Long-Run Patterns of Demand: The Expenditure System of the CDES Indirect Utility Function - Theory and Applications

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Abstract

In this paper, we unify and extend the analytical and empirical application of the "indirect addilog" expenditure system, introduced by Leser (1941), Somermeyer-Wit (1956) and Houthakker (1960). Using the Box-Cox transform, we present a parametric analysis of the Houthakker specification of the fundamental indirect utility function - called the CDES specification (constant differences of Allen elasticities of substitution) by Hanoch (1975). It is shown that the CDES demand system is less restrictive than implied by standard parameter restrictions in the literature, Hanoch (1975), Deaton & Muellbauer (1980), or else neither adequately indicated, Houthakker (1960), Silberberg & Suen (2001). Our parametric examination implies that Marshallian own-price elasticities are no longer restricted to being all larger than one in absolute value; hence CDES can now naturally exhibit both the inelastic and elastic own price elasticities of observable (Marshallian) demands. Furthermore, we argue that in computable general equilibrium models (CGE), the CDES compares favorably with other expenditure systems, e.g. the linear expenditure system (LES), since CDES and LES need the same outside information for calibration of the parameters, but CDES is not confined to constancy of marginal budget shares (linear Engel curves). Moreover, we show that the non-homothetic CDES preferences are a simple and natural extension of the homothetic CES (constant elasticities of substitution) preferences, and, accordingly, CDES can more realistically be used in specifying CGE models with a demand side of non-unitary income elasticities. A succint theoretical briefing of the CDES history with general and concise formulas is offered. We illustrate CDES estimation and the calculation of a comprehensive set of income and price elasticities by applying CDES to Danish budget survey data. With a large number budget items included, coherent numerical values for the income, own, and cross price elasticities, as shown here, seem nowhere calculated and available in the voluminous literature.

Keywords: CDES demand systems, non-homothetic preferences, general price elasticities, CGE modeling, budget data implementation.

JEL classification:

1. Introduction

In production and consumer theory, the common origin of the many applied specifications of production functions and utility functions are undoubtedly the CD, Cobb & Douglas (1928), and CES (constant elasticities of substitution) forms, Arrow et al (1961), also known as the Bergson family, Samuelson (1965, p.787). A major empirical restriction imposed by CD and CES utility functions was that such preferences are *homothetic* - implying *unitary income elasticities* (Engel curves that are straight lines through the origin). Moreover, the *budget shares* are always constant for any prices with CD, but are affected by price variation with the CES preferences.

It was subsequently proposed to generalize the CD production function by introducing positive *minimum amounts* of capital and labor, Tinbergen (1942, pp. 45-46) : "This function implies that capital and labour may replace each other completely: in principle one unit of product may be made with as little capital or labour as one likes, if only enough of the other factor is used. It seems more probable, however, that there is a certain limit below which no possibility of substitution exists. Graphically this would mean that the production curve in the (L,K) diagram does not approach the axes".

Shortly after the war, this idea had been introduced in the theory of consumption in a series of articles: Klein & Rubin (1948-1949), Samuelson (1948), Geary (1949-1950), and Stone (1954). This function is known as the Klein & Rubin or Stone & Geary utility function , and the derived demand system is the *linear expenditure system* (LES), where the *marginal budget shares* are constant. The income elasticities are not unitary, but the Engel curves are still straight lines, though not rays (from the origin).

In the early 1950s, this shortcoming of the LES model was recognized at Statistics Netherlands by Somermeyer and Wit, who wanted to compare the income elasticities in the Netherlands in the pre-war period [based on the last pre-war budget survey (1935/1936)] with those of the post-war period [based on the first post-war budget survey in 1951], cf. Wit (1957). In this period with shortage of data, it was important to use models that were parsimonious in parameters in order to obtain proper estimated values for these elasticities. Somermeyer & Wit (1956) introduced a *budget allocation model*, which had the same number of parameters to be estimated as the LES demand functions, but without the limitation of the constancy of marginal budget shares.

In computable general equilibrium (CGE) modeling, the LES utility function is often adopted for the description of household preferences. But de Boer & Missaglia (2005) proved that the budget allocation model proposed by Somermeyer & Wit needs exactly the same outside (calibration) data information - a social accounting matrix, the income elasticities, and the Frisch parameter - as the LES in order to assign a numerical value to its parameters.

It was discovered later that this budget allocation model had already been introduced by Leser (1941); a fact Somermeyer and Wit were unaware of. They published their results in Dutch, and hence their estimated expenditure model was not known to the outside world. Then Houthakker (1960), who was unaware of Somermeyer & Wit (1956) but knew the Leser (1941) contribution, actually related this budget allocation model to a preference ordering, specified by an *additive indirect utility* function. For a particular mathematical form of additive indirect utility functions, Houthakker (1960, p.252) used the name: *"indirect addilog preference ordering"*. After Houthakker's demonstration, Wit (1960) published the English translation of the 1956 and 1957 articles.

It is one of our *purposes* to rigorously review this indirect utility function of Houthakker (1960), who in fact suggested mostly *negative* reaction *parameters* (β_i, defined below). But Hanoch (1975) and others have since imposed such parameter restrictions that *all* (except for possibly one) the reaction *parameters* have to be *positive*, which empirically is too restrictive, as it prevents many own-price elasticities of being absolutely less than one (inelastic Marshallian demand curves).

We present a *parametric Box-Cox transform* of Houthakker's specification of the indirect utility function, which since the seminal contribution of Hanoch (1975) is wellknown as the CDES *indirect utility* function. We allow relevant reaction parameters to be negative as well as positive, allow complimentarity between commodities, allow inferior goods, and accomodates both elastic an inelastic demands with respect to own prices. Consequently, the basic CDES (Houthakker) indirect utility (expenditure) model is fortunately more versatile and general than the restricted form of Hanoch (1975) and others. As we shall emphasize, the CDES expenditure system constitutes - from the very beginning and later - a keystone in creating the foundations of analytical demand *theory*, and is still a suitable benchmark for many results of *empirical* demand analysis.

However, as time went on, more and more data became available and the CDES model was abandoned in favor of more general models, like the *Almost Ideal Demand System*, introduced by Deaton & Muellbauer (1980a,b). A drawback of this expenditure model, however, is that the fitted budget shares do not necessarily lie in the unit interval, and that another important restriction required by the economic theory of consumer optimization [e.g., a Slutsky matrix that is negative semi-definite with rank $(N-1)$] cannot be satisfied (imposed).

The organization of the paper is as follows: Section 2 presents the CDES indirect utility function, gives the parameter restrictions under which it satisfies the requirements of microeconomic theory (utility maximization for given prices and given total income/total expenditure); we derive the budget share expressions and show that they are a simple and natural generalization of the budget share formulas of CES. Section 3 is devoted to obtaining the income elasticities and to the shape of the Engel curves, whereas the price and substitution elasticities are given in section 4. It is shown that the Allen partial elasticities of substitution exhibit constant differences of elasticities of substitution - the CDES indirect utility function. In section 5, we show that the CDES expenditure system is a convenient empirical model for the estimation of income elasticities from budget survey data. Section 6 offers an estimation, based on Danish budget surveys, of the CDES *reaction parameters.* The data allow calculation of the income and price elasticities for a classification of 41 consumer goods and services. Final comments are found in section 7.

2. The CDES Indirect Utility and Expenditure System

Let $Y_i, i = (1, \ldots N), Y = (Y_1, ..Y_N)$, denote the quantity demanded of a commodity, and P_i its corresponding price. By assumption, consumers want to attain at least a utility level of U(Y), and the consumers minimize cost for this purpose. Let C denote the minimum cost (total expenditure) of attaining utility level U(Y) :

$$C = \sum_{i=1}^{N} P_i Y_i \tag{1}$$

A function $U = V(P, C)$ that gives the maximum utility as a function of - $P = (P_1, \cdots P_N)$, C - is an *indirect utility function* with : homogeneity, monotonicity, convexity and differentiability as *regularity* properties, cf. Diewert (1974, p.121; 1982, p.557), Katzner (1968), Varian (1992, p.102), Mas-Colell (1995, p.56),

- (i) homogeneous of degree zero in prices (P_i) and total expenditure (C)
- (ii) nonincreasing in prices (P_i), and nondecreasing in total expenditure (C)
- (iii) quasi-convex in prices (P_i) and total expenditure (C)
- (iv) differentiable in all prices $P_i > 0$ and in $C > 0$

The *indirect utility function* of the so-called *"indirect addilog"* form is given by

$$V^*(P, C) = \sum_{i=1}^{N} \alpha_i^* \, (C/P_i)^{\beta_i} \tag{2}$$

with the *standard parameter* restrictions,

$$\alpha_i^* > 0, \quad \beta_i > 0, \quad \sum_{i=1}^{N} \alpha_i^* = 1 \tag{3}$$

in Hanoch (1975, p.411), Deaton & Muellbauer (1980a, p.84), Chung (1991, p.42), Jensen & Larsen (2005, p.36), [ambiguously defined in Houthakker (1960, p.252,256) and unspecified in Silberberg & Suen (2001, p.360)].

The demand equations from (2) are obtained by using Roy's identity, i.e.:

$$Y_i \;=\; - \;\frac{\partial V^*(P, C)/\partial P_i}{\partial V^*(P, C)/\partial C} \tag{4}$$

giving here demand equations Y_i and next expenditure (budget) shares e_i as:

$$Y_i = \frac{\alpha_i^* \beta_i (C/P_i)^{\beta_i + 1}}{\sum_{j=1}^{N} \alpha_j^* \beta_j (C/P_j)^{\beta_i}} \quad ; \quad e_i = \frac{P_i Y_i}{C} = \frac{\alpha_i^* \beta_i (C/P_i)^{\beta_i}}{\sum_{j=1}^{N} \alpha_j^* \beta_j (C/P_j)^{\beta_i}} \quad ; \tag{5}$$

Mathematically, it is more general and convenient to rewrite (2) - and hence (5) - in the form of the Box-Cox transformation (technical details, cf. Appendix

A) by first *subtracting* from (2) the *constant*: $\sum_{i=1}^{N} \alpha_i^*$, and secondly, using the *reparameterization*:

$$\alpha_i = \alpha_i^* \beta_i \tag{6}$$

to obtain the *indirect utility function*,

$$V(P,C) \; = \; \sum_{i=1}^{N} \alpha_i \, [\frac{(C/P_i)^{\beta_i} - 1}{\beta_i}] \; ; \quad P_i > 0, \; C > 0 \tag{7}$$

with the imposed *normalization* restriction:

$$\sum_{i=1}^{N} \alpha_i = 1 \tag{8}$$

Our *specification* (7-8), with preference *parameter restrictions* , $\alpha_i > 0, \beta_i \geq -1$, (14), is called the *CDES indirect utility function*, as explained below, cf. (58).

Regarding the *general properties* (i)-(iv) above, it is immediately clear from the analytical expression, (7), (sum of power functions in C/P_i) that this function satisfies the properties (i) and (iv). In order to verify property (ii), we see that the derivative of the indirect utility function (7) with respect to C is:

$$\frac{\partial V(P,C)}{\partial C} = \sum_i \alpha_i P_i^{-\beta_i} C^{\beta_i - 1} = \sum_i \alpha_i (C/P_i)^{\beta_i} C^{-1} \; > 0 \tag{9}$$

i.e., $V(P,C)$ is *increasing* in total *expenditure* C for all $P_i > 0$, only if

$$\alpha_i > 0 \tag{10}$$

(a commodity for which the corresponding α_i would be zero does not belong to the consumption bundle of the consumer).

Secondly, we obtain from (7), that under (10):

$$\frac{\partial V(P,C)}{\partial P_i} = -\alpha_i P_i^{-\beta_i - 1} C^{\beta_i} = -\alpha_i (C/P_i)^{\beta_i} P_i^{-1} \; < 0 \tag{11}$$

i.e., the CDES indirect utility function $V(P,C)$ is *decreasing* in every *price*.

Hence, under the parameter restriction (10), the indirect utility function (7) satisfies the monotonicity property (ii).

In order to verify property (iii), we derive from (11):

$$\frac{\partial^2 V(P,C)}{\partial P_i \partial P_j} = \begin{cases} (\beta_i + 1)\alpha_i P_i^{-\beta_i - 2} C^{\beta_i} = (\beta_i + 1)\alpha_i (C/P_i)^{\beta_i} P_i^{-2} \; > 0 \; ; & i = j \\ 0 & i \neq j \end{cases} \tag{12}$$

Hence the CDES indirect utility function (7) is strictly *convex* in *prices* and consequently strictly *quasi-convex* in prices with (10), only if

$$\beta_i > -1 \tag{13}$$

Van Daal (1983) has shown that the indirect utility function (7) is strictly *quasi-convex*, if and only if

$$\alpha_i > 0, \quad \beta_i \geq -1 \tag{14}$$

where the last equality sign may apply for *at most* one commodity index of i.

Consequently, under the parameter restrictions, (10), (13), or (14), the CDES *indirect utility* function (7) satisfies the requirements, (i)-(iv), imposed by optimizing consumer theory. The underlying *dual CDES direct utility* function, however, does not have a closed analytical form, as discussed further below.

Thus, we have proven that the standard form (2-3) can fruitfully (larger empirical scope) be *generalized* such that *some* of the parameters β_i in the form (7) are *economically* allowed to be negative, i.e., to be more precise:

$$\exists\, i : -1 < \beta_i < 0 \tag{15}$$

(where we disregarded the special case that at most one β_i is allowed to be equal to -1, see (14), and that one, or more, even all $\beta_i = 0$, see the Appendix).

The CDES Expenditure System

Applying Roy's identity to CDES, (7), gives - using (9), (11) - the Marshallian demand functions:

$$Y_i \;=\; D_i(P,C) \;=\; -\;\frac{\partial V(P,C)/\partial P_i}{\partial V(P,C)/\partial C} \;=\; \frac{\alpha_i (C/P_i)^{\beta_i+1}}{\sum\limits_{j=1}^{N} \alpha_j (C/P_j)^{\beta_j}} \tag{16}$$

Nice asymptotic properties of these demand functions can be demonstrated as:

$$\lim Y_i = \begin{cases} \infty & \text{for } P_i \rightarrow 0 \\ 0 & \text{for } P_i \rightarrow \infty \\ 0 & \text{for } C \rightarrow 0 \end{cases} \tag{17}$$

The *non-satiation* for ($P_i \rightarrow 0$) is mathematically important in CGE context. Pre-multiplication of (16) with (P_i/C) gives the budget shares :

$$e_i = \frac{P_i Y_i}{C} = \frac{\alpha_i (C/P_i)^{\beta_i}}{\sum\limits_{j=1}^{N} \alpha_j (C/P_j)^{\beta_j}} \quad , \quad \sum_{i=1}^{N} e_i = 1 \tag{18}$$

In (7), (18), the parameters, α_i, may be called, *"intensity* (relative)*coefficients"*, and the parameters, β_i, *"reaction parameters"* (sensitivity to changes in, C/P_i), cf. Appendix A. The lower the value of β_i (i.e., the closer it is to -1), the more "urgent" is the consumption of this item, at least at lower income levels. For further discussion of the parameters, see Somermeyer & Langhout (1972).

It follows from the positive α_i (10) that every budget share, (e_i), (18), is bounded from below by zero, and together with (8) that every share, (e_i), is also bounded from above by one. Let $\bar{\beta}$ denote the *budget weighted* (e_i) sum of the *reaction parameters* (β_i):

$$\bar{\beta} \quad = \quad \sum_{j=1}^{N} e_j \beta_j \quad > \quad -1 \tag{19}$$

With Marshallian demands, (16), we get after some manipulations, cf. Appendix B, the basic *Marshallian first-order derivatives*, Barten & Boehm (1982, p.415):

$$\frac{\partial D_i}{\partial C} = (1 + \beta_i - \bar{\beta})(Y_i/C) \tag{20}$$

$$\frac{\partial D_i}{\partial P_i} = -(1+\beta_i)(Y_i/P_i)+\beta_i(Y_iY_i/C)\ ; \quad \frac{\partial D_i}{\partial P_j} = \beta_j(Y_iY_j/C); \ \frac{\partial D_j}{\partial P_i} = \beta_i(Y_iY_j/C) \tag{21}$$

Thus, by (20) and (21), the typical derivatives S_{ij} in the Slutsky matrix are:

$$\frac{\partial H_i(P,U)}{\partial P_i} = S_{ii}(P,C) = \frac{\partial D_i}{\partial P_i}+\frac{\partial D_i}{\partial C}Y_i = -(1+\beta_i)(Y_i/P_i)+(1+2\beta_i-\bar{\beta})\ (Y_iY_i/C) \tag{22}$$

$$\frac{\partial H_i(P,U)}{\partial P_j} = S_{ij}(P,C) = \frac{\partial D_i}{\partial P_j} + \frac{\partial D_i}{\partial C}Y_j = (1 + \beta_i + \beta_j - \bar{\beta})\ (Y_iY_j/C) = \ S_{ji} \tag{23}$$

i.e., the Slutsky ("substitution") matrix, $S(P, C)$, is *symmetric*, as it should be.

In view of (13), (19), the *off-diagonal* elements of the Slutsky matrix (23) may be *negative, zero* or *positive*; i.e, *Hicksian demand* functions, $Y_i = H_i(P, U)$, implied by the CDES indirect utility function allow for *complementarity, indifference*, and *substitutability* between commodities. As we permit β_i, cf.(15), to be in the interval: (-1,0), we easily allow for complementarity. With the restriction, $\beta_i > 0$, (3), substitutability is likely to be dominant; but we must fairly note, as stressed by Hanoch (1975, p.412) that all $\beta_i > 0$ also permit some pairs of complements, since (23) may be negative, if β_i and β_j are sufficiently small (relative to $\bar{\beta}$). Nevertheles, much more flexibility in many applications , as seen in the *empirical* section below, is obtained *without* this restriction, $\beta_i > 0$.
Finally, Van Driel (1974) has shown that the Slutsky matrix is *negative semi-definite* with *rank (N-1)*, if and only if (14) holds true.

The term $\bar{\beta}$, (19), increases *monotonically* with larger C, since

$$\frac{\partial \bar{\beta}}{\partial C} \ = \ [\sum_{j=1}^{N} e_j\ (\beta_j - \bar{\beta})^2]\ C^{-1} > \ 0 \tag{24}$$

cf. Appendix B, where also the price derivative of $\bar{\beta}$, (19), is derivedd as

$$\frac{\partial \bar{\beta}}{\partial P_j} = -\beta_j \, (\beta_j - \bar{\beta}) \frac{Y_j}{C} \tag{25}$$

Before turning to the income elasticities, price and substitution elasticities for particular commodities of the CDES expenditure system, we briefly consider budget share formulas of the wellknown CES *indirect utility* function,

$$V(P, C) = \left[\sum_{i=1}^{N} \alpha_i \, (C/P_i)^{\sigma - 1} \right]^{\frac{1}{\sigma - 1}} \tag{26}$$

and they are:

$$e_i = \frac{\alpha_i (C/P_i)^{\sigma-1}}{\sum_{j=1}^{N} \alpha_j (C/P_j)^{\sigma-1}} = \frac{\alpha_i (P_i)^{1-\sigma}}{\sum_{j=1}^{N} \alpha_j (P_j)^{1-\sigma}} = \frac{\alpha_i}{\sum_{j=1}^{N} \alpha_j (P_i/P_j)^{\sigma-1}} \tag{27}$$

where σ denotes the elasticity of substitution.

It follows from the comparison of (27) with (18) that when $\beta_i = \sigma - 1$ (constant) the CDES expenditure system reduces to CES. Therefore the CDES indirect utility and its expenditure system is a simple and *natural parametric generalization* of the CES preferences; cf. Jensen & Larsen (2005, p.36). Hence CDES can directly be seen as a tractable extension of relevance for many consumer and CGE models.

3. Income elasticities and the shape of Engel curves

Income elasticities of CDES Marshallian demand functions, $Y_i = D_i(P, C)$.

First we summarize the findings of Somermeyer & Langhout (1972). For ease of exposition, we define:

$$-1 < \beta_{min} = \min_j \beta_j \quad ; \quad \beta_{max} = \max_j \beta_j \tag{28}$$

Then it follows from (19) and (28) that:

$$\beta_{min} \;\leq\; \bar{\beta} \;\leq\; \beta_{max} \tag{29}$$

By definition, it follows from multiplying (20) with (C/Y_i) that the expenditure ("income") elasticities, $E(Y_i, C)$, of Marshallian demands $Y_i(P, C)$ become:

$$E(Y_i, C) = 1 + \beta_i - \bar{\beta} = 1 + \sum_{j=1}^{N} (\beta_i - \beta_j) e_j \tag{30}$$

or else

$$E(Y_j, C) = 1 + \beta_j - \bar{\beta} = 1 - \sum_{i=1}^{N} e_i (\beta_i - \beta_j) = 1 - \bar{\beta}_j \tag{31}$$

The higher β_i, the higher is $E(Y_i, C)$. The CEDS income (expenditure, C) elasticties of $Y_i(P, C)$ satisfy the condition of *Engel aggregation*:

$$\sum_{j=1}^{N} e_j E(Y_j, C) \;=\; \sum_{j=1}^{N} e_j(1 + \beta_j - \bar{\beta}) \;=\; 1 + \sum_{j=1}^{N} e_j\beta_j - \sum_{j=1}^{N} e_j\bar{\beta} \;=\; 1 \qquad (32)$$

Although every $E(Y_i, C)$, cf. $\bar{\beta}$, (19), declines with larger C, we see by (28) and (30), that the *whole set* ($i = 1...N$) of income elasticities are *lower-bounded* as well as *upper-bounded*:

$$-\beta_{max} \leq 1 + \beta_i - \beta_{max} \leq \; E(Y_i, C) \; \leq 1 + \beta_i - \beta_{min} \qquad (33)$$

By (33), the *maximal lower* boundary of the set is *at most one*, while the *minimal upper* boundary of the set is *at least one*. The lower boundary is zero or negative (allowing for *inferior* commodities), if:

$$\beta_i \quad \leq \quad \beta_{max} - 1 \qquad (34)$$

Because we economically permit β_i belonging to the interval (-1,0), cf.(15), our CDES demand system more easily allows for the existence of some inferior goods than the former restriction, $\beta_i > 0$, (3).

It follows from the formulas of the income elasticities (30) that a commodity is a *necessity* (inelastic), when $\beta_i < \bar{\beta}$, and a *luxury* (elastic), whenever $\beta_i > \bar{\beta}$. Although every $E(Y_i, C)$ like $\bar{\beta}$ is varying with C [changing between "income" groupes (rich/poor) and changing over time], we note from (30-31) that,

$$E(Y_i, C) - E(Y_j, C) = \beta_i - \beta_j \;\Leftrightarrow\; E(Y_i, C) = E(Y_j, C) + \beta_i - \beta_j \qquad (35)$$

i.e., the *differences* between the CDES *income elasticities* of various commodities (items) are invariant (*constant*). In other words, a *stable* hierarchy (*ranking*) exists between all the item groupes in the income (C) sensitivity of their demand/expenditures (Y_i/P_iY_i). The ranking of the income elasticities is robust, and it corresponds to the *ranking* of their *"reaction parameters "*, cf. (35).

Engel curves

Let $C_i = P_iY_i$ denote the *expenditure* on a commodity; locally, we have, (30),

$$E(Y_i, C) = E(C_i, C) = 1 + \beta_i - \bar{\beta}\;; \quad E(e_i, C) = \beta_i - \bar{\beta} \qquad (36)$$

Hence (35) also applies to $E(C_i, C)$ and $E(e_i, C)$, $i \neq j$. Using the Marshallian demand functions, cf. (16), (17), it is analogously proven by Somermeyer & Langhout (1972) that:

$$\lim_{C \to 0} C_i = P_i \lim_{C \to 0} Y_i = 0 \qquad (37)$$

and

$$\lim_{C \to \infty} C_i = P_i \lim_{C \to \infty} Y_i = \begin{cases} \infty & \text{if } \beta_i > \beta_{max} - 1 \\ \text{finite} & \text{if } \beta_i = \beta_{max} - 1 \\ 0 & \text{if } \beta_i < \beta_{max} - 1 \end{cases} \qquad (38)$$

Property (37) means that the *Engel* (item specific expenditure) *curves* - $C_i = g_i(C)$ - start from the origin, while properties (38) imply the possibility of three main types of Engel curves to occur with CDES Marshallian demands, viz.:

(1) unlimited monotonic increase
(2) monotonic increase to a maximum (saturation) level, and
(3) decrease towards zero after having reached a maximum level.

For more details, as well as an application to the Netherlands, we refer to Somermeyer & Langhout (1972).

4. Price and substitution elasticities

Price elasticities of CDES Marshallian demands, $Y_i = D_i(P, C)$.

The *own-price* elasticities, $E(Y_i, P_i)$, and the *cross-price* elasticities, $E(Y_i, P_j)$, of Marshallian demand functions are easily obtained by multiplying (21) with, respectively, (P_i/Y_i), (P_j/Y_i); we get the formulas as,

$$E(Y_i, P_i) = -(1+\beta_i) + \beta_i e_i = -1 - \beta_i(1-e_i) < 0 \ ; \qquad \beta_i > -1 \tag{39}$$

$$E(Y_i, P_j) \quad = \quad \beta_j e_j \quad = \quad \begin{cases} < 0 \\ > 0 \end{cases} \tag{40}$$

The higher β_i, the (absolutely) higher is $E(Y_i, P_i)$. The CDES price elasticities satisfy the restriction by the demand functions $D_i(P, C)$ meeting the *homogeneity condition* of degree zero in prices and income, cf. (39-40), (30) :

$$\sum_{j=1}^{N} E(Y_i, P_j) + E(Y_i, C) \ = \ 0 \ \ \Leftrightarrow \ \ \sum_{j=1}^{N} E(Y_i, P_j) \ = \ - \ E(Y_i, C) \tag{41}$$

$$-(1+\beta_i) + \beta_i e_i + \sum_{j \neq i} \beta_j e_j \ = \ - \ (1 + \beta_i - \bar{\beta}) \tag{42}$$

CDES price elasticities of $D_i(P, C)$ satisfy the condition of *Cournot aggregation*:

$$\sum_{j=1}^{N} e_j E(Y_j, P_i) \ = \ - \ e_i \ \Leftrightarrow \ e_i[-(1+\beta_i) + \beta_i e_i] + \sum_{j \neq i} e_j \beta_i e_i \ = \ - \ e_i \tag{43}$$

It follows from general properties of price elasticities that

$$E(C_i, P_i) = E(e_i, P_i) = -\beta_i(1-e_i) \tag{44}$$

$$E(Y_i, P_j) = E(C_i, P_j) = E(e_i, P_j) = \beta_j e_j \tag{45}$$

It is seen that the own-price elasticities (39) have rich possibilities of individual variation, but they are all *negative*, which a priori exclude "Giffen goods" (in practice of little relevance) from the CDES system. It is evident from (39) that with CDES: goods with (β_i) positive are *price elastic*, and those with (β_i) negative are *price inelastic*. The boundaries of (39) are: ($-1-\beta_i$ and -1).

Thus, the *sign* of β_i is the watershed, dividing the intervals of Marshallian price elasticities. In contrast to $\beta_i > -1$, (39), the former predominant *restriction* of $\beta_i > 0$, cf. (3) - by *excluding inelastic own-price* demand (39) - is *devestating* for empirical purposes [estimation/calibration on demand observations anywhere]. Moreover, with all $\beta_i > 0$, only *gross substitutes* (40) occur, but here also gross (Marshallian) *complements* are allowed for.

Thus the *cross-price elasticities* (40) depend on which commodity price (P_j) is actually changing. But (40) also means that cross elasticities of all goods ($i = 1, ...N$) with respect to a particular price, (P_j), are *all* the *same* (equal size). The price response implied by (40) is the following. If the price increase of (P_j) refers to some urgent commodity, (β_j, negative), then the expenditure on all other commodities will *decrease* with a given percentage, $\beta_j e_j$. If the price increase of (P_j) refers to a demanded item of lesser urgency (β_j, positive), then all other expenditures will *increase* with the percentage, $\beta_j e_j$. Thus both positive and negative cross price effects are distributed neutrally over all other commodities.

In many circumstances such proportional *cross-effects* do not seem to be unreasonable price responses within a fully coherent (consistent) and complete expenditure system, operating rigorously under the total budget constraint, (C). More on this in the empirical section. To give another example: in many developing countries there is hardly any information on price responses. Thus, the assumption implied by CDES that price effects are proportional is quite neutral and convenient. Such use of CDES to describe household preferences in a CGE model is easily defendable; see de Boer & Missaglia (2005).

Price elasticities (Slutsky) of CDES Hicksian demands, $Y_i = H_i(P,U)$.

The Slutsky - “compensated (Hicksian) demand” - elasticities, $E_S(Y_i, P_j)$, follow directly from the Slutsky elements (derivatives), (22-23), and (30), (39-40) as:

$$E_S(Y_i,P_i) = (P_i/Y_i)S_{ii} = E(Y_i,P_i)+e_iE(Y_i,C) = -(1+\beta_i)+(1+2\beta_i-\bar{\beta})e_i < 0 \tag{46}$$

$$E_S(Y_i,P_j) = (P_j/Y_i)S_{ij} = E(Y_i,P_j)+e_jE(Y_i,C) = (1+\beta_i+\beta_j-\bar{\beta})e_j = \begin{cases} < 0 \\ > 0 \end{cases} \tag{47}$$

A *homogeneity* of degree zero in prices for $H_i(P,U)$ is met by CDES elasticities:

$$\sum_{j=1}^{N} P_j S_{ij} \; = \; 0 \;\; \Leftrightarrow \;\; \sum_{j=1}^{N} E_S(Y_i,P_j) \; = \; E_S(Y_i,P_i) + \sum_{j\neq i} E_S(Y_i,P_j) \; = \; 0 \tag{48}$$

$$-(1+\beta_i) + (1+2\beta_i-\bar{\beta})e_i + \sum_{j\neq i}(1+\beta_i+\beta_j-\bar{\beta})e_j \; = \; 0 \tag{49}$$

The *utility level* constraint associated with Hicksian demands $H_i(P,U)$ is satisfied by CDES Slutsky elasticities, (46-47):

$$\sum_{j=1}^{N} P_j S_{ji} \;=\; 0 \;\;\Leftrightarrow\;\; \sum_{j=1}^{N} e_j E_S(Y_j,P_i) \;=\; e_i E_S(Y_i,P_i) + \sum_{j\neq i} e_j E_S(Y_j,P_i) \;=\; 0 \tag{50}$$

$$-e_i(1+\beta_i) + e_i(1+2\beta_i-\bar{\beta})e_i + \sum_{j\neq i} e_j(1+\beta_i+\beta_j-\bar{\beta})e_i \;=\; 0 \tag{51}$$

As far the *"pure"* (Hicksian) price responses (47) are concerned, there is considerable flexibility, both respect to the *signs* (complementarity, indifference, substitutability), and the *numerical* values of these "pure" cross-price elasticities. Behind the *gross* price elasticity values (39-40), there clearly exist freedom for distinct *"income effects"* and *"substitution effects"*, (46-47), of any price changes wihin the CDES demand/expenditure/budget system, (16), (18). Gross complements (40) may be Hicksian substitutes (47).

CDES Allen partial elasticities of substitution

As gradually became clearer with the progress in economic science, there were equivalent ("dual") ways to describe consumer preferences and to obtain consumer demand systems. The neoclassical postulate underlying the derivation of demand (Marshallian) systems was constrained utility maximization - Max $U(Y)$, sub $PY = C$ - which is here also a maintained hypothesis. However, even for analytically tractable parametrizations of the direct utility, $U(Y)$, the *explicit form* of the Marshallian demand functions, $D_i(P,C)$, could seldom be given, as $D_i(P,C)$ were only implicitly determined by solving the set of Lagrangian first-order conditions. Hence it turned out to be more convenient to assume this consumer max-problem actually solved and draw (check) some of its implications, i.e parametrize instead an *indirect* utility function $U = V(P,C)$ and use Roy's identity to get the consumer (Marshallian) demand, $D_i(P,C)$ - as done above, (7), (16) with CDES. Moreover, besides explicitly obtaining Marshallian demand functions $D_i(P,C)$ and the observable budget shares, $e_i(P,C)$ in terms of the indirect utility parameters (e.g., the *"reaction parameters"*, β_i), we can still analyze the *substitution* properties [implied by of $U(Y)$] by also calculating all the relevant (Allen-partial) elasticities of substitution, (σ_{ij}) in terms of the parameters of the indirect utility function, $V(P,C)$. The key is the Slutsky equations of the derivatives (elasticities) connecting Marshallian and Hicksian demands. The Slutsky equations must hold ("integrability conditions") irrespective of the alternative procedures (dual) for obtaining the Marshallian/Hicksian demands. The Slutsky elements, S_{ij}, (23) are symmetric, while the elasticities, $E_S(Y_i,P_j)$, (47), are not symmetric, but substitution elasticities (σ_{ij}) are to be symmetric.

The *Allen partial elasticities of substitution*, (σ_{ij}), are obtained by "normalizing" the Slutsky derivatives/elasticities, (22-23), (46-47), as:

$$\sigma_{ii} = (C/Y_iY_i)\; S_{ii} = E_S(Y_i,P_i)/e_i = E(Y_i,P_i)/e_i + E(Y_i,C) \;< 0 \tag{52}$$

$$\sigma_{ij} = (C/Y_iY_j)\ S_{ij} = E_S(Y_i, P_j)/e_j = E(Y_i, P_j)/e_j + E(Y_i, C) = \begin{cases} < 0 \\ > 0 \end{cases} \quad (53)$$

$$\sum_{j=1}^{N} e_j\sigma_{ij} = \sum_{j=1}^{N} E_S(Y_i, P_j) = \sum_{j=1}^{N} E(Y_i, P_j) + E(Y_i, C) \ = \ 0 \quad (54)$$

where the restriction (54) follows from the homogeneity condition, cf. (41), (48).

As wellknown, the Allen elasticities of substitution, (σ_{ij}) were originally given by S_{ij}, as the (ij)-elements of the inverse bordered Hessian matrix of the *direct* utility $U(Y)$, and later from the Hessian matrix of the *expenditure* function, $C = e(P, U)$,

$$\frac{\partial H_i(P, U)}{\partial P_j} \ = \ S_{ij}(P, C) \ = \ \frac{\partial^2 e(P, U)}{\partial P_i \partial P_j} \quad (55)$$

which fits the "normalization" and the proper interpretation in (52-53), see Hanoch (1978, p. 290); cf. Takayama (1985, p. 144). The great analytical-economic *advantage* of *indirect* utility function $U = V(P, C)$ is that also the *substitution* properties can be expressed *explicitly* in the *parameters* of $V(P, C)$.

Using (52-53) and (46-47) or (22-23), the elasticity of substitution (utility constant) between the i'th good and *all* others, (σ_{ii}),(54), and the *specific* elasticity of substitution between the i'th good and the j'th good, (σ_{ij}), become:

$$\sigma_{ii} = -(1 + \beta_i)(1/e_i - 1) + \beta_i - \bar{\beta} \quad (56)$$

$$\sigma_{ij} = 1 + \beta_i + \beta_j - \bar{\beta} = \sigma_{ji} \quad (57)$$

The higher β_i, the higher are *all* σ_{ij} $(i \neq j)$. The CDES *substitution elasticities* (56-57) are *variable*, changing with utility (income) levels and prices via $\bar{\beta}$ and e_i. Evidently, $\sigma_{ij} > 0\,(< 0)$, if goods i and j are substitutes (complements) for each other. Thus like $E_S(Y_i, P_j)$, the sign of σ_{ij} decides whether a particular pair of goods are Hicksian substitutes or complements.

Incidentally, it should be noted that Chung (1994, p.44-45) incorrectly states that the "indirect addilog" implies that goods are independent in the net concept, i.e., $\sigma_{ij} = 0$, (57). " The Allen-Uzawa cross-partial elasticities of substitution are zero. This result is a severe restriction". He uses the indirect utility function in the definition of the Allen partial elasticity of substitution, instead of the expenditure function, $C = e(P, U)$, that generates Hicksian demands (Shephard‘s lemma), (55). But this incident may also illustrate the fact that the two *dual functions*, $C = e(P, U)$, and $U(Y)$ - corresponding to the "indirect addilog", $U = V(P, C)$, (2), (7) - have *no closed analytical forms*, and that (56-57) were never obtained by such 'Hicksian/Shepard" procedure, (55).

The property that the *differences* of the *elasticities* (partial) of *substitution* are *constant* - hence the Hanoch name to (2), (7): CDES - follows directly from (57),

$$\sigma_{ij} - \sigma_{kl} = (\beta_i + \beta_j) - (\beta_k + \beta_l)\ ; \quad \sigma_{ik} - \sigma_{jk} = \beta_i - \beta_j \quad (58)$$

Such *invariance* relations among the *Allen* elasticities, σ_{ij}, (58), do not exist at all among the corresponding *Slutsky* elasticities, $E_S(Y_i, P_j)$, (47).

The *constant differences*, (58), of CDES are *zero* in the *special* case,

$$\forall_i \ \beta_i \ = \ \beta \ = \ \sigma - 1 \ (constant) \tag{59}$$

i.e., the case of *dual* CES direct utility and CES indirect utility functions. Indeed with CES, (59), the CDES formulas, (56-57), (46-47), (39-40), (30), (36), are reduced to the simple CES versions (of N goods):

$$\sigma_{ii} = -\sigma(1 - e_i)/e_i \ ; \quad \sigma_{ij} = \sigma \tag{60}$$

$$E_S(Y_i, P_i) \ = \ -\sigma(1 - e_i) \ ; \quad E_S(Y_i, P_j) \ = \ \sigma e_j \tag{61}$$

$$E(Y_i, P_i) = -\sigma(1 - e_i) - e_i \ = \ -\sigma + (\sigma - 1)e_i \ ; \quad E(Y_i, P_j) \ = \ (\sigma - 1)e_j \tag{62}$$

$$E(Y_i, C) \ = \ E(C_i, C) \ = \ E(e_i, C) + 1 \ = \ 1 \tag{63}$$

If the pattern of income-, price-, and substitution elasticities in the CDES demand (expenditure) system may theoretically be viewed as rather restrictive, the range and scope in the CES system, (60-63), is evidently much narrower. In addtion to (63), the empirical drawback of CES is that the own-price elasticities (62) are all either larger or smaller than one, cf. $\beta_i > 0$ above, (39). In relation to household budget survey data, the CES demand system makes no sense.

For the specification of consumer demands in applied (computable) general equilibrium (CGE) models, the functional forms commonly used are CD, LES and CES, Shoven & Whalley (1992, p.95). A nesting structure for homothetic CES preferences can increase the scope for (60-62), but not at all for (63). For several purposes (consumer theory, interpretability, tractability), the more general CDES demand system has certainly merits vis-a-vis alternative specifications currently used in CGE models.

Literature comments

A few comments on the genesis of CDES demand add some further insights into its system character and its natural place among other demand systems. How did it come about and what were the motivations ?

In this journal, Leser (1941) wanted to *measure* the price and income *elasticities* of the *demand* for various commodities. But the complete set of these elasticities should satisfy the restrictions imposed by the theory of utility maximization under a budget constraint. The *budget constraint* implied that the set of income- and price elasticities of Marshallian demands must meet the restriction of Engel aggregation, (32), and Cournot aggregation, (43).

Then the symmetry imposed in terms of the *Hicks-Allen relations* (substitution elasticities), gave the right-hand side of the equations (52-54), Leser (1941, p. 45) - cleverly adapted from Hicks & Allen (1934, p. 201). But like many others, he had little interest in the cross-price elasticities of demand, as the relevant number of alternative goods or composites to consider is an empirically difficult issue. However, the cross-price elasticities could not be ignored by just putting them equal zero, since then the Cournot aggregation (43) is violated, except for the trivial case with also the own-price elasticities of minus one. Compatible with maximizing utility and a budget constraint, Leser (1941, p.43) assumed: " The cross-elasticities of demand depend only on the nature of the good whose price changes, and not on the nature of the good for which the effect is studied" - i.e., like (40), which together with (39) satisfies the *Cournot aggregation* (43).

Income elasticities that are all constant would not comply with Engel aggregation (32), except for unitary elasticity. Hence the traditional "double-logarithmic" expenditure functions (Engel curves) - cf. the power function in numerator of (16), (18), widely used and estimated for single commodities - could not be applied to all commodities in the budget and satisfy (32). However, by combining these numerators with the suitable common denominator expression, cf. (16), (18), the parameter (β_i) is no longer a constant income elasticity, but only a "*reaction parameter*" that particularly affects the demand elasticities of this good (i). Furthermore, the proper demand and budget functions, (16), (18), now generate varying income elasticities (30) that satisfy *Engel aggregation* (32). Evidently, the "reaction parameter" (β_i) naturally stands out in the $E(Y_i, C)$ formula and indeed, differences between the income elasticities of goods are solely determined the differences of their respective "reaction coefficients", (35). By working from the budget formula (18) and through partial derivations, Leser (1941, p.49) obtained the income- and own- price elasticity formulas, stated in (30),(39). As he was not, beyond Cournot aggregation, interested in cross-price effects, he did not work out the CDES substitution elasticities involved: (56-57). For the record, we mention that Leser in fact obtained the lower boundary value ($\beta_i > -1$), (39); Leser (1941, p. 49).

Houthakker (1960) studied the implications of assuming *additivity* of direct and indirect utility functions for demand elasticities. The topic of this seminal paper was continued in Samuelson (1965, 1969), Houthakker (1965), Hicks (1969) and Hanoch (1975) with important issues for the understanding of the CDES system.

The additivity property of any *indirect utility* function was shown, Houthakker (1960, p.250), generally to imply that the *cross-price* elasticies, $E(Y_i, P_j), \forall_i$, are *equal* - i.e., it is not a special property of the additive CDES form (7), which just implied the distinct CDES *parameter* expression, (40). Regarding income elasticities, indirect additivity has the consequences, Hanoch (1975, p.410):

$$E(Y_i, C) - E(Y_j, C) = \sigma_{ik} - \sigma_{jk} \tag{64}$$

The CDES parametric version of (64) was seen above in (58), (35).

The implication of additivity of any *direct utility* function was that *ratio* of *cross-price* elasticities is *equal* to the *ratio* of their *income* (C) elasticities, Houthakker (1960, p.248), cf. Deaton & Muellbauer (1980a, p. 138) :

$$E(Y_i, P_j)/E(Y_k, P_j) \; = \; E(Y_i, C)/E(Y_k, C)\,; \quad \forall_i \;\; E(Y_i, P_j) \neq \; 0\,;\; N \geq 3 \tag{65}$$

Generally, Marshallian cross-price elasticities are again here severely restricted by additivity of the *direct* utility function, $U(Y)$. If some particular *preference* *orderings* were *both* directly and indirectly *additive*, then Houthakker noted that *all* income elasticities must be *equal*, and hence *unitary*, i.e.,

$$[U(Y) = V(P,C) \ \wedge \ additive] \ \Rightarrow \ E(Y_i, C) \ = \ 1 \ , \ i = (1, ..., N) \qquad (66)$$

The restrictions, $(\forall_i \ E(Y_i, P_j) \neq \ 0 \ ; \ N \geq 3)$, in (65) for obtaining (66) were added by Samuelson (1969, p. 357) in response to Hicks (1969). Cross-price elasticities of zero would render the left-hand side of (65) meaningless and further deductions from it vacuous. This *exception* of *zero* cross price elasticities is the CD case with both (dual) additive direct and indirect utility functions. The CD case - where (66) holds - must formally be treated separately to avoid indeterminate expressions, cf. (95), Appendix. Henceforth, the CD case is by the restriction in (65) excluded from (66) and (67-72) below.

If the functional form with *power* functions was adopted for the so-called *"direct addilog"* utility function (with b_i replacing β_i), cf. (2), and Houthakker (1960, p.252),

$$U^*(Y) = \sum_{i=1}^{N} a_i^* \, (Y_i)^{b_i} \, ; \qquad \forall_i \ \ b_i \neq \ 0 \qquad (67)$$

then the *ratio* (65) becomes *constant* :

$$E(Y_i, P_j)/E(Y_k, P_j) \ = \ E(Y_i, C)/E(Y_k, C) \ = \ b_i/\, b_k \qquad (68)$$

where the last equality easily followed from a consideration of the income elasticities, Houthakker (1960, p. 253).

Allen-partial *elasticities* of *substitution* for the pair of additive power specifications were not considered in Houthakker (1960). The CES form (Arrow-Chenery-Minhas-Solow, 1961) had not yet appeared, and the multi-good CES version came in Uzawa (1962). Regarding (67), we easily get by (53) and (68) the following expressions (the second by using symmetry):

$$\sigma_{ij}/\sigma_{kj} = b_i/\, b_k \ ; \quad \sigma_{ij}/\sigma_{kl} \ = \ (\sigma_{ij}/\sigma_{kj})(\sigma_{jk}/\sigma_{lk}) \ = \ (b_i/\, b_k)(b_j/\, b_l) \qquad (69)$$

The property that the *ratios* of the *elasticities* (partial) of *substitution* are *constant* made - CRES - the proper *name* for the Houthakker *direct* utility function (67). Gorman (1965) studied the general class of CRES functions, and particular CRES subclasses were analyzed in Mukerji (1963) and Hanoch (1971, 1975).

The constant ratios, (69), are equal to *one* in the special case,

$$\forall_i \ \ b_i \ = \ b \ = \ (\sigma - 1)/\sigma \ \ (constant) \ , \qquad \sigma \neq \ 1 \qquad (70)$$

and the CRES specification, (67), is then reduced the CES direct utility function - with the set of CES elasticities given above, (60-63). In short, the *special cases* of CDES and CRES being *dual* CES preference orderings are, cf. (59), (70),

$$CDES = CES = CRES \ \Leftrightarrow \ [U^*(Y) : b_i = b] = [V^*(P,C) : \beta_i = \beta] \qquad (71)$$

or

$$[\, U(Y) = V(P,C) \;\wedge\; additive \,] \;\Leftrightarrow\; [\, \sigma_{ij} \;=\; \sigma \,] \tag{72}$$

The CES *form* of the direct/indirect utility functions is the *only* functional form with the property of *self-duality.* The only way indirect additivity (including CDES) and direct additivity (including CRES) can be dual (equivalent preferences) is by both additivities collapsing to CES form, (72). A nice and important parametric pair of CES forms occurs with the value: $\sigma = 2$, cf. (70), (67), (95), Samuelson (1965, p.795), Solow (1956, p.77), Jensen et al (2005, p.78):

$$U(Y) = \sum_{i=1}^{N} (\alpha_i Y_i)^{\frac{1}{2}} \quad \Leftrightarrow \quad V(P,C) = \sum_{i=1}^{N} \alpha_i \,(C/P_i) \tag{73}$$

Since constant returns to scale is a predominant property of long-run production functions, CES and CRESH (homogeneity restricted CRES), Hanoch (1971), are natural elements of the supply side in CGE models. For consumption (demand side) however, CES and any homothetic preferences, (66) would jettison solid empirical and observable evidence on budget shares from budget studies since Engel (1857). CDES represents *nonhomothetic* preferences - C can formally not be separated in (7), except for (71) - which is one of its fundamental merits among tractable preference orderings.

The approach to demand analysis and expenditure systems by Somermeyer et al (1956, 1972) was that of "flexible functional form" under the maintained hypothesis of constrained utility maximization. By experimenting with linear, hyperbolic, power, and exponential specifications and their respective parameter restrictions, it turned that the nonlinear power specification was desirable from several criteria, parsimony of parameters, ease of interpretation, computational ease and sensible robustness outside the range of observed data. That the power specification is flexible and hard to replace for demand functions to be derivable from a utility function was noted by Arrow (1961, p. 177). Thus the demand functions (budget shares) (16), (18) came into focus, and their relevant parameter intervals was to be scrutinzed. The non-negativity of demand (Y_i) for every non-negative price and income (C) was ensured by *positive* (α_i), (16); this restriction is involved with the monotonicity requirements (9-11). The second-order Slutsky conditions (negative semi-definiteness of the substitution matrix),(9-11) implied the restrictions on (β_i) - as obtained by Van Driel (1974) mentioned above. Essentially these restrictions imposed by the maintained hypothesis of an underlying (*unknown*) strictly quasi-concave *direct* utility function is by duality reflected in condition (14) of strictly quasi-convexity of the *explicit indirect* utility function of CDES. A hard and long story of *empirical* work and *experimentation* is now *codified* in the proper *intervals* of the *parameters* in (7).

5. Estimation of reaction parameters from budget surveys

The consumer prices here do not vary, and they are the same for all households; we add an *index* ($h = 1, \cdots H$) to denote *individuals/households.* Hence the

CDES budget share equations (18) of the households are :

$$e_{ih} = \frac{\alpha_i (C_h/P_i)^{\beta_i}}{\sum_{j=1}^{N} \alpha_j (C_h/P_j)^{\beta_j}} \tag{74}$$

Wit (1957, 1960) proposed selecting a *reference commodity* (which without loss of generality is the first one), and using the following transformation of (74):

$$\log\left(\frac{e_{ih}}{e_{1h}}\right) = \log e_{ih} - \log e_{1h} = \gamma_i + (\beta_i - \beta_1)\log(C_h) + \varepsilon_{ih} \quad i = 2, \cdots, N \tag{75}$$

where ε_{ih} is the disturbance term, and where the constant term is,

$$\gamma_i = \log \alpha_i - \log \alpha_1 - (\beta_i \log P_i - \beta_1 \log P_1) \tag{76}$$

It follows from (75) that with this procedure we can only estimate the *differences* of the "reaction parameters" of interest, $\beta_i - \beta_1$.

By defining,

$$y_i = \begin{bmatrix} (\log e_{i1} & - & \log e_{11}) \\ & \vdots & \\ (\log e_{iH} & - & \log e_{1H}) \end{bmatrix}; \; X = \begin{bmatrix} 1 & \log C_1 \\ \vdots & \vdots \\ 1 & \log C_H \end{bmatrix}; \tag{77}$$

$$\beta_i^* = \begin{bmatrix} & \gamma_i & \\ \beta_i & - & \beta_1 \end{bmatrix}; \varepsilon_i = \begin{bmatrix} \varepsilon_{i1} \\ \vdots \\ \varepsilon_{iH} \end{bmatrix}$$

we can rewrite (75) as:

$$y_i = X\beta_i^* + \varepsilon_i \quad i = 2, \cdots, N \tag{78}$$

i.e. to a *seemingly unrelated regression* (SUR) model with *identical explanatory variables* - for which it is known that ordinary least squares (applied to each equation separately) is efficient, (Heij et al., 2004, p.687).

Due to problems of data availability, the reference commodity will vary between countries (and periods); but irrespective of the particular choice of reference commodity, comparison of results is simple, as each set of parameter estimates can consistently be transformed to another reference by,

$$(\beta_i - \beta_1) \; - \; (\beta_m - \beta_1) \; = \; \beta_i - \beta_m \tag{79}$$

Hence the actual choice of reference commodity does neither affect the calculation of income (C) elasticities. Moreover, if just *one* estimated *price elasticity* was known from other sources, then we can calculate the corresponding reaction parameter estimate from (39-40) - which together with the estimated differences (78) will give us the *absolute values* of all reaction parameters (β_i). Thereby, the *complete set* of CDES *price elasticities* can be derived, as we shall demonstrate.

6. Budget Shares, Estimates of $\beta_i - \beta_2$ and $E(Y_i, C)$

Classification of consumer goods(services) and social strata

The universal human wants (needs) have, in various amounts, included at least three main categories of consumer goods: *1. Food, 2. Clothing, 3. Shelter.*

As to the provision for other wants and fancies, Adam Smith (p.34) says: "Every man is rich or poor according to the degree in which he can afford to enjoy the *necessaries, conveniences*, and *amusements* of human life."

In more detail on these wants and fancies, Smith (pp.182-183) continues: "Cloathing and lodging, household furniture, and what is called equipage, are the principal objects of the greater part of those wants and fancies. The rich man consumes no more food than his poor neighbor. In quality it may be very different, and to select and prepare it may require more labor and art; but in quantity it is very nearly the same.

But compare the spacious palace and great wardrobe of the one, with the hovel and the few rags of the other, and you will be sensible that the difference between their cloathing, lodging, and household furniture, is almost as great in quantity as it is in quality. The desire for *food* is *limited* in every man by the narrow capacity of the human stomach; but the desire of the *conveniencies* and *ornaments* of building, dress, equipage, and the household furniture, seems to have *no limit* or certain *boundary*."

Budget surveys have until 1970's often exclusively focused on the expenditure patterns ("standard of living costs") of *employee* households (wage and salary earners), since a subsidiary purpose was to obtain "weights" (e_i) for the calculation of various official price indexes that regulated nominal wage contracts. Hence, less variation in ornaments of lodging and dress, etc., ("life styles") are expected for such households than those quoted from Adam Smith. But his classification of goods into: *necessaries, conveniences*, and *amusements* resembles modern measurements by *income elasticties*, $E(Y_i, C)$: below, around, larger than unity, as the budget constraint (Engel aggregation) implies.

The *classification* of *consumer goods and services* that was used in the Danish Consumer Survey (employee households) of 1971, is shown in **Table 1**. It covers *seven* main categories: 1. Food and Beverages 2. Clothing and Footwear 3. Housing 4. Dwelling Operations 5. Medical Care and Health 6. Leisure 7. Transport. Each category consists of a varied set of sub-items; the complete expenditure pattern is described by a total number (N) of *41 items*. The coresponding 41 *budget shares*, (e_i), for two life-cycle groups (junior-/senior families), is seen in Table 1.

The sampling design and random selection of around 1000 household in the 1971 survey from 1 million *employee* households may briefly be described. The households were interviewed, made detailed expenditure accounting for 1 month, and they were successsively chosen throghout the year to offset seasonal influence on spending patterns. The stratifications of household units were based on several criteria: geography (metropolitan, provincial), age of members and number of children, social groupings (occupational status in private and public sector). We have here chosen to use the budget shares from a *demographic* type of the household stratification into 8 life-cycles groupes; but only two groupes are shown in Table 1. The *junior* families have only children below *7 years*, whereas the *senior* families include no longer any children and the house wife is above *45 years* - these senior households are not retired (pensioners), as at least 50 percent of the total income of any household in this consumer survey must be factor (wage, salary) income.

Factor Income and Transfers (welfare, children allowances, unemployment benefits) gives *Gross Income*. The latter with deductions of Direct Taxes (income, real estate, social insurance) gives *Disposable Income*, which is split between *Total Consumption* (C) and *Gross Saving* (S). The published budget shares of various consumer expenditure usually have disposable income as the denominator. Since the relationship between Disposable Income and Gross Saving varies across life-cycle groupes, the differences in consumer expenditure patterns are, theoretically and empirically, more adequately described (and recalculated by us) as budget shares with the total consumption (C) as denominator. Table 1 shows the *average* budget *shares* (e_i) of the life-cycle groupes with their respective *average* (yearly) *totals* (C) : 42557 DKK and 40831 DKK.

Regarding the expenditure pattern of the junior and senior families in table 1, we cannot be surprised to see that budget shares of (milk products) and (beer,wine) are significantly higher (lower) for the junior families; the seniors enjoys higher quality foods at home (meat,9) and outside (restaurants,35). The housing cost (gross rents,17) takes a higher toll on the young families. Health expenditures (medical products,29) are slightly higher for the seniors. Apart from the items mentioned, the average *budget shares* of the two life-cycles groupes are overall remarkly *similar*. Cars (transport equipment, insurance,auto repairs, gasoline,41,39-37) became significant budget items for both the juniors and seniors in the years around 1971. The expenditures on transport equipment (41) [like housing,(19)] refer "user cost" of these physical assets [i.e.,not to their asset (purchase) price]. The "user costs" are not calculated as "imputed rents (services)", but made up of cash payments on down-payments, consumer credit, mortgage interest that are recorded with renting (owning) these durable assets.

Within the two life-cycle groupes, several sampling units will have *zero expenditures* on a number of the items listed in Table 1; there are non-smokers, some are vegetarians, other eschew alcholic beverages, and many have no automobiles. In terms of consumer modelling, the nonzero budget shares in Table 1 each refer to budget shares of a *representative consumer*, evaluated at, respectively, ($C = 42557, C = 40831$). The main categories (subtotals, Table 1) in the Danish expenditure pattern for employee household in 1971 are similar to those for employees in other Scandinavian countries at that time (Sweden,1969, Norway, 1973), cf. SU (1977, p.226).

Sampling surveys of budget data from employee household have been collected and compared by government agencies for a long time. A single table of budget shares from the first and most famous, Engel (1857), of all family expenditure studies is quoted here in Table 1 A, because it has acted as *benchmark* for later inquiries, and because it is seldom seen anywhere in the literature, except

Table 1: Classification of Consumer Goods and Services and Budget Shares ($100\, e_i\,,\; i = 1, \ldots 41$) of Social Strata.

		Junior Families	Senior Families
1. Food and Beverages:			
1	Bread (cereals)	2.44	2.39
2	Butter	0.64	0.76
3	Margarine (fats, oils)	0.64	0.65
4	Sugar (confectionary)	0.85	0.98
5	Milk (cream, yoghurt)	2.23	1.63
6	Cheese (curd)	0.64	0.76
7	Other foods	1.81	2.17
8	Vegetables (fruits)	3.08	3.04
9	Meat	5.63	7.27
10	Fish	0.64	0.98
11	Coffee (tea, cocoa)	1.38	1.95
12	Soft drinks (mineral water)	0.53	0.76
13	Beer	1.27	1.84
14	Wine and spirits	0.96	1.74
15	Tobacco products	3.50	3.80
	Sum 1-15:	**26.24**	**30.72**
2. Clothing and Footwear:			
16	Clothing	5.73	5.86
17	Footwear (shoes, boots)	1.38	0.98
	Sum 16-17:	**7.11**	**6.84**
3. Housing:			
18	Fuel (gas, liquids) and light	4.25	4.45
19	Gross rents (water rates, mortgage)	17.62	12.04
	Sum 18-19:	**21.87**	**16.49**

		Junior Families	Senior Families
4. Dwelling Operations:			
20	Glassware (tableware, utensils)	0.96	0.76
21	Household textiles (furnishings)	0.85	0.98
22	Household machines (appliances)	1.70	1.08
23	Furniture (fixtures, carpets)	3.61	3.47
24	Non-durable household goods	2.12	2.17
25	Household services (domestic services)	0.85	0.54
26	Communication (post, telephone)	0.96	1.41
27	Radio and television sets	1.17	1.74
28	Miscellanous (services n.e.c.)	4.46	2.82
	Sum 20-28:	**16.68**	**14.97**
5. Medical Care and Health:			
29	Medical and pharmaceutical products	0.53	1.08
30	Medical services (physicians, nurses)	0.85	0.87
31	Personal care (barber, beauty shops)	1.27	1.63
	Sum 29-31:	**2.65**	**3.58**
6. Leisure:			
32	Personal goods (jewellery, watches, rings)	0.96	0.87
33	Leisure equipment (camera, musical, boats)	2.65	3.15
34	Entertainment (comfort, cultural services)	2.23	3.15
35	Restaurants (cafes, hotels, lodging services)	1.49	3.80
36	Books and papers (magazines)	2.12	1.95
	Sum 32-36:	**9.45**	**12.92**
7. Transport:			
37	Gasoline (oil)	3.61	3.15
38	Auto repairs (transport equipment)	2.55	2.17
39	Other transport expenses (insurance, taxes)	2.44	2.17
40	Transport services (bus, train, taxi, rent)	1.70	1.95
41	Transport equipment (vehicle purchases)	5.73	5.10
	Sum 37-41:	**16.03**	**14.54**
Sum 1-41:		**100.00**	**100.00**
Total Consumption Expenditures (average) DDK		**42557**	**40831**

Source: Statistiske Undersgelser No. 34, Copenhagen (1977, p. 238-45).

Marshall. We have changed Marshall's item descriptions a little bit - in accordance with Engel's original expenditure classification; cf. Stigler (1954, p. 98) on the origin of the Engel data and the early collections of budgetary data in Europe and USA. By induction and the evidence of Table 1A, Engel (1857, p. 28) felt justified to state as a general *empirical law*: " The poorer a family is, the larger share of the total expenditures must be allotted to the provsion of nourishment (nahrung, food)". Engel's law has never been refuted anywhere, and is confirmed in all subsequent surveys, Houthakker (1957), irrespective of climatic or cultural conditions. As stressed by Engel (1857, p. 33), the general law refers to the budget *share* and not to the *level* (absolute size) of the *food expenditures* - which in fact were 66% higher for family type ($150-200$£) than the level in family type ($45-60$£).

The variations of the *budget shares* with income for other items in Table 1A also reflect some *basic patterns* and historical *tendencies* (with exceptions and gradual shifts) observed ever since in many empirical consumer demand studies. The structural consistency exhibited by consumption patterns is of great importance from the viewpoints of demand theory, empirical methods, and applications. In accordance with these assumptions, a thorough investigation of family budget data and market statictics was carried out by Wold & Juren (1952) so as to obtain a unified *historical description* of the *demand structure* in Sweden; one picture of their budget data is shown in Table 1B. Finally, the average budget shares of income groupes in Leser's paper are given in Table 1C.

A remarkable feature of the Tables: 1 - 1C, is the stability of the budget share for the item, Fuel & Light, in 150 years. The budget shares of Clothing in Table 1A-1B imply for a long time an income elasticity slightly below one. The high income elasticity of housing is a characteristic feature of demand in Scandinavia as is already evident from budget shares in Table 1B.

For the income (disposable) variable, the frequency distribution of the sampling units (households) within the two life-cycle groupes of Table 1 are shown in Table 1D. For each of these income intervals for Junior and Senior families, we could as in Table 1 calculate all the average budget shares; their pattern would look similar to Table 1A-1C - with also Engel's law always confirmed.

However, instead of tabulating 41 budget shares by discrete income (total expenditure) intervals, the entire sample of the respective Junior and Senior families are now used to *estimate* the CDES *parameters*, cf.(75), with here *butter* ($j = 2$) as *reference* commodtiy. The estimated parameters, ranked according to size for both life-cycle groupes, are shown in Table 2. We have given the parameter estimates without indicating the standard errors (or t-values) in Table 2; they are availlable in Jensen (1980, p.283-85), but here omitted as such standard deviations carry little weight as economic-statistical significance indicators, on groundes explained by Wold (1952, p. 260). Statistically, estimates that are closer to zero (middle of Table 2) are less significant than large negative (positive) parameter estimates at the end of the rankings. Economically, the range of the reaction parameters (β_i), (13), place many item values meaningful close to zero.

Table 1A. Budget Shares of Social Strata : Saxony 1857.

Items of Expenditure		Proportions of the Expenditure of the Family of-		
		Workman with an yearly income of 45 - 60 £	Workman with an yearly income of 90 - 120 £	Workman with an yearly income of 150 - 200 £
1	Food (beverage, tobacco, taverns)	62.0	55.0	50.0
2	Clothing (footwear, jewellery)	16.0	18.0	18.0
3	Lodging (rent, utensils, furniture)	12.0	12.0	12.0
4	Light and Fuel (wood,coal, gas,oil)	5.0	5.0	5.0
5	Education (culture, church)	2.0	3.5	5.5
6	Public Protection (taxes)	1.0	2.0	3.0
7	Care of Health (medical, pharma)	1.0	2.0	3.0
8	Personal Services	1.0	2.5	3.5
Totals		100.0	100.0	100.0

Source: Marshall (1920, p. 115); Engel (1857, 1895, p. 30).

Table 1B. Budget Shares of Social Strata : Sweden 1913-1933.

Item group		Industrial worker families			Middle class families	
		1913	1923	1933	1923	1933
1	Nourishment	50.4	47.8	40.0	31.5	28.5
2	Clothing	14.2	16.0	14.6	14.5	14.4
3	Fuel, Light, Laundry	6.4	6.5	6.1	6.2	6.0
4	Housing	13.3	11.6	16.9	13.5	18.2
5	Furniture (furnishings)	4.7	4.8	5.2	7.1	6.4
6	Personal Services (domestic)	1.0	0.5	0.4	4.2	3.3
7	Hygiene (medical care)	1.8	2.3	3.2	2.8	3.3
8	Education (culture, travel)	5.0	6.2	7.0	9.2	9.7
9	Other Expenditures	3.1	4.2	6.6	11.0	10.2
All expenditures		100.0	100.0	100.0	100.0	100.0

Source: Wold (1952, p. 20); Recalculation with all expenditures = C.

Table 1C. Budget Shares : Families in the U.S.A. 1918-19

Item group		$100\, e_i$
1	Food	38.9
2	Clothing	16.2
3	Fuel, Light	5.4
4	Rent	13.6
5	Furnishings	5.0
6	Miscellaneous	20.9
Total		100.0

Source: Leser (1941, p. 53).

In judging the validity and reliability of the parameter estimates, a main criterion will the overall consistency obtained for the different budget items within and between social strata, and proper comparisons and checks with specific evidence from other sources and periods. Looking at the *point estimates* in Table 2 for the two family types, a main feature of the results is that a *stable hierarchy* exists between the item groupes in their *preference* and *demand sensitivity*, as was evident in budget shares of Table 1 discussed above.

The CDES *expenditure* ("income") *elasticities* of the 41 consumer goods and services, satisfying Engel aggregation, are shown in Table 3. The methods and formulas employed in calculating the elasticities in Table 3 from Table 1 and Table 2 have been explained above in section 3. Hence with butter ($j = 2$) as reference commodity of the estimates in Table 2 and (31), we get for butter

$$E(Y_2, C) = 1 - \sum_{i=1}^{41} e_i(\beta_i - \beta_2) = 1 - \bar{\beta}_2 = 1 - 0.48614 = 0.51386 \tag{80}$$

$$E(Y_2, C) = 1 - \bar{\beta}_2 = 1 - 0.19075 = 0.80925 \tag{81}$$

of Junior/ Senior families - evaluated at, respectively, ($C = 42557, C = 40831$). All the other elasticities in Table 3 are then calculated from (35), (80),(81), and Table 2 as, respectively,

$$E(Y_i, C) = E(Y_2, C) + \beta_i - \beta_2 = 0.51386 + \beta_i - \beta_2 \tag{82}$$

$$E(Y_i, C) = 0.80925 + \beta_i - \beta_2 \tag{83}$$

The main feature of the results obtained in Table 3 (with the same ranking as in Table 2) is the overall *similarity* that subsists between the commodity classes in their expenditure (income) sensitivity for the two life-cycle groupes (apart from some obvious differences already mentioned, cf. Table 1). Since the sample average of C was higher for the Junior groupe, the item elasticities $E(Y_i, C)$ should ceteris caribus be smaller for the Junior families, cf. (24), (31), and their size in Table 3 comply with such tendency.

Several food items are among the necessities with the lowets elasticities, although fancy food also exists among the luxuries for both groupes (rank 31, 35). Clothing, footwear, and goods and services associated with dwelling operations have elasticities around one. Housing, furniture, and cars are the high elasticity goods for Juniors, and latter good also tops the list for the Senior families. The CDES elasticities in Table 3 display a pattern and numerical values that conform with the general picture of abundant empirical demand studies.

Table 2 : Ranking of the Reaction Parameter Estimates: $(\beta_i - \beta_2)$

Junior Families			Senior Families		
Rank	**Commodity Class**	$\beta_i - \beta_2$	**Rank**	**Commodity Class**	$\beta_i - \beta_2$
1	Bread	-0.29404	1	Margarine	-0.56234
2	Margarine	-0.25465	2	Bread	-0.39438
3	Sugar	-0.24830	3	Coffee	-0.33723
4	Coffee	-0.21253	4	Miscellaneous	-0.25552
5	Tobacco	-0.20813	5	Fuel and light	-0.23122
6	Milk	-0.12385	6	Tobacco	-0.24924
7	Fuel and light	-0.06171	7	Soft drinks	-0.23892
8	Butter	-	8	Personal goods	-0.18789
9	Radio, tv sets	0.01635	9	Meat	-0.18665
10	Medical products	0.04118	10	Medical products	-0.10276
11	Other nondurable goods	0.09349	11	Other nondurable goods	-0.09355
12	Cheese	0.09450	12	Household machines	-0.07920
13	Household textiles	0.10560	13	Milk	-0.05749
14	Vegetables	0.11840	14	Medical services	-0.04911
15	Meat	0.12885	15	Beer	-0.04888
16	Household machines	0.15462	16	Books and papers	-0.02345
17	Glassware	0.20301	17	Fish	-0.00829
18	Fish	0.23152	18	Butter	-
19	Soft drinks	0.27881	19	Cheese	0.03200
20	Leisure equipment	0.34423	20	Entertainment	0.06120
21	Beer	0.34631	21	Household textiles	0.07244
22	Entertainment	0.37195	22	Personal care	0.08888
23	Auto repairs	0.37212	23	Furniture	0.09684
24	Clothing	0.38976	24	Sugar	0.09990
25	Gasoline	0.42036	25	Auto repairs	0.12164
26	Footwear	0.43999	26	Vegetables	0.12429
27	Personal goods	0.54508	27	Glassware	0.13912
28	Medical services	0.58257	28	Gross rents	0.17936
29	Wine and spirits	0.58607	29	Footwear	0.24733
30	Personal care	0.60113	30	Other transport	0.29736
31	Other foods	0.64302	31	Transport services	0.34186
32	Other transport	0.66149	32	Clothing	0.37099
33	Books and papers	0.68632	33	Radio, tv sets	0.39254
34	Gross rents	0.71112	34	Gasoline	0.42218
35	Restaurants	0.74708	35	Other foods	0.45335
36	Household services	0.78325	36	Wine and spirits	0.64572
37	Miscellaneous	0.86186	37	Leisure equipment	0.69500
38	Transport services	0.91800	38	Household services	0.70973
39	Furniture	1.04451	39	Communication	0.89911
40	Communication	1.33410	40	Restaurants	1.02420
41	Transport equipment	1.78225	41	Transport equipment	1.32967
	$\bar{\beta}_2 = \sum_{i=1}^{41} e_i(\beta_i - \beta_2)$	0.48614		$\bar{\beta}_2 = \sum_{i=1}^{41} e_i(\beta_i - \beta_2)$	0.19075

Table 3: Estimates of Expenditure ("Income") Elasticities Goods and Services : $E(Y_i, C) = E(P_iY_i, C)$

	Junior Families			**Senior Families**	
Rank	**Commodity Class**	$E(Y_i, C)$	**Rank**	**Commodity Class**	$E(Y_i, C)$
1	Bread	0.21982	1	Margarine	0.24691
2	Margarine	0.25921	2	Bread	0.41487
3	Sugar	0.26556	3	Coffee	0.47202
4	Coffee	0.30133	4	Miscellaneous	0.55373
5	Tobacco	0.30573	5	Fuel and light	0.57803
6	Milk	0.39001	6	Tobacco	0.56001
7	Fuel and light	0.45215	7	Soft drinks	0.57033
8	Butter	0.51386	8	Personal goods	0.62136
9	Radio, tv sets	0.53021	9	Meat	0.62260
10	Medical products	0.55504	10	Medical products	0.70649
11	Non-durable goods	0.60735	11	Non-durable goods	0.71570
12	Cheese	0.60836	12	Household machines	0.73005
13	Household textiles	0.61946	13	Milk	0.75176
14	Vegetables	0.63226	14	Medical services	0.76014
15	Meat	0.64271	15	Beer	0.76037
16	Household machines	0.66848	16	Books and papers	0.78580
17	Glassware	0.71687	17	Fish	0.80096
18	Fish	0.74538	18	Butter	0.80925
19	Soft drinks	0.79267	19	Cheese	0.84125
20	Leisure equipment	0.85809	20	Entertainment	0.87045
21	Beer	0.86017	21	Household textiles	0.88169
22	Entertainment	0.88581	22	Personal care	0.89813
23	Auto repairs	0.88598	23	Furniture	0.90609
24	Clothing	0.90362	24	Sugar	0.90915
25	Gasoline	0.93422	25	Auto repairs	0.93089
26	Footwear	0.95385	26	Vegetables	0.93354
27	Personal goods	1.05894	27	Glassware	0.94837
28	Medical services	1.09643	28	Gross rents	0.98861
29	Wine and spirits	1.09993	29	Footwear	1.05658
30	Personal care	1.11499	30	Other transport	1.10661
31	Other foods	1.15688	31	Transport services	1.15111
32	Other transport	1.17535	32	Clothing	1.18024
33	Books and papers	1.20018	33	Radio,, tv sets	1.20179
34	Gross rents	1.22498	34	Gasoline	1.23143
35	Restaurants	1.26094	35	Other foods	1.26260
36	Household services	1.29711	36	Wine and spirits	1.45497
37	Miscellaneous	1.37572	37	Leisure equipment	1.50425
38	Transport services	1.43186	38	Household services	1.51898
39	Furniture	1.55837	39	Communication	1.70836
40	Communication	1.84796	40	Restaurants	1.83345
41	Transport equipment	2.29611	41	Transport equipment	2.13892
	$\sum_{i=1}^{41} e_i E(Y_i, C)$	1.00000		$\sum_{i=1}^{41} e_i E(Y_i, C)$	1.00000

7. Reaction Parameters, Price and Substitution Elasticities

The information about the estimated differences $(\beta_i - \beta_2)$ in Table 2 were sufficient for calculating income elasticities of CDES demand, (16), (82); adding an arbitrary constant to all the β_i - parameters would not change the elasticity with the respect to C in (16) - whereas any particular price elasticity would clearly be affected by such additive constant in every β_i. Thus CDES price elasticities cannot be calculated from Table 2; moreover, the 'reaction parameters'(β_i) cannot be fully recovered from just family budget surveys (expenditure data). But they are as 'invariances' (basic parameters) - in contrast to continuosly changing income and price elasticities - the ultimate empirical objective to obtain for the CDES indirect utility function and demand system, (7), (16).

As mentioned at the end of section 5, *extranous* information (literature surveys, benchmark observations, calibration procedures) on any single price elasticty will in combination with Tabele 2 allow full identification and estimation of all the parameters, (β_i). Our parameter calibrations will be based on the price elasticity of butter that is a well-defined homogenoues good and a classic article of many demand studies - convenient also as being our reference commodity.

Hence by (16), the reaction parameter for butter will be *calibrated* as

$$\beta_2 = [E(Y_2, P_2) - 1]/(1 - e_2) \tag{84}$$

Market statistics in the form of time series are often seen as the principal material for the estimation of direct price and cross price elasticties. However, our elasticity and parameter in (84) do not refer to total butter market demand, but instead to butter demand elasticities segmented to our life-cycle groups. Accordingly information from specialized demand studies are called for ; such food demand analyses are available, since our life cycle categories have been standard for a long time in many countries.

Calculations of β_2 by (84) - for different butter price elasticities and with e_2 from Table 1 - are collected in Table 4a. The starred price elastities appear as the most plausible ones for several reasons. They are partly in line with butter elasticities seen for our family types in Wold & Juren (1952, p. 266-69, 285-88), conform properly with the income elasticities, (80),(81), and fit in appropriately with the wider implications of the corresponding β_2 (and $\overline{\beta}$) values upon all the other price elasticities, as explained below.

Table 4a. Reaction Parameter β_2, $\overline{\beta}$, and $E(Y_2, P_2)$.

Junior Families			Senior Families		
$E(Y_2, P_2)$	β_2	$\overline{\beta}$	$E(Y_2, P_2)$	β_2	$\overline{\beta}$
-0.9	-0.10064	0.38550	-0.9*	-0.10066	0.09009
-0.8	-0.20128	0.28486	-0.8	-0.20131	-0.01056
-0.7*	-0.30192	0.18422			
-0.6	-0.40256	0.08358			

Table 4: Ranking and Size of Reaction Parameter Estimates: (β_i)

Junior Families			Senior Families		
Rank	**Commodity Class**	β_i	**Rank**	**Commodity Class**	β_i
1	Bread	-0.59596	1	Margarine	-0.66300
2	Margarine	-0.55657	2	Bread	-0.49504
3	Sugar	-0.55022	3	Coffee	-0.43789
4	Coffee	-0.51445	4	Miscellaneous	-0.35618
5	Tobacco	-0.51005	5	Fuel and light	-0.33188
6	Milk	-0.42577	6	Tobacco	-0.34990
7	Fuel and light	-0.36363	7	Soft drinks	-0.33958
8	Butter	-0.30192	8	Personal goods	-0.28855
9	Radio, tv sets	-0.28557	9	Meat	-0.28731
10	Medical products	-0.26074	10	Medical products	-0.20342
11	Non-durable goods	-0.20843	11	Non-durable goods	-0.19421
12	Cheese	-0.20742	12	Household machines	-0.17986
13	Household textiles	-0.19632	13	Milk	-0.15815
14	Vegetables	-0.18352	14	Medical services	-0.14977
15	Meat	-0.17307	15	Beer	-0.14954
16	Household machines	-0.14730	16	Books and papers	-0.12411
17	Glassware	-0.09891	17	Fish	-0.10895
18	Fish	-0.07040	18	Butter	-0.10066
19	Soft drinks	-0.02311	19	Cheese	-0.06866
20	Leisure equipment	0.04231	20	Entertainment	-0.03946
21	Beer	0.04439	21	Household textiles	-0.02822
22	Entertainment	0.07003	22	Personal care	-0.01178
23	Auto repairs	0.07020	23	Furniture	-0.00382
24	Clothing	0.08784	24	Sugar	-0.00076
25	Gasoline	0.11844	25	Auto repairs	0.02098
26	Footwear	0.13807	26	Vegetables	0.02363
27	Personal goods	0.24316	27	Glassware	0.03846
28	Medical services	0.28065	28	Gross rents	0.07870
29	Wine and spirits	0.28415	29	Footwear	0.14667
30	Personal care	0.29921	30	Other transport	0.19670
31	Other foods	0.34110	31	Transport services	0.24120
32	Other transport	0.35957	32	Clothing	0.27033
33	Books and papers	0.38440	33	Radio,, tv sets	0.29188
34	Gross rents	0.40920	34	Gasoline	0.32152
35	Restaurants	0.44516	35	Other foods	0.35269
36	Household services	0.48133	36	Wine and spirits	0.54506
37	Miscellaneous	0.55994	37	Leisure equipment	0.59434
38	Transport services	0.61608	38	Household services	0.60907
39	Furniture	0.74259	39	Communication	0.79845
40	Communication	1.03218	40	Restaurants	0.92354
41	Transport equipment	1.48033	41	Transport equipment	1.22901
	$\bar{\beta} = \sum_{i=1}^{41} e_i\beta_i$	0.18422		$\bar{\beta} = \sum_{i=1}^{41} e_i\beta_i$	0.09009

This selection of (β_2) from Table 4a together with Table 2 finally give the *sizes* for the complete set of reaction parameters, listed in Table 4. Meaningful interpretation and comparison between the numbers for junior/senior families were not possible (despite correct ranking) in Table 2, as β_2 was unknown and might be widely different for such life-cycle groupes. Overall the numbers in Table 4 now look more similar (min, range) for the two family types, although natural differences in their preferences still exist. As to Junior families, the six items at the top of the table are more 'urgent necessities' than those of the Seniors. On the other hand, the average urgency (rigidity) level ($\overline{\beta}$) is somewhat lower for the Juniors, respectively. The location and different sizes of item specific reaction parameters (β_i) will affect the pattern of price elasticities of demand for the two family types.

The Marshallian & Slutsky price elasticities (own, cross), and the Allen substitution elasticities - calculated from Table 4 and Table 1 according to their parametric formulas given in section 3 - are together with income elasticities exhibited for the respective life-cycle groupes in Tables 5A-5B, Tables 6A-6B. For ease of discussion and numerical accuracy evaluations, the item budget shares ($100e_i$) are included in these tables - as are the exact consistency checks by the Engel and Cournot aggregations and homogeneity restrictions.

As seen from the listed *own-(direct) price elasticities* of commodities in Table 5A and Table 6A, the Marshall and Slutsky own-price elaticities have nearly the same size (differing in most cases only on second decimals). Evidently, the *"income effect"* of own-price changes upon Marshallian price elastcities are very *small*, when a large number of commodities (and hence individually small budget shares) are involved, cf. (46). Since no inferior goods were actually observed (estimated) for neither Junior nor Senior families in Table 3, all the Marshallian price elasticties in Tables 5A and Table 6A are abslolutely larger than the corresponding Slutsky ("compensated") elasticity.

We see for CDES income and price elasticities, Tables 5A, 6A, cf. (30), (39),

$$E(Y_i, C) + E(Y_i, P_i) \ = \ -\bar{\beta} + \beta_i e_i \ ; \quad \bar{\beta} = 0.18422, \ \ \bar{\beta} = 0.09009 \qquad (85)$$

and for the differences of price elasticities, cf. (39),

$$E(Y_i, P_i) - E(Y_j, P_j) \ = \ -(\beta_i - \beta_j) + \beta_i e_i - \beta_j e_j \qquad (86)$$

Compared to (35), the sum/differences in (85), (86) are not exactly constant, but nearly so, when large number of items are involved (small e_i). On this CDES background and and its elasticity numbers, we next search for some clues about the magnitudes and general numerical pattern among various price elasticities.

Whereas Engel aggregation implies that some income elasticties are to be larger and some other must be smaller than 1, no such rule are implied by the budget constraint for the own-price elastcities. They can all exceed one (elastic) or all be inelastic without violating any conditions imposed by consumer demand theory or budget constraints. The Cournot aggregation implied by the budget constraint, however, does give some indication about the mutual relationships

Table 5 A. Junior Families : Income- and Own-Price Elasticities.

R	Commodity Class	$100e_i$	$E(Y_i, C)$	$E(Y_i, P_i)$	$E_S(Y_i, P_i)$	σ_{ii}
1	Bread	2.44161	0.21982	-0.41859	-0.41322	-16.92409
2	Margarine	0.63694	0.25921	-0.44697	-0.44532	-69.91539
3	Sugar	0.84926	0.26556	-0.45445	-0.45219	-53.24590
4	Coffee	1.38004	0.30133	-0.49265	-0.48849	-35.39660
5	Tobacco	3.50318	0.30573	-0.50782	-0.49710	-14.19008
6	Milk	2.22930	0.39001	-0.58372	-0.57502	-25.79394
7	Fuel and light	4.24628	0.45215	-0.65181	-0.63261	-14.89792
8	Butter	0.63694	0.51386	-0.70000	-0.69673	-109.38614
9	Radio, tv sets	1.16773	0.53021	-0.71776	-0.71157	-60.93629
10	Medical products	0.53079	0.55504	-0.74064	-0.73769	-138.98171
11	Non-durable goods	2.12314	0.60735	-0.79599	-0.78310	-36.88389
12	Cheese	0.63694	0.60836	-0.79390	-0.79002	-124.03364
13	Household textiles	0.84926	0.61946	-0.80534	-0.80008	-94.20982
14	Vegetables	3.07856	0.63226	-0.82213	-0.80266	-26.07269
15	Meat	5.62633	0.64271	-0.83666	-0.80050	-14.22782
16	Household machines	1.69851	0.66848	-0.85520	-0.84384	-49.68135
17	Glassware	0.95541	0.71687	-0.90203	-0.89518	-93.69581
18	Fish	0.63694	0.74538	-0.93005	-0.92530	-145.27174
19	Soft drinks	0.53079	0.79267	-0.97701	-0.97280	-183.27594
20	Leisure equipment	2.65393	0.85809	-1.04118	-1.01841	-38.37373
21	Beer	1.27389	0.86017	-1.04382	-1.03286	-81.07982
22	Entertainment	2.22930	0.88581	-1.06847	-1.04872	-47.04251
23	Auto repairs	2.54777	0.88598	-1.06841	-1.04584	-41.04905
24	Clothing	5.73248	0.90362	-1.08280	-1.03100	-17.98525
25	Gasoline	3.60934	0.93422	-1.11416	-1.08044	-29.93462
26	Footwear	1.38004	0.95385	-1.13616	-1.12300	-81.37416
27	Personal goods	0.95541	1.05894	-1.24083	-1.23072	-128.81499
28	Medical services	0.84926	1.09643	-1.27826	-1.26895	-149.41910
29	Wine and spirits	0.95541	1.09993	-1.28143	-1.27092	-133.02330
30	Personal care	1.27389	1.11499	-1.29540	-1.28119	-100.57355
31	Other foods	1.80467	1.15688	-1.33494	-1.31406	-72.81457
32	Other transport	2.44161	1.17535	-1.35079	-1.32209	-54.14822
33	Books and papers	2.12314	1.20018	-1.37624	-1.35075	-63.62052
34	Gross rents	17.62208	1.22498	-1.33709	-1.12122	-6.36259
35	Restaurants	1.48620	1.26094	-1.43854	-1.41980	-95.53232
36	Household services	0.84926	1.29711	-1.47724	-1.46622	-172.64781
37	Miscellaneous	4.45860	1.37572	-1.53497	-1.47363	-33.05150
38	Transport services	1.69851	1.43186	-1.60561	-1.58129	-93.09859
39	Furniture	3.60934	1.55837	-1.71578	-1.65954	-45.97895
40	Communication	0.95541	1.84796	-2.02232	-2.00466	-209.82105
41	Transport equipment	5.73248	2.29611	-2.39547	-2.26384	-39.49149
		100.00000 $100 \sum e_i$	1.00000 $\sum e_i E(Y_i, C)$			

Table 5 B. Junior Families : Own Price and Cross Price Elasticities.

R	Commodity Class	$100e_i$	$E(Y_1, P_i)$	$E_S(Y_1, P_i)$	$E_S(Y_i, P_1)$	$\sigma_{1i} = \sigma_{1i}$
1	Bread	2.44161	-0.41859	-0.41322	-0.41322	-16.92409
2	Margarine	0.63694	-0.00355	-0.00214	-0.00822	-0.33675
3	Sugar	0.84926	-0.00467	-0.00281	-0.00807	-0.33040
4	Coffee	1.38004	-0.00710	-0.00407	-0.00719	-0.29463
5	Tobacco	3.50318	-0.01787	-0.01017	-0.00709	-0.29023
6	Milk	2.22930	-0.00949	-0.00459	-0.00503	-0.20595
7	Fuel and light	4.24628	-0.01544	-0.00611	-0.00351	-0.14381
8	Butter	0.63694	-0.00192	-0.00052	-0.00200	-0.08210
9	Radio, tv sets	1.16773	-0.00333	-0.00077	-0.00161	-0.06575
10	Medical products	0.53079	-0.00138	-0.00022	-0.00100	-0.04092
11	Non-durable goods	2.12314	-0.00443	0.00024	0.00028	0.01139
12	Cheese	0.63694	-0.00132	0.00008	0.00030	0.01240
13	Household textiles	0.84926	-0.00167	0.00020	0.00057	0.02350
14	Vegetables	3.07856	-0.00565	0.00112	0.00089	0.03630
15	Meat	5.62633	-0.00974	0.00263	0.00114	0.04675
16	Household machines	1.69851	-0.00250	0.00123	0.00177	0.07252
17	Glassware	0.95541	-0.00095	0.00116	0.00295	0.12091
18	Fish	0.63694	-0.00045	0.00095	0.00365	0.14942
19	Soft drinks	0.53079	-0.00012	0.00104	0.00480	0.19671
20	Leisure equipment	2.65393	0.00112	0.00696	0.00640	0.26213
21	Beer	1.27389	0.00057	0.00337	0.00645	0.26421
22	Entertainment	2.22930	0.00156	0.00646	0.00708	0.28985
23	Auto repairs	2.54777	0.00179	0.00739	0.00708	0.29002
24	Clothing	5.73248	0.00504	0.01764	0.00751	0.30766
25	Gasoline	3.60934	0.00427	0.01221	0.00826	0.33826
26	Footwear	1.38004	0.00191	0.00494	0.00874	0.35789
27	Personal goods	0.95541	0.00232	0.00442	0.01130	0.46298
28	Medical services	0.84926	0.00238	0.00425	0.01222	0.50047
29	Wine and spirits	0.95541	0.00271	0.00481	0.01230	0.50397
30	Personal care	1.27389	0.00381	0.00661	0.01267	0.51903
31	Other foods	1.80467	0.00616	0.01012	0.01370	0.56092
32	Other transport	2.44161	0.00878	0.01415	0.01415	0.57939
33	Books and papers	2.12314	0.00816	0.01283	0.01475	0.60422
34	Gross rents	17.62208	0.07211	0.11085	0.01536	0.62902
35	Restaurants	1.48620	0.00662	0.00988	0.01624	0.66498
36	Household services	0.84926	0.00409	0.00595	0.01712	0.70115
37	Miscellaneous	4.45860	0.02497	0.03477	0.01904	0.77976
38	Transport services	1.69851	0.01046	0.01420	0.02041	0.83590
39	Furniture	3.60934	0.02680	0.03474	0.02350	0.96241
40	Communication	0.95541	0.00986	0.01196	0.03057	1.25200
41	Transport equipment	5.73248	0.08486	0.09746	0.04151	1.70015
		100.00000	-0.21982	0.00000	0.00000	0.00000
		$100 \sum e_i$	$\sum E(Y_1, P_i)$	$\sum E_S(Y_1, P_i)$	$\sum e_i E_S(Y_i, P_1)$	$\sum e_i \sigma_{1i}$

between Marshallian price elastcities, as mentioned by Marschak (1943, p. 27), by rewriting (43) as,

$$e_i \,[\, 1 + E(Y_i, P_i) \,] \;=\; -\,[\, \sum_{j \neq i}^{N} e_j \, E(Y_j, P_i) \,] \tag{87}$$

Thus, unless $E(Y_i, P_i) = -1$, the *cross* elasticities of the other commodities (Y_j) with respect to own price (P_i) *cannot* all *vanish*; moreover, if $E(Y_i, P_i)$ is elastic (inelastic), then at least one of the Marshallian cross price elasticities must be posetive (negative), i.e. be a gross (Marshallian) substitute (complement).

By the *homogeneity* of degree zero in prices and income, cf. (41), we have

$$E(Y_i, P_i) \;=\; -\,[\, \sum_{j \neq i}^{N} E(Y_i, P_j) \,] - \; E(Y_i, C) \tag{88}$$

$$= \; E_S(Y_i, P_i) \; - \; e_i \, E(Y_i, C) \tag{89}$$

Assuming no inferior goods, it is immediately seen from (88) that if the commodity (Y_i) has a larger number of Marshallian complements (substitutes), then the absolute smaller (larger) becomes the Marshallian own-price elasticity of (Y_i). The homogeneity equation (88) also shows very clearly the fallacy in taking the income elasticty, $E(Y_i, C)$, (with opposite sign) as an approximation to the Marshallian own price elasticity, $E(Y_i, P_i)$. For, even if *each* of the cross price elasticties, $E(Y_i, P_j)$, is negligible, their *sum* is not necessarily negligible, since there is many of them. That the sum in (85) is close to a nonzero constant is also a particular property of CDES demand that obviously calls for some empirical relaxation in a further generalization of the indirect utility function, (7).

Regarding the *homogeneity* (88), it seems worth stressing that Slutsky (1952, equation (56); 1915) demonstrated equation (88) - on derivative form instead of elasticities - as a *consequence* of the the *symmetry* relations (23); Schultz (1935, p.458) saw, (88), "economically as probably the most signicant consequence" of Slutsky symmetry. Thus, as is now wellknown, symmetry together with Engel- and Cournot aggregation imply the Marshallian homogeneity property (88).

Next, the ability of the so-called Hicksian (Slutsky) own-price elasticty, $E_S(Y_i, P_i)$, (46), to separate and properly absorb in (89) all the cross-price elasticities of (88) is nontrivial either. But cross-price effects in (89) are only concealed and reappear by homogeneity of Hicksian $Y_i(P, U)$, (52), (54), as

$$E_S(Y_i, P_i) \;=\; e_i \sigma_{ii} \;=\; -\,[\, \sum_{j \neq i}^{N} e_j \sigma_{ij} \,] \;=\; -\,[\, \sum_{j \neq i} E_S(Y_i, P_j) \,] \tag{90}$$

By (88), (89), (90), we may also state the homogeneity property of Marshallian demands by alternatively expressing (decomposing) the own-price elasticities as,

$$E(Y_i, P_i) \;=\; e_i \,[\, \sigma_{ii} \; - \; E(Y_i, C) \,] \tag{91}$$

Table 6 A. Senior Families : Income- and Own-Price Elasticities.

R	Commodity Class	$100e_i$	$E(Y_i, C)$	$E(Y_i, P_i)$	$E_S(Y_i, P_i)$	σ_{ii}
1	Margarine	0.65076	0.24691	-0.34132	-0.33971	-52.20252
2	Bread	2.38612	0.41487	-0.51678	-0.50688	-21.24279
3	Coffee	1.95228	0.47202	-0.57066	-0.56145	-28.75864
4	Miscellaneous	2.81996	0.55373	-0.65387	-0.63825	-22.63347
5	Fuel and light	4.44685	0.57803	-0.68288	-0.65718	-14.77844
6	Tobacco	3.79610	0.56001	-0.66339	-0.64213	-16.91551
7	Soft drinks	0.75922	0.57033	-0.66300	-0.65867	-86.75665
8	Personal goods	0.86768	0.62136	-0.71396	-0.70857	-81.66237
9	Meat	7.26681	0.62260	-0.73357	-0.68833	-9.47224
10	Medical products	1.08460	0.70649	-0.79879	-0.79113	-72.94206
11	Non-durable goods	2.16920	0.71570	-0.81001	-0.79448	-36.62565
12	Household machines	1.08460	0.73005	-0.82210	-0.81418	-75.06717
13	Milk	1.62690	0.75176	-0.84443	-0.83220	-51.15240
14	Medical services	0.86768	0.76014	-0.85153	-0.84494	-97.37921
15	Beer	1.84382	0.76037	-0.85322	-0.83920	-45.51438
16	Books and papers	1.95228	0.78580	-0.87832	-0.86298	-44.20359
17	Fish	0.97614	0.80096	-0.89212	-0.88430	-90.59162
18	Butter	0.75922	0.80925	-0.90000	-0.89397	-117.74799
19	Cheese	0.75922	0.84125	-0.93187	-0.92548	-121.89884
20	Entertainment	3.14534	0.87045	-0.96179	-0.93441	-29.70771
21	Household textiles	0.97614	0.88169	-0.97206	-0.96345	-98.70050
22	Personal care	1.62690	0.89813	-0.98842	-0.97380	-59.85654
23	Furniture	3.47072	0.90609	-0.99632	-0.96487	-27.80030
24	Sugar	0.97614	0.90915	-0.99925	-0.99038	-101.45870
25	Auto repairs	2.16920	0.93089	-1.02053	-1.00034	-46.11553
26	Vegetables	3.03688	0.93354	-1.02292	-0.99457	-32.74966
27	Glassware	0.75922	0.94837	-1.03817	-1.03097	-135.79384
28	Gross rents	12.03905	0.98861	-1.06923	-0.95021	-7.89274
29	Footwear	0.97614	1.05658	-1.14524	-1.13493	-116.26723
30	Other transport	2.16920	1.10661	-1.19244	-1.16843	-53.86478
31	Transport services	1.95228	1.15111	-1.23650	-1.21402	-62.18496
32	Clothing	5.85683	1.18024	-1.25450	-1.18538	-20.23922
33	Radio,, tv sets	1.73536	1.20179	-1.28682	-1.26596	-72.95120
34	Gasoline	3.14534	1.23143	-1.31141	-1.27268	-40.46243
35	Other foods	2.16920	1.26260	-1.34504	-1.31766	-60.74394
36	Wine and spirits	1.73536	1.45497	-1.53561	-1.51036	-87.03433
37	Leisure equipment	3.14534	1.50425	-1.57565	-1.52834	-48.59058
38	Household services	0.54230	1.51898	-1.60577	-1.59753	-294.58537
39	Communication	1.40998	1.70836	-1.78720	-1.76311	-125.04515
40	Restaurants	3.79610	1.83345	-1.88849	-1.81889	-47.91468
41	Transport equipment	5.09761	2.13892	-2.16636	-2.05733	-40.35870
		100.00000 $100 \sum e_i$	1.00000 $\sum e_i E(Y_i, C)$			

Table 6 B. Senior Families : Own Price and Cross Price Elasticities.

R	Commodity Class	$100e_i$	$E(Y_1, P_i)$	$E_S(Y_1, P_i)$	$E_S(Y_i, P_1)$	$\sigma_{1i} = \sigma_{i1}$
1	Margarine	0.65076	-0.00431	-0.00161	-0.00592	-0.24813
2	Bread	2.38612	-0.51678	-0.50688	-0.50688	-21.24279
3	Coffee	1.95228	-0.00855	-0.00045	-0.00055	-0.02302
4	Miscellaneous	2.81996	-0.01004	0.00166	0.00140	0.05869
5	Fuel and light	4.44685	-0.01476	0.00369	0.00198	0.08299
6	Tobacco	3.79610	-0.01328	0.00247	0.00155	0.06497
7	Soft drinks	0.75922	-0.00258	0.00057	0.00180	0.07529
8	Personal goods	0.86768	-0.00250	0.00110	0.00301	0.12632
9	Meat	7.26681	-0.02088	0.00927	0.00304	0.12756
10	Medical products	1.08460	-0.00221	0.00229	0.00505	0.21145
11	Non-durable goods	2.16920	-0.00421	0.00479	0.00527	0.22066
12	Household machines	1.08460	-0.00195	0.00255	0.00561	0.23501
13	Milk	1.62690	-0.00257	0.00418	0.00613	0.25672
14	Medical services	0.86768	-0.00130	0.00230	0.00633	0.26510
15	Beer	1.84382	-0.00276	0.00489	0.00633	0.26533
16	Books and papers	1.95228	-0.00242	0.00568	0.00694	0.29076
17	Fish	0.97614	-0.00106	0.00299	0.00730	0.30592
18	Butter	0.75922	-0.00076	0.00239	0.00750	0.31421
19	Cheese	0.75922	-0.00052	0.00263	0.00826	0.34621
20	Entertainment	3.14534	-0.00124	0.01181	0.00896	0.37541
21	Household textiles	0.97614	-0.00028	0.00377	0.00923	0.38665
22	Personal care	1.62690	-0.00019	0.00656	0.00962	0.40309
23	Furniture	3.47072	-0.00013	0.01427	0.00981	0.41105
24	Sugar	0.97614	-0.00001	0.00404	0.00988	0.41411
25	Auto repairs	2.16920	0.00046	0.00945	0.01040	0.43585
26	Vegetables	3.03688	0.00072	0.01332	0.01046	0.43850
27	Glassware	0.75922	0.00029	0.00344	0.01082	0.45333
28	Gross rents	12.03905	0.00948	0.05942	0.01178	0.49357
29	Footwear	0.97614	0.00143	0.00548	0.01340	0.56154
30	Other transport	2.16920	0.00427	0.01327	0.01459	0.61157
31	Transport services	1.95228	0.00471	0.01281	0.01565	0.65607
32	Clothing	5.85683	0.01583	0.04013	0.01635	0.68520
33	Radio,, tv sets	1.73536	0.00507	0.01226	0.01686	0.70675
34	Gasoline	3.14534	0.01011	0.02316	0.01757	0.73639
35	Other foods	2.16920	0.00765	0.01665	0.01831	0.76756
36	Wine and spirits	1.73536	0.00946	0.01666	0.02291	0.95993
37	Leisure equipment	3.14534	0.01869	0.03174	0.02408	1.00921
38	Household services	0.54230	0.00330	0.00555	0.02443	1.02394
39	Communication	1.40998	0.01126	0.01711	0.02895	1.21332
40	Restaurants	3.79610	0.03506	0.05081	0.03194	1.33841
41	Transport equipment	5.09761	0.06265	0.08380	0.03923	1.64388
		100.00000	-0.41487	0.00000	0.00000	0.00000
		$100 \sum e_i$	$\sum E(Y_1, P_i)$	$\sum E_S(Y_1, P_i)$	$\sum e_i E_S(Y_i, P_1)$	$\sum e_i \sigma_{1i}$

8. Final Comments

To summarize, the expenditure system of the CDES indirect utility function:

Appendix A. The Box-Cox transformation of a variable

In mathematics, the Box-Cox transformation of a variable x is defined as:

$$x(\lambda) = \frac{x^{\lambda} - 1}{\lambda} \tag{92}$$

If $\lambda = 1$, then (apart from the constant -1) the variable is linear.
If $\lambda \to 0$, the limits of the numerator and the denominator are equal to zero. Hence we apply de l'Hôpital's rule. We first take the derivative of numerator and denominator with respect to λ, and take the limit again:

$$\lim_{\lambda \to 0} \frac{x^{\lambda} - 1}{\lambda} = \lim_{\lambda \to 0} \frac{x^{\lambda} \log(x)}{1} = \log(x) \tag{93}$$

where log denotes the natural logarithm.

In econometrics, the Box-Cox transformation is used when we wish to estimate a model, but we are not sure, whether the variables should be included, or whether their logarithms should be used. The model is estimated in the form of the Box-Cox transformation, and it is tested, whether the parameter λ is equal to one, corresponding to the inclusion of the variables, or whether λ is equal to zero, in which case the logarithm of the variables should be used; see Heij et. al. (2005, pp. 297-301) for more details.

In the context of economic theory, the Box-Cox transformation can also advantageously be used in the definition of the indirect utility function of the CDES specification and associated expenditure systems:

$$V(P, C) = \sum_{i=1}^{N} \alpha_i \frac{(C/P_i)^{\beta_i} - 1}{\beta_i} \tag{94}$$

If $\beta_i = 0$, then the term, $\frac{(C/P_i)^{\beta_i} - 1}{\beta_i}$, has to be replaced by : $\log(C/P_i)$.
If all $\beta_i = 0$, then CDES (94) can be reduced to,

$$V(P, C) = \sum_{i=1}^{N} \alpha_i \log(C/P_i) \tag{95}$$

i.e., to the indirect utility function of the Cobb-Douglas transform.
If all $\beta_i = \sigma - 1$, then CDES (94) becomes the CES indirect utility function,

$$V(P, C) = \left[\sum_{i=1}^{N} \alpha_i \left(C/P_i\right)^{\sigma - 1} \right]^{\frac{1}{\sigma - 1}} \tag{96}$$

In practice, when the CDES model (94) is econometrically estimated, the event - that an *estimated coefficient*, β_i, is *zero* - is not impossible, but occurs with a probability of measure zero. That is why (95) is not treated in our main text.

In our exposition, we mentioned the result of Van Driel (1974) and of Van Daal (1982) that at most one β_i-parameter is allowed to be: -1. It means that for this particular commodity the corresponding term in the indirect utility function should be replaced by : $(C/P_i)^{-1}$. But in practice, this special case is again not impossible, but the probability of occurrence is of measure zero.

Appendix B. Derivation of Marshallian demand derivatives

Derivation of the Marshallian demand relation (16) with respect to C gives,

$$\begin{aligned}\frac{\partial Y_i}{\partial C} &= \frac{\partial \alpha_i (C/P_i)^{\beta_i+1}/\partial C}{\sum \alpha_j (C/P_j)^{\beta_j}} - \frac{\alpha_i (C/P_i)^{\beta_i+1}\partial\left(\sum \alpha_j (C/P_j)^{\beta_j}/\partial C\right)}{\left[\sum \alpha_k (C/P_k)^{\beta_k}\right]^2} \\ &= (1+\beta_i)\frac{Y_i}{C} - Y_i \frac{\partial\left(\sum \alpha_j (C/P_j)^{\beta_j}/\partial C\right)}{\sum \alpha_k (C/P_k)^{\beta_k}} \\ &= (1+\beta_i)\frac{Y_i}{C} - \frac{Y_i}{C}\sum \beta_j e_j = (1+\beta_i-\bar{\beta})\frac{Y_i}{C} \end{aligned} \tag{97}$$

where use has been made of (16), (18), (19). Hence (97) establishes (20).

Regarding price derivatives, we first consider the case, $i \neq j$. Then, P_j only appears in thedenominator, and the derivative of (16) with respect to P_j becomes [simmilar result with interchanging the index (i) and (j)],

$$\begin{aligned}\frac{\partial Y_i}{\partial P_j} &= \alpha_i (C/P_i)^{\beta_i+1} \frac{\partial\left(\sum \alpha_k (C/P_k)^{\beta_k}\right)^{-1}}{\partial P_j} \\ &= -\frac{\alpha_i (C/P_i)^{\beta_i+1}}{\sum \alpha_k (C/P_k)^{\beta_k}} \times \frac{\partial\left(\alpha_j (C/P_j)^{\beta_j}\right)/\partial P_j}{\sum \alpha_k (C/P_k)^{\beta_k}} = \beta_j\left(\frac{Y_i Y_j}{C}\right) \end{aligned} \tag{98}$$

Secondly, we consider the case, $i = j$. Then, the term P_i appears in the numerator as well as in the denominator of (16). Derivation with respect to P_i gives, after reduction,

$$\frac{\partial Y_i}{\partial P_i} = \frac{\partial\left(\alpha_i (C/P_i)^{\beta_i+1}\right)/\partial P_i}{\sum \alpha_k (C/P_k)^{\beta_k}} + \beta_i \frac{(Y_i)^2}{C} = -(1+\beta_i)\frac{Y_i}{P_i} + \beta_i\frac{(Y_i)^2}{C} \tag{99}$$

Hence (98) and (99) establish (21).

Regarding the proof of the derivative (24) of $\bar{\beta}$, (19), we have

$$\begin{aligned}\frac{\partial \bar{\beta}}{\partial C} &= \sum \frac{\partial e_j}{\partial C}\beta_j = \sum \frac{\partial (P_j Y_j C^{-1})}{\partial C} = \sum \left[P_j C^{-1}\frac{\partial Y_j}{\partial C} - C^{-2}P_j Y_j\right]\beta_j \\ &= C^{-1}\sum\left[P_j \frac{\partial Y_j}{\partial C} - e_j\right]\beta_j \end{aligned} \tag{100}$$

The last expression can, using (97), be rewritten as:

$$C^{-1}\sum e_j(\beta_j - \bar{\beta})\beta_j = C^{-1}\sum e_j(\beta_j-\bar{\beta})^2 > 0 \tag{101}$$

where the latter equality is obtained using: $\sum e_j(\beta_j - \bar{\beta}) = 0$. Hence (101) establishes (24).

As to the price derivative of $\bar{\beta}$, (19), we get, using (98) and (99):

$$\frac{\partial \bar{\beta}}{\partial P_j} = \sum_i \beta_i \frac{\partial e_i}{\partial P_j}$$

$$
\begin{aligned}
&= \sum_i \beta_i \frac{\partial(P_i Y_i / C)}{\partial P_j} = \frac{Y_j \beta_j}{C} + \sum_i \frac{P_i \beta_i}{C} \left[\beta_j \frac{Y_i Y_j}{C} - (1+\beta_i) \frac{Y_i}{P_i} \delta_{ij} \right] \\
&= -\frac{Y_j \beta_j^2}{C} + \frac{Y_j \beta_j}{C} \sum_i \frac{P_i Y_i}{C} \beta_i = -\beta_j \, (\beta_j - \bar{\beta}) \frac{Y_j}{C} \qquad (102)
\end{aligned}
$$

Hence (102) establishes (25). From (102) we next obtain:

$$
\frac{\partial \bar{\beta}}{\partial \ln P_j} = P_j \frac{\partial \bar{\beta}}{\partial P_j} = -\beta_j \, (\beta_j - \bar{\beta}) e_j \qquad (103)
$$

Dedication. This paper is dedicated to: Professor W.H. Somermeyer, Director of the Econometric Institute from 1966 until his untimely death on 31 May 1982; visiting professor at the Copenhagen Business School, Fall 1979.

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References

[1] Allen, R. G. D. (1936), Professor Slutsky's theory of consumer's choice, *Review of Economic Studies*, 3, 120-129.

[2] Allen, R. G. D. and A.L. Bowley (1935),*Family Expenditure*, P.S. King & Son, London.

[3] Allen, R. G. D. (1950), The Substitution Effect in Value Theory, *Economic Journal*, 60, 675-685.

[4] Arrow, K. J. (1961), Additive logarithmic demand functions and the Slutsky relations, *Review of Economic Studies*, 28, 176-181.

[5] Arrow, K.J., H.B. Chenery, B. Minhas, R.M. Solow (1961), Capital-labor substitution and economic efficiency, *Review of Economics and Statistics*, 42, 225-250.

[6] Arrow, K.J. and M.D. Intriligator (1982), ed. *Handbook of Mathematical Economics*, Vol.2, North-Holland, Amsterdam.

[7] Barnett, W. A. (1979), Theoretical Foundations for the Rotterdam Model, *Review of Economic Studies*, 46, 109-130.

[8] Barten, A.P. (1967), Evidence on the Slutsky Conditions of Demand Equations, *Review of Economics and Statistics*, 49, 77-84.

[9] Barten, A.P. (1968), Estimating Demand Equations, *Econometrica*, 36, 213-251.

[10] Barten, A.P. (1969), Maximum Likelihood Estimation of a Complete System of Demand Equations, *European Economic Review*, 1, 7-73.

[11] Barten, A.P. and E. Geyskens (1974), The Negativity Condition in Consumer Demand, *European Economic Review*, 6, 227-260.

[12] Barten, A.P. (1977), The System of Consumer Demand Functions Approach: A Review, *Econometrica*, 45, 23-51.

[13] Barten, A.P. and V. Boehm (1982), *Consumer Theory*, Chapter 9, in Arrow and Intriligator (1982).

[14] Cobb, C.W. and P.H. Douglas (1928), A theory of production, *American Economic Review*, Papers and Proceedings, XVIII, 139-165.

[15] Chung, J.W. (1994), *Utility and Production Functions*, Blackwell, Oxford.

[16] Cramer, J.S. (1970), Interaction of price and income in consumer demand, *European Economic Review*, 1, 428-436.

[17] Deaton, A.S. (1974), A reconsideration of the empirical implications of additive preferences, *Economic Journal*, 84, 338-348.

[18] Deaton, A.S. (1975), The Measurement of Income and Price Elasticities, *European Economic Review*, 6, 261-273.

[19] Deaton, A.S. and J. Muellbauer (1980a), *Economics and Consumer Behavior*, Cambridge University Press, Cambridge.

[20] Deaton, A.S. and J. Muellbauer (1980b), An almost ideal demand system, *American Economic Review*, 70, 312-326.

[21] De Boer, P.M.C. and M. Missaglia (2005), Introducing the indirect addilog system in a computable general equilibrium model: a case study for Palestine, Report EI 2005-25 of the Econometric Institute, Erasmus University Rotterdam, the Netherlands.

[22] Diewert, W.D. (1974), *Applications of Duality Theory*, Chapter 3, in Intriligator and Kendrick (1974).

[23] Diewert, W.D. (1982), *Duality Approaches to Microeconomic Theory*, Chapter 12, in Arrow and Intriligator (1982).

[24] Engel, E. (1857), Die Productions- and Consumptionsverhltnisse des KnigreichsSachsen, *Zeitschrift des Statistischen Bureaus des Kniglich Schsishen Ministerium des Inneren*, No. 8-9, Sonntag, November 22, 1857; reprint, Bulletin of the International Statistical Institute,(1895), 9, 1-54.

[25] Engel, E. (1895), Die Lebenshaltung belgischer Arbeiterfamilien fruher and jetzt, *Bulletin of the International Statistical Institute*, 9, 1-124, with an Appendix, 1-54.

[26] Frisch, R. (1959), A Complete Scheme for Computing All Direct and Cross Demand Elasticities in a Model wirh Many Sectors, *Econometrica*, 27, 177-196.

[27] Fuss, M. and D. McFadden (1978), ed. *Production Economics: A Dual Approach to Theory and Applications*, Vol.1, North-Holland, Amsterdam.

[28] Geary, R.C. (1949-1950), A Note on a Constant Utility Index of the Cost of Living, *Review of Economic Studies*, 18, 65-66.

[29] Goldberger, A.S. (1970), A Cross-Country of Consumer Expenditure Patterns, *European Economic Review*, 2, 357-400.

[30] Hanoch, G. (1971), CRESH production functions, *Econometrica*, 39, 695-712.

[31] Hanoch, G. (1975), Production and demand models with direct and indirect implicit additivity, *Econometrica*, 43, 395-419.

[32] Hanoch, G. (1978), Polar functions with constant two factors - one price elasticities, Chapter II.3, in Fuss and McFadden (1978).

[33] Heij, C., P.M.C. de Boer, P.H. Franses, T.Kloek and H.K. van Dijk (2004), *Econometric Methods with Applications in Business and Economics*, Oxford University Press, Oxford.

[34] Hicks, J.R. and R.G.D. Allen (1934), A reconsideration of the theory of value. Part I-II, *Economica*, 1, 52-57, 196-219.

[35] Hicks, J.R. (1969), Direct and indirect additivity, *Econometrica*, 37, 353-354.

[36] Houthakker, H.S. (1952), Compensated changes in quantities and qualities consumed, *Review of Economic Studies*, 19, 155-64.

[37] Houthakker, H.S. (1957), An international comparison of household expenditure patterns commemorating the centenary of Engel's law , *Econometrica*, 25, 532-51.

[38] Houthakker, H.S. (1960), Additive preferences, *Econometrica*, 28, 244-257.

[39] Houthakker, H.S. (1965), A note on self-dual preferences, *Econometrica*, 33, 797-801.

[40] Houthakker, H.S. (1965), New evidence on demand elasticities, *Econometrica*, 33, 277-288.

[41] Intriligator, M.D. and D.A. Kendrick (1974), ed., *Frontiers of Quantitative Economics*, Vol.2, North-Holland, Amsterdam.

[42] Jensen, B.S. (1980),*Expenditure System for the Danish Household Sector*, Symposium i Annvendt Statistik, NEUCC, Lyngby, Denmark, 263-296.

[43] Jensen, B.S. and M. E. Larsen (2005), General Equilibrium Dynamics of Multi-Sector Growth Models, *Journal of Economics*, Suppl. 10, 17-56.

[44] Joergensen, E. (1965), *Income-Expenditure Relations of Danish Wage and Salary Earners*, Statistical Inquires No. 14, Statistical Department, Copenhagen.

[45] Katzner, D.W. (1968), A Note on the Differentiability of Consumer Demand Equations, *Econometrica*, 36, 415-418.

[46] Klein, L.R. and H. Rubin (1948-1949), A Constant-Utility Index of the Cost of Living, *Review of Economic Studies*, 15, 84-87.

[47] Kloek, T. and H. Theil (1965), International Comparisons of Prices and Quantities Consumed, *Econometrica*, 33, 535-556.

[48] Kreps, D.M. (1995), *A Course in Microeconomic Theory*, Harvester Wheatsheaf, New York.

[49] Leser, C.E.V. (1941), Family budget data and price-elasticities of demand, *Review of Economic Studies*, 9, 40-57.

[50] Leser, C.E.V. (1964), Forms of Engel curves, *Econometrica*, 32, 443.

[51] Lluch, C. and A. Powell (1975), International Comparisons of Expenditure Patterns, *European Economic Review*, 5, 275-303.

[52] Marschak, J. (1943), Demand Elasticities Reviewed, *Econometrica*, 11, 25-34.

[53] Mas-Colell, A., M.D. Whinston and J.R. Green (1995), *Microeconomic Theory*, Oxford University Press, New York.

[54] Marshall, A. (1920), *Principles of Economics*, 8.ed., Macmillan, London.

[55] Mukerji, V. (1963), A generalized SMAC function with constant ratios of elasticity of substitution, *Review of Economic Studies*, 30, 233-236.

[56] Parks, R.W. (1969), System of Demand Equations: An Empirical Comparison of Alternative Functional Forms, *Econometrica*, 37, 629-650.

[57] Pollak, R.A. and T.J. Wales (1969), Estimation of the Linear Expenditure System, *Econometrica*, 37, 611-628.

[58] Pollak, R.A. (1972), Generalized Separability, *Econometrica*, 40, 431-453.

[59] Prais, S.J and H.S Houthakker (1971), *Analysis of Family Budgets*, 2.ed, Cambridge University Presss, London.

[60] Roy, R. (1947), La distribution du revenue entre les divers biens, *Econometrica*, 15, 205-225.

[61] Sato, K. (1972), Additive Utility Functions with Double-Log Demand Functions, *Journal of Political Economy*, 80, 102-124.

[62] Samuelson, P.A. (1948), Some Implications of Linearity, *Review of Economic Studies*, 30, 88-90.

[63] Samuelson, P.A. (1950), The Problem of Integrability in Utility Theory, *Economica*, 17, 355-385.

[64] Samuelson, P.A. (1965), Using full duality to show that simultaneously additive direct and indirect utilities implies unitary price elasticity of demand, *Econometrica*, 33, 781-796.

[65] Samuelson, P.A. (1969), Corrected formulation of direct and indirect additivity, *Econometrica*, 37, 355-359.

[66] Silberberg, E. and W. Suen (2001), *The Structure of Economics, A Mathematical Analysis*, 3. edition, McGraw-Hill, Singapore.

[67] Smith, Adam (Edwin Cannan, 6.ed.), *The Wealth of Nations*, Vol. 1, Methuen, London, 1961.

[68] Somermeyer, W.H. and J.W.W.A. Wit (1956), Een verdeelmodel (an allocation model), Report M14, Statistics Netherlands, Voorburg.

[69] Somermeyer, W.H. and A. Langhout (1972), Shapes of Engel curves and demand curves: implications of the expenditure model, applied to Dutch data, *European Economic Review*, 3, 351-386.

[70] Stone, R. (1954), Linear expenditure system and demand analysis: an application to the British pattern of demand, *Economic Journal*, 64, 511-532.

[71] SU, Nr. 34 (1977), *Oekonomiske Forhold i Loenmodtagerhusstande: Forbrugsundersoegelsen 1971*, Danmarks Statistik, Koebenhavn.

[72] Theil, H. (1965), The Information Approach to Demand Analysis, *Econometrica*, 33, 67-87.

[73] Tinbergen, J. (1942), Professor Douglas' production function, *Revue de l'Institut International de Statistique*, 10, 37-48.

[74] Van Daal, J. (1983), On the utility functions of the indirect addilog budget allocation model, Report 8304, Econometric Institute, Erasmus University Rotterdam, The Netherlands.

[75] Van Driel, J. (1974), Consumer expenditure allocation models, Report 7405, Econometric Institute, Erasmus University Rotterdam, The Netherlands.

[76] Varian, H.R. (1992), *Microeconomic Analysis*, 3. edition, Norton, New York.

[77] Wit, J.W.W.A. (1957), Inkomenselasticiteiten in 1935/'36 and 1951 voor Nederland: toepassing van een model van inkomensbesteding, *Statistische en Econometrische Onderzoekingen*, Statistics Netherlands, Voorburg, 27-41.

[78] Wit, J.W.W.A. (1960), Income elasticities in 1935/'36 and 1951 in the Netherlands: an application of a model for income spending, *Statistical Studies*, Vol. 10, Statistics Netherlands, Voorburg.

[79] Wold, H. (1940), Efterfrgan p jordbruksprodukter och dets knslighet for pris- og indkomstforndringer, *S.O.U.: 16*, Stockholm.

[80] Wold, H. (1952), *Demand Analysis*, John Wiley & Sons, New York.

[81] Wold, H. (1970), Giffen's Paradox and the Marshallian Demand Curve, *The Manchester School*, 38, 65-71.

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