum s_2 / c_2]$*:*

$$\sigma_i \qquad k \to 0 \qquad\qquad k \to \infty$$

$$\sigma_i = \sigma < 1: l_i \to \bar{l}_i \quad \lambda_{K_i} \to s_i \quad \delta_K \to 1 \qquad l_i \to s_i \quad \lambda_{K_i} \to \overline{\lambda}_{K_i} \quad \delta_K \to 0 (147)$$

$$\sigma_i = \sigma > 1: \quad l_i \to s_i \quad \lambda_{K_i} \to \overline{\lambda}_{K_i} \quad \delta_K \to 0 \qquad l_i \to \bar{l}_i \quad \lambda_{K_i} \to s_i \quad \delta_K \to 1 (148)$$

Corollary A. *In the case of the CES and the Indirect Addilog Utility functions (72), (87) with the GDP shares* $s_i(\omega)$ *depending on* ω*, the same conclusions (143-148) hold with* $\overline{e}_i$, $\underline{e}_i^{**}$, $\overline{s}$, $\underline{e}_i^{**}$, $\underline{s}$*, given by (84) and (85).*

Proof. From Lemma 1, (29), (30), (45), we get,

$$\delta_K = \sum_{\sigma_i < 1} \frac{s_i}{1 + c_i \omega^{1-\sigma_i}} + \sum_{\sigma_i > 1} \frac{s_i}{1 + c_i \omega^{1-\sigma_i}} \quad (149)$$

For $\omega \to \infty$, the first sum goes to 0, and the second goes to $\overline{s}$.
For $\omega \to 0$, the second sum goes to zero, and the first goes to $\underline{s}$.
From (45), we get $\epsilon_{K_i} \to 0$ for $\omega \to \infty$ and $\sigma_i < 1$, and for $\omega \to 0$ and $\sigma_i > 1$. Furthermore, we get $\epsilon_{K_i} \to 1$ for $\omega \to \infty$ and $\sigma_i > 1$, and for $\omega \to 0$ and $\sigma_i < 1$. Together with (37) we get the limits for λ_{K_i} and together with (36) we get the limits for l_i. □

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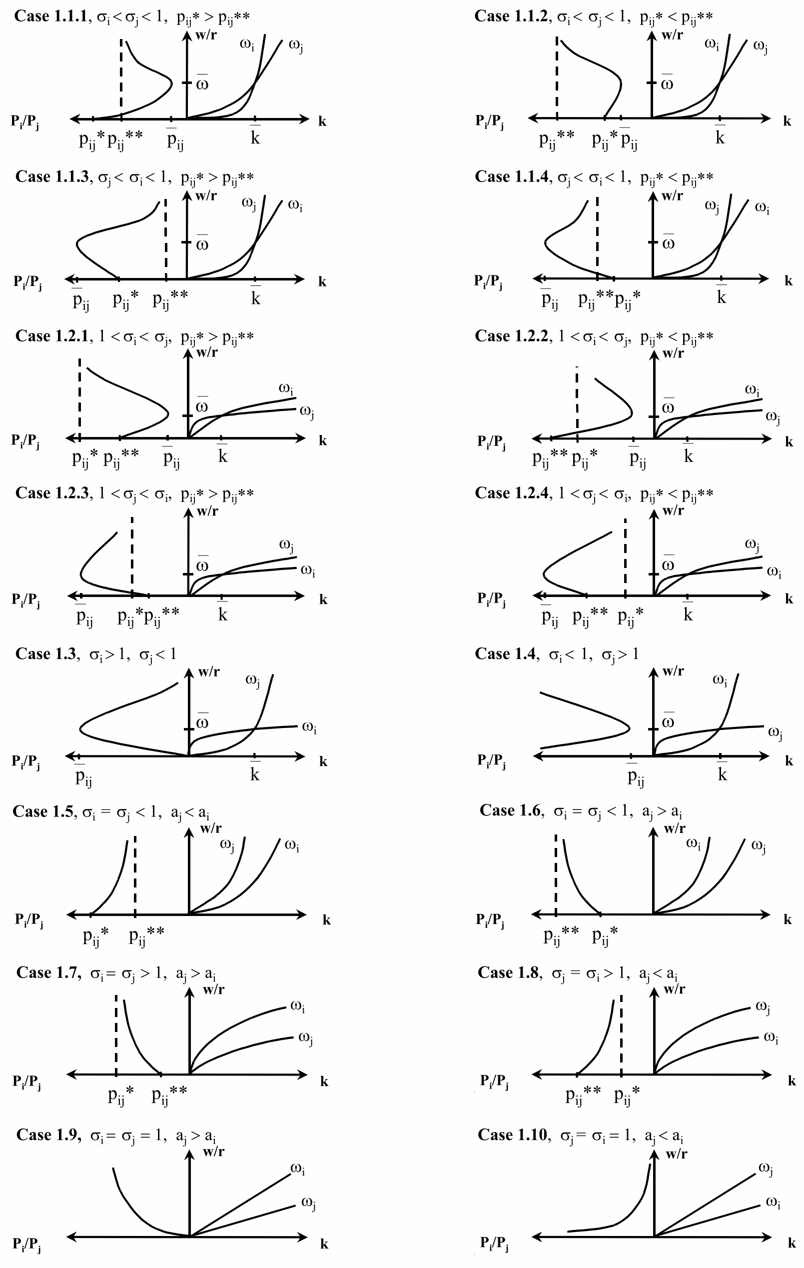

Figure 1: FPCP correspondence $p(\omega)$, (47,49,50)

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