# The Expenditure System of CDES Indirect Utility Functions

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## 1. Introduction

In production and consumer theory, the common origin of the many applied specifications of production function and utility function are undoubtedly the CD (1928) and CES (1961) forms. A major shortcoming of the CD and CES utility functions is that these preferences are homothetic, implying **unitary income elasticities** so that *Engel curves* are straight lines through the origin. Moreover, the *budget shares* are always constant with CD; but they do change with price variation for CES preferences.

Tinbergen (1942) in an article, that never received a proper attention due to the Second World War, proposed to generalize the CD production function by introducing positive *minimum amount*s of capital and labor: "This function implies that capital and labour may replace each other completely: in principle one unit of product may be made with as little capital or labour as one likes, if only enough of the other factor is used. It seems more probable, however, that there is a certain limit below which no possibility of substitution exists. Graphically this would mean that the production curve in the (L,K) diagram... does not approach the axes..... This type of function was mentioned also by Kaldor." (Tinbergen, 1942, pp. 45-46; Kaldor, 1936-1937, p.162).

Shortly after the war this idea has been introduced in the theory of consumption in a series of articles: Klein and Rubin (1948-1949), Samuelson (1948) and Stone (1954). This function is known as the Stone-Geary utility function and the ensuing demand model as the **Linear Expenditure System (LES).** In case of the LES function the *marginal budget shares* are constant; the income elasticities are not unitary, but the Engel curves are still straight lines, though not through the origin.

In the early fifties of the last century this shortcoming of the LES function was recognized at Statistics Netherlands by Somermeyer and Wit who wanted to compare the income elasticities in the Netherlands in the pre-war period based on the last pre-war budget survey (1935/1936) with those of the post-war period for which the first budget survey was conducted in 1951 (Wit, 1957). In this period of shortage of data, one had to rely upon simplifying assumptions in order to be able to (gu)estimate the values of these elasticities. Somermeyer and Wit (1956) introduced a budget allocation model that has the same data requirements as the LES function, but that does not has the limitation of constancy of marginal budget shares. It has been discovered later that this model had already been introduced by Leser (1941); a fact of which Somermeyer and Wit were unaware of.

They published their results in Dutch so that this model was not known in the outside world and a fellow country man, Houthakker (1960), who was unaware of the Leser (1941) and Somermeyer-Wit (1956) contribution, (re)discovered this model departing from an **implicitly indirectly additive** utility function. It is due to Houthakker that this model is named "**indirect addilog model**". After Houthakker's discovery, Wit (1960) published the English translation of the path breaking 1956 and 1957 articles.

As time went on, more and more data became available and the indirect addilog model was abandoned in favor of more general models, like the **Almost Ideal Demand System**, introduced by Deaton and Muellbauer (1980). A drawback of this model is that the fitted budget shares do not necessarily lie in the unit interval and that negativity can not be imposed.

In section 3 we pay attention to the *range* of **income elasticities** and the *shapes* of the *Engel curves* implied by a parametric extension of the indirect addilog model - the **CDES Indirect Utility** function - and show that it is "well behaved". In section 4 we deal with the price and substitution elasticities and show that the *differences* of the cross *elasticities* of *substitution* are *constant* (CDES). It is a simple and natural parametric extension CES preferences. Finally, section 6 offer an empirical estimation of the CDES *preference (reaction) parameters*, based on Danish family budget surveys. The data allow calculation of the income elasticities for a classification of 41 household goods and services.

## 2. The CDES Expenditure System

We assume that a consumer desires to attain at least a utility level of U and that the consumer minimizes the cost for this purpose. Let C denote the minimum cost of attaining utility level U, and let $Y_i$ denote the quantity that a consumer demands from commodity i (1,...,n) and $P_i$ the corresponding price :

$$C = \sum_{i=1}^{N} P_i Y_i \tag{1}$$

Usually, the *indirect utility* function of the *addilog expenditure system* is given by

$$V^*(P, C) = \sum_{i=1}^{N} \alpha_i^* \left(C/P_i\right)^{\beta_i} \tag{2}$$

with the *standard parameter* restrictions, (cf. Deaton and Muellbauer (1980, p. 84), Chung (1991), p.42), Silberberg & Suen (2001, p. 360), Jensen and Larsen (2005), 36):

$$\alpha_i^* > 0, \quad \beta_i > 0, \quad \sum_{i=1}^{N} \alpha_i^* = 1 \tag{3}$$

The demand equations are obtained using Roy's identity, i.e.:

$$Y_i = -\frac{\partial V^*(P,C)/\partial P_i}{\partial V^*(P,C)/\partial C} \tag{4}$$

to yield the demand equations $Y_i$ and the expenditure (budget) shares $e_i$:

$$Y_i = \frac{\alpha_i^* \beta_i (C/P_i)^{\beta_i - 1}}{\sum\limits_{j=1}^{N} \alpha_j^* \beta_j (C/P_j)^{\beta_i}} \quad ; \quad e_i = \frac{\alpha_i^* \beta_i (C/P_i)^{\beta_i}}{\sum\limits_{j=1}^{N} \alpha_j^* \beta_j (C/P_j)^{\beta_i}} \quad ; \tag{5}$$

From a mathematical point of view it is more convenient use to rewrite (2) in the form of the Box-Cox transformation - or alternative to that end, we may first *subtract* from (2) the *constant*: $\sum_{i=1}^{N} \alpha_i^*$, and secondly, use the *reparameterization*:

$$\alpha_i = \alpha_i^* \beta_i \tag{6}$$

to obtain the more generalized parametric form of the indirect utility:

$$V(P,C) = \sum_{i=1}^{n} \alpha_i \frac{(C/P_i)^{\beta_i} - 1}{\beta_i} \tag{7}$$

where we impose the *normalization* restriction:

$$\sum_{i=1}^{N} \alpha_i = 1 \tag{8}$$

As explained below, specification (7-8) is called the **CDES Indirect Utility** function. For technical details on the Box-Cox transformation, we refer to Appendix. Alternatively in (3), the normalization restriction:

$$\sum_{i=1}^{N} \alpha_i^* \beta_i = 1 \tag{9}$$

could have been adopted as well, which, in view of (6) corresponds to (8).

Derivation of the indirect utility function (7) with respect to C gives:

$$\frac{\partial V(P,C)}{\partial C} = \sum_i \alpha_i P_i^{-\beta_i} C^{\beta_i - 1} = \sum_i \alpha_i (C/P_i)^{\beta_i} C^{-1} \tag{10}$$

which is only increasing in C for **all** $P_i > 0$, if

$$\alpha_i > 0 \tag{11}$$

Secondly, we obtain from (7) that under (11):

$$\frac{\partial V(P,C)}{\partial P_i} = -\alpha_i P_i^{-\beta_i - 1} C^{\beta_i} < 0 \tag{12}$$

i.e. the CDES indirect utility function is decreasing in prices, as it should be.

Next from (12), we derive:

$$\frac{\partial^2 V(P,C)}{\partial P_i \partial P_j} = \begin{cases} \beta_i + 1)\alpha_i P_i^{-\beta_i - 2} C^{\beta_i} & fori = j \\ 0 & fori \neq j \end{cases} \tag{13}$$

Hence the indirect utility function (7) is strictly convex, and consequently strictly quasi-convex, only if

$$\beta_i > -1 \tag{14}$$

Van Daal (1983) has shown that the indirect utility function (7) is strictly quasi-convex, if and only if,

$$\alpha_i > 0, \quad \beta_i \geq -1 \tag{15}$$

where the last equality sign may apply for *at most* one value of i.

Consequently, under the parameter restrictions (13), the dual *direct* utility function is strictly quasi concave; thus the maximization of the dual direct utility function underlying the CDES indirect utility function indeed gives rise to a maximum. The underlying dual direct utility function, however, does not have an analytical form.

**Proposition**. *The generalization of the indirect addilog utility function from (2) to the CDES indirect utility function (7) is that* $\beta_i$ *is also allowed to be negative, precisely :* $\beta_i \in (-1, 0)$.

Application of Roy's identity to CDES, (7), gives the demand functions :

$$Y_i = -\frac{\partial V(P,C)/\partial P_i}{\partial V(P,C)/\partial C} = \frac{\alpha_i (C/P_i)^{\beta_i - 1}}{\sum_{j=1}^{N} \alpha_j (C/P_j)^{\beta_j}} \tag{16}$$

Hence pre-multiplication with $P_i/C$, gives the expenditure (budget) shares :

$$e_i = \frac{\alpha_i (C/P_i)^{\beta_i}}{\sum_{j=1}^{N} \alpha_j (C/P_j)^{\beta_k}} \quad , \quad \sum_{i=1}^{N} e_i = 1 \tag{17}$$

It follows from the positive $\alpha_i$ in (15), that every budget share, ($e_i$), (17), is bounded from below by zero, and together with (8) that every share, ($e_i$), is also bounded from above by one.

From (16) we obtain after some manipulations

$$\frac{\partial Y_i}{\partial C} = (1 + \beta_i - \bar{\beta})(Y_i/C) \quad ; \quad \bar{\beta} = \sum_{j=1}^{N} e_j \beta_j \tag{18}$$

and:

$$\frac{\partial Y_i}{\partial P_j} = \beta_j Y_i Y_j C^{-1} - (1 + \beta_i) P_i^{-1} Y_i \delta_{ij} \tag{19}$$

where the Kronecker delta

$$\delta_{ij} = \begin{cases} 1 & \text{for } \ i = j \\ 0 & \text{for } \ i \neq j \end{cases} \tag{20}$$

Consequently, using (18) and (19), the typical element $S_{ij}$ of the Slutsky matrix:

$$S_{ij} = \frac{\partial Y_i}{\partial P_j} + \frac{\partial Y_i}{\partial C} Y_j \quad = \quad (1 + \beta_i + \beta_j - \bar{\beta}) \, Y_i Y_j C^{-1} = \ S_{ji}, \ \ i \neq j. \tag{21}$$

Consequently, the Slutsky matrix is *symmetric*, as it should be.

In view of (16), the *off-diagonal* elements of the Slutsky matrix may be *negative, zero* or *positive*; thus, the demand system generated by the CDES indirect utility function allows for *complementarity, indifference* and *substitutability* between commodities. Since we permit, $\beta_i$, to be in the interval: (-1,0), we do allow for complementarity. With the restriction, $\beta_i > 0$ , only substitutability was allowed for. Finally, Van Driel (1974) has shown that the Slutsky matrix is *negative semi-definite with rank (n-1)*, if and only if (13) holds true.

The parameters $\alpha_i$ - with indeterminate level – may be interpreted as *"preference coefficients"*, and the parameters $\beta_i$ as *"reaction parameters"*; the lower the value of $\beta_i$ (i.e. the closer it is to -1), the more "urgent" the consumption of i may be considered to be, at least at lower income levels, For the general discussion of the parameters, we refer to Somermeyer and Langhout (1972).

## 3. Income elasticities and the shapes of Engel curves

*Income elasticities*

Firtst we summarize the findings of Somermeyer and Langhout (1972). For ease of exposition we order the commodities such that:

$$\beta_1 = \min_j \beta_j \quad \text{and} \quad \beta_n = \max_j \beta_j \tag{22}$$

so that it follows from (18) that:

$$\beta_1 \ \leq \ \bar{\beta} \ \leq \ \beta_n \tag{23}$$

It is easily derived from (18) that the income elasticities are equal to:

$$E(Y_i, C) = 1 + \beta_i - \bar{\beta} = 1 + \sum_{j=1}^{N} (\beta_i - \beta_j) e_j \tag{24}$$

In view of (16) and (24), the income elasticities are lower-bounded as well as upper-bounded:

$$-\beta_n < 1 + \beta_i - \beta_n \leq \ E(Y_i, C) \ \leq 1 + \beta_i - \beta_1 \tag{25}$$

They are approaching their lower and upper limits according as C tends to infinity and to zero respectively. In view of (13) the lower bound is at most one, while the upper bound is at least equal to one. The lower bound is zero or negative, allowing for *inferior* commodities if:

$$\beta_i \quad \leq \quad \beta_n - 1 \tag{26}$$

Because we allow $\beta_i$ to be in the interval (-1,0), we allow for the existence of inferior commodities; a possibility that was excluded by the restriction: $\beta_i > 0$. It follows from (24) that commodity i is *necessary*, when $\beta_i < \bar{\beta}$ and *luxury*, when $\beta_i > \bar{\beta}$.

*Engel curves*

Let $C_i = P_i Y_i$ denote the *expenditure* on commodity $i$; using the demand functions, (16), it is analogously proven by Somermeyer and Langhout (1972) that:

$$C_i = P_i Y_i; \qquad \lim_{C \to 0} C_i = P_i \lim_{C \to 0} Y_i = 0 \tag{27}$$

and

$$\lim_{C \to \infty} C_i = P_i \lim_{C \to \infty} Y_i = \begin{cases} \infty & \text{if } \beta_i > \beta_n - 1 \\ \text{finite} & \text{if } \beta_i = \beta_n - 1 \\ 0 & \text{if } \beta_i < \beta_n - 1 \end{cases} \tag{28}$$

It follows from (25) that the criterion $\beta_n - 1$ also rules the sign of the lower bound of the income elasticity. Equation (27) means that the Engel curve arises from the origin; while equations (28) imply the possibility of three main types of Engel curves, viz.:

(1) unlimited monotonic increase
(2) monotonic increase to a maximum (saturation) level, and
(3) decrease towards zero after having reached a maximum level.

It should be noted that types (2) and (3) cannot occur if $\beta_n < 0$. In practice, however, this case is not likely to occur, since this implies that all budget items are of a fairly urgent nature. For more details, as well as an application to the Netherlands, we refer to Somermeyer and Langhout (1972).

## 4. Price and substitution elasticities

*Marshallian price elasticities*

From (19) we derive that the own price elasticities to be:

$$E(Y_i, P_i) = \beta_i(e_i - 1) - 1 < 0 \tag{29}$$

so that Giffen goods are excluded. The cross price elasticities are:

$$E(Y_i, P_j) = \beta_j e_j \tag{30}$$

which means that all cross elasticities of a particular price, $P_j$, are the same.

The price response implied by (30) is the following. If the price increase of $P_j$ refers to a necessary commodity, i.e. $\beta_j$ negative and rather close to minus one, then the expenditure on all other commodities will *decrease* with a given percentage $\beta_j e_j$. If the price increase of $P_j$ refers to a luxury commodity, i.e. $\beta_j$ positive, and of considerable magnitude, then all other expenditures will *increase* with $\beta_j e_j$. It follows that luxuries are price elastic and necessities inelastic. Thus both positive and negative cross price effects are distributed neutrally over all other commodities. In many circumstances such proportional effects do not seem to be an unreasonable price response.

*Slutsky elasticities*

The Slutsky (compensated) elasticities easily follow from (24), (26) and (29) as:

$$E_S(Y_i, P_i) = E(Y_i, P_i) + e_i E(Y_i, C) = (1 + 2\beta_i - \bar{\beta})e_i - (1 + \beta_i) < 0 \qquad (31)$$

$$E_S(Y_i, P_j) = E(Y_i, P_j) + e_j E(Y_j, C) = (1 + \beta_i + \beta_j - \bar{\beta})e_i = \begin{cases} < 0 \\ > 0 \end{cases} \qquad (32)$$

*Allen elasticities of substitution*

Having derived the income and price elasticities, the Allen partial elasticities of *substitution*, $(\sigma_{ij})$ , easily follow from the well-known relationship:

$$E(Y_i, P_j) = [\sigma_{ij} - E(Y_i, C)]e_j \qquad (33)$$

Using (24), (26) and (29), the own elasticities become:

$$\sigma_{ii} = -(1 + \beta_i)/e_i + (1 - \bar{\beta}) \qquad (34)$$

and similarly, the cross-elasticities of substitution, $(\sigma_{ij})$ become :

$$\sigma_{ij} = \beta_i + \beta_j + (1 - \bar{\beta}) \qquad (35)$$

The property that the *differences* of the cross-elasticies of *substitution* are *constant* - CDES, (7) - follows dirctly from (35)(35):

$$\sigma_{ij} \; - \; \sigma_{ik} \; = \; \beta_j \; - \beta_k \qquad (36)$$

It should be noted that Chung (1994, p.44, (3.71)) incorrectly states that the differences, (36), are equal to zero. He seems to use the indirect utility function in the definition of the Allen partial elasticity of substitution instead of the expenditure/cost function.

## 5. Empirical values of CDES reaction parameters

We briefly report estimated parameters - $(\beta_i)$ - of the CDES indirect utility function, (7).

### Table 1: Classification of Household Goods and Services

**Food and Beverages**

1. Bread (cereals)
2. Butter
3. Margarine (other fats and oil)
4. Sugar (chocolate)
5. Milk (cream and yoghurt)
6. Cheese (curd)
7. Other foods
8. Vegetables (fruits)
9. Meat
10. Fish
11. Coffee (tea and cocoa)
12. Soft drinks (mineral water)
13. Beer
14. Wine and spirits
15. Tobacco

**Clothing and footwear**

16. Clothing
17. Footwear

**Household operations**

18. Glassware (tableware)
19. Household textiles (other furnishings)
20. Household machines

21. Furniture
22. Non-durable household goods
23. Household services
24. Radio and television
25. Services n.e.c.

**Medical care and health**

26. Medical and pharmaceutical products
27. Medical services
28. Personal care

**Leisure**

29. Jewellery
30. Leisure equipment
31. Entertainment
32. Restaurant (hotels)
33. Books and papers (magazines)

**Housing**

34. Fuel (gas, liquid fuels)
35. Electricity
36. Gross rents

**Transport and communication**

37. Gasoline
38. Repairs (transport)
39. Other transport (expenditures)
40. Purchased transport
41. Communication
42. Personal transport equipment

## Table 2: Ranking of "Reaction Parameters"

| Families with children under 7 years | | | Families without children, (wife over 45 years) | | |
|---|---|---|---|---|---|
| Rank | Commodity Class | $\beta_i - \beta_2$ | Rank | Commodity Class | $\beta_i - \beta_2$ |
| 1 | Fuel | -0.33348 | 1 | Margarine | -0.56234 |
| 2 | Bread | -0.29404 | 2 | Fuel | -0.48517 |
| 3 | Margarine | -0.25465 | 3 | Electricity | -0.42657 |
| 4 | Sugar | -0.24830 | 4 | Bread | -0.39438 |
| 5 | Coffee | -0.21253 | 5 | Coffee | -0.33723 |
| 6 | Tobacco | -0.20813 | 6 | Services n.e.c. | -0.25552 |
| 7 | Milk | -0.12385 | 7 | Tobacco | -0.24924 |
| 8 | Electricity | -0.01895 | 8 | Soft drinks | -0.23892 |
| 9 | Radio, tv | 0.01635 | 9 | Jewellery | -0.18789 |
| 10 | Medical products | 0.04118 | 10 | Meat | -0.18665 |
| 11 | Other non-durable goods | 0.09349 | 11 | Medical products | -0.10276 |
| 12 | Cheese | 0.09450 | 12 | Other non-durable goods | -0.09355 |
| 13 | Household textiles | 0.10560 | 13 | Household machines | -0.07920 |
| 14 | Vegetables | 0.11840 | 14 | Milk | -0.05749 |
| 15 | Meat | 0.12885 | 15 | Medical services | -0.04911 |
| 16 | Household machines | 0.15462 | 16 | Beer | -0.04888 |
| 17 | Glassware | 0.20301 | 17 | Books and papers | -0.02345 |
| 18 | Fish | 0.23152 | 18 | Fish | -0.00829 |
| 19 | Soft drinks | 0.27881 | 19 | Cheese | 0.03200 |
| 20 | Leisure equipment | 0.34423 | 20 | Entertainment | 0.06120 |
| 21 | Beer | 0.34631 | 21 | Household textiles | 0.07244 |
| 22 | Entertainment | 0.37195 | 22 | Personal care | 0.08888 |
| 23 | Repairs | 0.37212 | 23 | Furniture | 0.09684 |
| 24 | Clothing | 0.38976 | 24 | Sugar | 0.09990 |
| 25 | Gasoline | 0.42036 | 25 | Repairs | 0.12164 |
| 26 | Footwear | 0.43999 | 26 | Vegetables | 0.12429 |
| 27 | Jewellery | 0.54508 | 27 | Glassware | 0.13912 |
| 28 | Medical services | 0.58257 | 28 | Gross rents | 0.17936 |
| 29 | Wine and spirits | 0.58607 | 29 | Footwear | 0.24733 |
| 30 | Personal care | 0.60113 | 30 | Other transport | 0.29736 |
| 31 | Other foods | 0.64302 | 31 | Purchased transport | 0.34186 |
| 32 | Other transport | 0.66149 | 32 | Clothing | 0.37099 |
| 33 | Books and papers | 0.68632 | 33 | Radio and television | 0.39254 |
| 34 | Gross rents | 0.71112 | 34 | Gasoline | 0.42218 |
| 35 | Restaurant | 0.74708 | 35 | Other foods | 0.45335 |
| 36 | Household services | 0.78325 | 36 | Wine and spirits | 0.64572 |
| 37 | Services n.e.c. | 0.86186 | 37 | Leisure equipment | 0.69500 |
| 38 | Purchased transport | 0.91800 | 38 | Household services | 0.70973 |
| 39 | Furniture | 1.04451 | 39 | Communication | 0.89911 |
| 40 | Communication | 1.33410 | 40 | Restaurant | 1.02420 |
| 41 | Transport equipment | 1.78225 | 41 | Transport equipment | 1.32967 |

## Table 3: Estimates of Expenditure (Income) Elasticities of Goods and Services : $E(Y_i, C) = E(P_iY_i, C)$

| Families with children under 7 years | | | Families without children, (wife over 45 years) | | |
|---|---|---|---|---|---|
| Rank | Commodity Class | $E(Y_i, C)$ | Rank | Commodity Class | $E(Y_i, C)$ |
| 1 | Fuel | 0.18039 | 1 | Margarine | 0.24691 |
| 2 | Bread | 0.21982 | 2 | Fuel | 0.32408 |
| 3 | Margarine | 0.25921 | 3 | Electricity | 0.38268 |
| 4 | Sugar | 0.26556 | 4 | Bread | 0.41487 |
| 5 | Coffee | 0.30133 | 5 | Coffee | 0.47202 |
| 6 | Tobacco | 0.30573 | 6 | Services n.e.c. | 0.55373 |
| 7 | Milk | 0.39001 | 7 | Tobacco | 0.56001 |
| 8 | Electricity | 0.49491 | 8 | Soft drinks | 0.57033 |
| 9 | Radio, tv | 0.53021 | 9 | Jewellery | 0.62136 |
| 10 | Medical products | 0.55504 | 10 | Meat | 0.62260 |
| 11 | Other non-durable goods | 0.60735 | 11 | Medical products | 0.70649 |
| 12 | Cheese | 0.60836 | 12 | Other non-durable goods | 0.71570 |
| 13 | Household textiles | 0.61946 | 13 | Household machines | 0.73681 |
| 14 | Vegetables | 0.63226 | 14 | Milk | 0.75176 |
| 15 | Meat | 0.64271 | 15 | Medical services | 0.76014 |
| 16 | Household machines | 0.66848 | 16 | Beer | 0.76037 |
| 17 | Glassware | 0.71687 | 17 | Books and papers | 0.78580 |
| 18 | Fish | 0.74538 | 18 | Fish | 0.80096 |
| 19 | Soft drinks | 0.79267 | 19 | Cheese | 0.84125 |
| 20 | Leisure equipment | 0.85809 | 20 | Entertainment | 0.87045 |
| 21 | Beer | 0.86017 | 21 | Household textiles | 0.88169 |
| 22 | Entertainment | 0.88581 | 22 | Personal care | 0.89813 |
| 23 | Repairs | 0.88598 | 23 | Furniture | 0.90609 |
| 24 | Clothing | 0.90362 | 24 | Sugar | 0.90915 |
| 25 | Gasoline | 0.93422 | 25 | Repairs | 0.93089 |
| 26 | Footwear | 0.95385 | 26 | Vegetables | 0.93354 |
| 27 | Jewellery | 1.05894 | 27 | Glassware | 0.94837 |
| 28 | Medical services | 1.09643 | 28 | Gross rents | 0.98861 |
| 29 | Wine and spirits | 1.09993 | 29 | Footwear | 1.05658 |
| 30 | Personal care | 1.11499 | 30 | Other transport | 1.10661 |
| 31 | Other foods | 1.15688 | 31 | Purchased transport | 1.15111 |
| 32 | Other transport | 1.17535 | 32 | Clothing | 1.18024 |
| 33 | Books and papers | 1.20018 | 33 | Radio and television | 1.20179 |
| 34 | Gross rents | 1.22498 | 34 | Gasoline | 1.23143 |
| 35 | Restaurant | 1.26094 | 35 | Other foods | 1.26260 |
| 36 | Household services | 1.29711 | 36 | Wine and spirits | 1.45497 |
| 37 | Services n.e.c. | 1.37572 | 37 | Leisure equipment | 1.50425 |
| 38 | Purchased transport | 1.43186 | 38 | Household services | 1.51898 |
| 39 | Furniture | 1.55837 | 39 | Communication | 1.70836 |
| 40 | Communication | 1.84796 | 40 | Restaurant | 1.83345 |
| 41 | Transport equipment | 2.29611 | 41 | Transport equipment | 2.13892 |

## Appendix. The Box-Cox transformation of a variable

In mathematics the Box-Cox transformation of a variable x is defined as:

$$x(\lambda) = \frac{x^{\lambda} - 1}{\lambda} \tag{37}$$

If $\lambda = 1$, then (apart from the constant -1) the variable is linear.

If $\lambda \to 0$, then the limit of the numerator, as well as the limit of the denominator are equal to zero, so that we apply de l'Hôpital's rule. We first take the derivative of numerator and denominator with respect to $\lambda$, and take the limit again:

$$\lim_{\lambda \to 0} \frac{x^{\lambda} - 1}{\lambda} = \lim_{\lambda \to 0} \frac{x^{\lambda} \log(x)}{1} = \log(x) \tag{38}$$

where log denotes the natural logarithm.

In econometrics the Box-Cox transformation of a variable is used when one wishes to estimate a model and one is not sure whether the variable x should be included as such or whether its logarithm should be used. The model is estimated in the form of the Box-Cox transformation and it is the tested whether the parameter of a certain variable x is equal to one, corresponding to the inclusion of the variable as such, or equal to zero, in which case the logarithm of the variable should be used. See Heij et. al. (2005, pp. 297-301) for more details.

In the context of economic theory the Box-Cox transformation can advantageously be used in the definition of the indirect utility function of the generalized addilog expenditure system:

$$V(P, C) = \sum_{i=1}^{n} \alpha_i \frac{(C/P_i)^{\beta_i} - 1}{\beta_i} \tag{39}$$

If $\beta_i = 0$, then the term, $\frac{(C/P_i)^{\beta_i} - 1}{\beta_i}$, has to be replaced by : $\log(C/P_i)$. In practice, when the model is econometrically estimated, the event that the estimated coefficient is zero is not impossible, but occurs with probability zero. That is the reason why we do not treat this case in the main text.

In the main text the result of Van Driel (1974) and of Van Daal (1982) that at most one $\beta_i$is allowed to be minus one is mentioned. It means that for that particular commodity the corresponding term in the indirect utility function should be replaced by : $(C/P_i)^{-1}$. In practice this special case is again not impossible, but the probability of occurrence is zero again.

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