Transitional Dynamics in the Uzawa-Lucas Model of Endogenous Growth

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Abstract

In this paper we solve an $N \in \mathbb{N}$ players differential game with logarithmic objective functions. The optimization problem considered here is based on the Uzawa Lucas model of endogenous growth. Agents have logarithmic preferences and own two capital stocks. Since the number of players is an arbitrary fixed number $N \in \mathbb{N}$ the model's solution is more realistic than the idealized concepts of the social planer or the competitive equilibrium. We show that the symmetric Nash equilibrium is completely described by the solution to one single ordinary differential equation. The numerical results imply that the influence of the externality along the balanced growth path vanishes rapidly as the number of players increases. Off the steady state the externality is of great importance even for a large number of players.

Key words: Value Function Approach, Nash-Equilibrium, Open-loop Strategies, Ordinary Differential Equation.

JEL Classifications: C72, C61, O41.

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1 Introduction

This paper studies an N players differential game that is based on the Uzawa (1965) and Lucas (1988) model of endogenous growth. Each player owns a goods producing firm and has access to educational services. The game's time horizon is infinite and the number of players is fixed. The goods sector technology is Cobb-Douglas in human and physical capital while the schooling technology is linear in human capital only. At the outset of the game, the players' initial endowments are identical and their task is to choose simultaneously the optimal consumption and the optimal allocation of human capital between the two sectors. Since the goods sector productivity is influenced by the economy-wide average level of human capital, the representative player has to know the decisions of his co-players in order to determine optimally his own controls. Furthermore, the game is symmetric in the players' constraints and their objective functions. These facts motivate us to restrict our search for a solution to symmetric Nash equilibria.

By generalizing the solution method introduced in Bethmann and Reiß (2004) we present the game's open-loop solution. The main idea is to exploit the model's inherent homogeneity (cf. Caballé and Santos, 1993) in order to reduce the complexity of the optimization problem. This is done by using the geometric mean when defining the economy-wide average level of human capital such that we are able to subsume the information of the relevant state variables in a weighted product, with the model's inherent homogeneity determining the respective weights. Following Mulligan and Sala-i-Martin (1991) we refer to the result of this transformation as the state-like variable. The multiplicative structure of this variable together with the logarithmic utility function allows us to rewrite the value function as the sum of the 'value-function-like function' and expressions representing the influence of the human capital stocks of each single agent on his life-time utility. The introduction of the value-function-like function then allows us to reduce the complexity of the representative players Hamilton-Jacobi-Bellman (HJB) equation to an implicit partial differential equation. In fact we are able to state an explicit solution to this equation. However, it is the generalization of the unstable solutions to the resulting HJB equations in the social planer's and the infinitely many agents case (cf. Bethmann and Reiß, 2004). Nevertheless, the knowledge of this function allows us to determine the model's steady state. Using a standard transformation (cf. Bronstein and Semendjajew, 1999) we get an equivalent explicit partial differential equation, which can be further reduced to an ordinary differential equation by inserting the symmetric Nash equilibrium condition. Finally, we use the steady state in order to determine an initial value problem for this ordinary differential equation.

The numerical results show that the number of agents does indeed influence the optimal human capital allocation of the representative agent. However this influence of N decreases very fast. Moreover, the strength of this phenomenon heavily depends on the degree of the external effect of human capital on the goods sector productivity. In fact we show that both parameters work in opposite directions. An increase in the degree of the external effect resembles the effect that a lowering of the number of players has on the outcome of the differential game. Although, the conjecture that both effects are reciprocal suggests itself, we show that this only holds for the steady state allocation. Off the balanced growth path the influence of number of players also vanishes, the degree of the external effect however is still driving the agents' decisions.

The paper is organized as follows. Section 2 introduces the model. Section 3 states the differential game implied by the players' coupled optimization problems. Section 4 presents the mathematical solution strategy. Section 5 examines the numerical results showing the above mentioned antagonistic effects of the two parameters. Section 6 concludes. The Appendix contains statements that are omitted in the main paper for clarity of the exposition.

2 The model

We assume a closed economy populated by $N \in \mathbb{N}$ identical and infinitely-lived oligopolists. They are producing a single good and have access to a schooling sector providing educational services. The representative oligopolist[1] $A^{(i)}$, $i \in \mathbb{N}^N$, has logarithmic preferences over consumption streams

$$U^{(i)} = \int_{t=0}^{\infty} e^{-\rho t} \ln c_t^{(i)} dt, \tag{1}$$

where $c_t^{(i)}$ is the agent's level of consumption at time t and $\rho > 0$ is the subjective discount rate. The logarithmic utility function implies that the intertemporal elasticity of substitution is equal to one. Agents have a fixed endowment of time, which is normalized as a constant flow of one unit. The variable $u_t^{(i)}$ denotes the fraction of time allocated to goods production at time t. The fraction $1 - u_t^{(i)}$ of time is spent in the schooling sector. As agents do not benefit from leisure, the whole time budget is allocated to the two sectors. Hence, in any solution the condition

$$u_t^{(i)} \in [0, 1] \tag{2}$$

has to be fulfilled. The variables $c_t^{(i)}$ and $u_t^{(i)}$ are the agent's control variables. Human capital production is determined by a linear technology in human capital

$$\dot{h}_t^{(i)} = B(1 - u_t^{(i)})h_t^{(i)}, \tag{3}$$

where we assume that B is positive. This technology together with constraint (2) implies that human capital will never shrink, i.e. the growth rate $\dot{h}^{(i)}$ is non-negative. If we set $u_t^{(i)}$ equal to zero, we get the potential growth rate of the oligopolist's human capital stock. If we set $u_t^{(i)}$ equal to one, a stagnation of his human capital follows. The schooling technology implies that the realized

[1]Henceforth we use oligopolist, agent, and player interchangeably.

marginal and average product are equal to $B(1-u_t^{(i)})$. Note that we abstract from depreciation.

The economy's goods supply is given by N identical producers manufacturing a single good. They are using a Cobb-Douglas technology in the two inputs physical and human capital. The level of human capital utilized in goods production equals the total level of the stock of human capital multiplied by the fraction of time spent in the goods sector at time t. Total factor productivity A is enhanced by the external effect γ of the economy's average stock of human capital, $h_{a,t}$. Hence, the oligopolist's output y_t is determined by

$$y_t^{(i)} = A(k_t^{(i)})^\alpha (u_t^{(i)} h_t^{(i)})^{1-\alpha} h_{a,t}^\gamma.$$

The parameter α is the output elasticity of physical capital and we assume $\alpha \in (0,1)$. The exponent γ is assumed to be nonnegative. If we set u_t equal to one, we get the potential output of the representative oligopolist. We assume that the economy-wide average level of human capital is defined by the geometric mean of the individual human capital stocks

$$h_a = \Big(\prod_{n=1}^{N} h^{(n)}\Big)^{\frac{1}{N}}. \tag{4}$$

Since all agents are homogeneous, the economy's average level of human capital must equal the representative agent's level of human capital at any point in time

$$h_t^{(i)} = h_{a,t}, \quad \forall t \geq 0. \tag{5}$$

The oligopolist can either consume or invest his output $y_t^{(i)}$, i.e. his budget constraint reads as follows

$$y_t^{(i)} = c_t^{(i)} + \dot{k}_t^{(i)}, \qquad \forall t \geq 0.$$

The right-hand side describes the spending of the oligopolist's earnings, where $\dot{k}_t^{(i)}$ is the rate of change of his physical capital stock $k_t^{(i)}$. Since we abstract from depreciation, this rate corresponds to his net investment in physical capital. The left-hand side collects the streams of income stemming from the agent's physical capital stock and from his work effort $u_t^{(i)} h_t^{(i)}$. We assume that the initial values k_0 and h_0 are strictly positive. Note that by consuming more than current production it is possible to disinvest in physical capital, i.e. the growth rate of physical capital turns negative.

Informational assumptions

In order to quantify the influence of the external effect and of the number of players on the game's outcome, we compute the model's open-loop equilibrium strategies. The open-loop equilibrium concept is characterized by the fact that the agents commit themselves at the outset to entire temporal paths of human capital allocation and consumption that maximize the discounted utility given the decision paths of all other agents. The next section discusses the optimization problem at hand as a differential game.

3 The N-players game

We consider a non-cooperative differential game with $N \in \mathbb{N}$ players. The game extends over the unbounded time interval $[0, \infty)$. The state of the game at each instant $t \in [0, \infty)$ is described by a vector $s_t := \{k_t^{(1)}, h_t^{(1)}; ..; k_t^{(N)}, h_t^{(N)}\} \in \mathbb{R}_{++}^{2N}$, where $\mathbb{R}_{++}^{2N}$ is the state space of the game. The entries $k^{(i)}$ and $h^{(i)}$ denote the physical and human capital stocks of agent $A^{(i)}$, $i \in \mathbb{N}^N$, i.e. we denote player-specific variables and functions by upper indices. Let us define the set $\bar{S}$ as the subset of $\mathbb{R}_{++}^{2N}$ with elements of the form $\{z, y; z, y; ..; z, y\}$, where $y, z \in \mathbb{R}_{++}$. Then the symmetric initial state of the game is a fixed vector $\bar{s}_0 = \{k_0, h_0; ..; k_0, h_0\} \in \bar{S} \subset \mathbb{R}_{++}^{2N}$. At each point in time $t \in [0, \infty)$, each player $i \in \mathbb{N}^N$ chooses the control variables $c^{(i)}$ and $u^{(i)}$ from his set of feasible controls χ, with

$$\chi = \left\{(f, g) : \mathbb{R}_{++}^{2N} \mapsto [0, \infty) \times [0, 1] \,|\text{locally bounded and measurable}\right\}.$$

The state of the game evolves according to the differential equations

$$\begin{aligned} \dot{k}_t^{(i)} &= A\big(k_t^{(i)}\big)^{\alpha}\big(u_t^{(i)}\big)^{1-\alpha}\big(h_t^{(i)}\big)^{1-\alpha}\big(h_{a,t}\big)^{\gamma} - c_t^{(i)}, \\ \dot{h}_t^{(i)} &= B\big(1 - u_t^{(i)}\big)h_t^{(i)}. \end{aligned}$$

These equations hold for all $t \in [0, \infty)$ and all $i \in \mathbb{N}^N$. Note that the assumption (4) allows us to rewrite the term h_a^{γ}, which influences total factor productivity in the goods sector, as follows

$$h_a^{\gamma} = \left(h^{(i)}\right)^{\frac{\gamma}{N}} \left(\prod_{j \neq i} h^{(j)}\right)^{\frac{\gamma}{N}},$$

where $j \neq i$ is a shortcut and stands for $j \in \mathbb{N}^N \setminus \{i\}$. We assume that agents make their choices simultaneously and try to solve the following dynamic optimization problems (DOPs). The representative agent $A^{(i)}$ seeks to maximize his lifetime utility

$$U^{(i)} = \max_{c_t^{(i)}, u_t^{(i)}} \int_{t=0}^{\infty} e^{-\rho t} \ln c_t^{(i)} dt,$$

subject to the state dynamics

$$\begin{aligned} &\dot{k}_t^{(i)} = A(k_t^{(i)})^{\alpha}\big(u_t^{(i)}\big)^{1-\alpha}(h_t^{(i)})^{1-\alpha+\frac{\gamma}{N}}\left(\prod_{j \neq 1} h_t^{(j)}\right)^{\frac{\gamma}{N}} - c_t^{(i)}, && \forall\, t \geq 0, \\ &\dot{h}_t^{(i)} = B\big(1 - u_t^{(i)}\big)h_t^{(i)}, && \forall\, t \geq 0, \\ &k_t^{(i)} \geq 0 \quad \text{and} \quad h_t^{(i)} \geq 0, && \forall\, t \geq 0. \end{aligned}$$

Since for positive γ all these optimization problems are coupled via the values of the individual human capital stocks $h^{(j)}$, we interpret this optimization problem as a multiple players' non-cooperative game $\Gamma(t, s_t)$. Note that the above DOPs are homogeneous in the initial conditions (cf. Bethmann and Reiß, 2004), a fact that we will exploit extensively in the next section. The symmetry in the agents' initial endowments together with the symmetry in preferences and technology causes us to look for symmetric Nash equilibria. Applying the definition (e.g. Dockner et al. (2000), Chapter 4) to our game $\Gamma(0, \bar{s}_0)$, the policy functions $u^{(i*)}$ and $c^{(i*)}$ form a Nash equilibrium if

$$U^{(i)}\left((u^{(i*)}, c^{(i*)}); (u^{(j*)}, c^{(j*)}), j \neq i\right) \geq U^{(i)}\left((u^{(i)}, c^{(i)}); (u^{(j*)}, c^{(j*)}), j \neq i\right)$$

holds for all feasible controls $u^{(i)}$ and $c^{(i)}$ and for all agents $A^{(i)}$, $i \in \mathbb{N}^N$. In the following section we assume that agents commit themselves to entire courses of action at the outset of the game and are not allowed to revise them at any subsequent date.

4 The open-loop solution

In this section we solve the non-cooperative game under the assumption that the agents are not allowed to revise their action paths once they have made their choices, i.e. we consider that agents play open-loop strategies. In Section 4.1 we exploit the game's homogeneity in the initial conditions in order to reduce the Hamilton-Jacobi-Bellman equation to an implicit partial differential equation. Along the balanced growth path this equation is an implicit differential equation and we are able to give an explicit solution which is indeed the representative agent's value function. However, outside this path the application of this function leads to non-feasible controls. In Section 4.2 we start with the implicit partial differential equation describing the solution to our problem. We define the symmetric Nash equilbrium and reduce the problem to an initial value problem for a single explicit ordinary differential equation.

4.1 The Hamilton Jacobi Bellman equation

The representative agent defines the value function as the solution to his optimization problem:

$$V^{(i)}(s_0) := \max_{(c^{(i)},u^{(i)}) \in \mathcal{X}} \begin{cases} \int_0^\infty e^{-\rho t} \ln c_t^{(i)}\, dt, & \tau = \infty \\ -\infty, & \tau < \infty, \end{cases}$$

where τ denotes the stopping time $\tau := \inf\{t \geq 0 \,|\, k_t = 0\}$. In order to determine the value function, we write down the Hamilton-Jacobi-Bellman (HJB) equation for the value function $V^{(i)} := V^{(i)}(s_t)$:

$$\rho V^{(i)} = \max_{(c^{(i)},u^{(i)}) \in X} \left\{ \ln c^{(i)} + V_k^{(i)} \dot{k}_t^{(i)} + V_h^{(i)} \dot{h}_t^{(i)} + \sum_{j \neq i} V_{h^{(j)}}^{(i)} \dot{h}_t^{(j)} + V_t^{(i)} \right\} \quad (6)$$

Here, $V_k^{(i)}$, $V_h^{(i)}$, and , $V_{h^{(j)}}^{(i)}$ denote the partial derivatives with respect to the agent's own capital stocks $k^{(i)}$ and $h^{(i)}$ and with respect to the other agent's human capital stocks $h^{(j)}$. These derivatives can be interpreted as the shadow prices of relaxing the corresponding constraints. $V_t^{(i)}$ denotes the partial derivative with respect to time t. Since the representative agent has no influence on the evolution of his co-players stocks of human capital $h^{(j)}$, it does not matter whether the braces include the last two terms or not. The first order necessary conditions are:

$$c^{(i*)} = \tfrac{1}{V_k^{(i)}}, \tag{7}$$

$$u^{(i*)} = \left(\frac{A(1-\alpha)V_k^{(i)}}{BV_h^{(i)}}\right)^{\frac{1}{\alpha}} \frac{k^{(i)}\left(\prod_{j\neq 1} h^{(j)}\right)^{\frac{\gamma}{\alpha N}}}{\left(h^{(i)}\right)^{1-\frac{\gamma}{\alpha N}}}. \tag{8}$$

The representative agent chooses the consumption stream such that the marginal utility is equal to the marginal change of wealth with respect to physical capital. The optimal allocation of human capital between the two sectors is determined by the weighted ratio of the marginal changes in goods production and schooling due to a marginal shifting of the human capital allocation. The respective weights are the agent's shadow prices of the corresponding state variable. Since the value function $V^{(i)}$ is obviously increasing in its arguments, the relation (7) ensures that the consumption rate is positive. Equally, $u^{(i*)} \in (0, \infty)$ holds, but $u^{(i*)} > 1$ may well occur. For the moment, let us suppose that the controls $(c^{(i*)}, u^{(i*)})$ found above are feasible. We continue with the insertion of our findings into the HJB equation (6). We obtain:

$$\begin{aligned}&\rho V^{(i)} + 1 + \ln V_k^{(i)} - V_t^{(i)} \\ &= \alpha\left(AV_k^{(i)}\right)^{\frac{1}{\alpha}}\left(\tfrac{1-\alpha}{BV_h^{(i)}}\right)^{\frac{1-\alpha}{\alpha}} k^{(i)}\left(\prod_{n=1}^{N} h^{(j)}\right)^{\frac{\gamma}{\alpha N}} + BV_h^{(i)}h^{(i)} + \sum_{j\neq i} BV_{h^{(j)}}^{(i)}(1-u_t^{(j)})h^{(j)}.\end{aligned}$$

The homogeneity in the initial conditions allows us to generalize Mulligan and Sala-i-Martin's (1991) idea to introduce the representative agent's state-like variable $x_t^{(i)}$ and his control-like variable $q_t^{(i)}$. Here, we define them as follows:

$$x_t^{(i)} = \frac{k_t^{(i)}}{h_t^{(i)}\left(\prod_{n=1}^{N} h_t^{(n)}\right)^{\frac{\gamma/N}{1-\alpha}}} \quad \text{and} \quad q_t^{(i)} = \frac{c_t^{(i)}}{h_t^{(i)}\left(\prod_{n=1}^{N} h_t^{(n)}\right)^{\frac{\gamma/N}{1-\alpha}}}. \tag{9}$$

The evolution of $x^{(i)}$ can be determined by taking the derivative with respect to time and inserting the state dynamics for $k^{(i)}$ and $h^{(n)}$ respectively, $n \in \mathbb{N}^N$:

$$\dot{x}_t^{(i)} = A\left(x_t^{(i)}\right)^{\alpha}\left(u_t^{(i)}\right)^{1-\alpha} - q_t^{(i)} - \frac{(1-\alpha+\gamma-\frac{\gamma}{N}\sum_{j\neq i}u_t^{(j)})Bx_t^{(i)}}{1-\alpha} + \frac{(1-\alpha+\frac{\gamma}{N})Bu_t^{(i)}x_t^{(i)}}{1-\alpha}. \tag{10}$$

We see that the evolution of $x_t^{(i)}$ is completely described by three groups of variables. First, by the state-like variable $x^{(i)}$ itself, second by $A^{(i)}$'s controls $u^{(i)}$ and $q^{(i)}$, and third by all other agent's human capital allocation decision paths $u_t^{(j)}$ with $j \neq i$. The homogeneity in the initial conditions implies that we are led to apply the same controls $\tilde{u}_t^{(i)} = u_t^{(i)}$ and $\tilde{q}_t^{(i)} = q_t^{(i)}$ for any symmetric initial state $\tilde{s}_0$ with $\tilde{x}_0^{(i)} = x_0^{(i)}$. The only difference is that the consumption rate $\tilde{c}_t^{(i)}$ differs from $c_t^{(i)}$, by the factor

$$\frac{\tilde{h}_0^{(i)}}{h_0^{(i)}} \left(\prod_{n=1}^{N} \frac{\tilde{h}_0^{(n)}}{h_0^{(n)}} \right)^{\frac{\gamma/N}{1-\alpha}} = \left(\frac{\tilde{h}_0^{(i)}}{h_0^{(i)}} \right)^{\frac{1-\alpha+\gamma}{1-\alpha}} .$$

Following Bethmann and Reiß (2004) we deduce the symmetric solution from the value function-like function $f^{(i)}$ with $V^{(i)}(\bar{s}_t) = f^{(i)}(x^{(i)}, t; u_t^{(j)}, j \neq i)$ via

$$V^{(i)}(\bar{s}_t, t; u_t^{(j)}, j \neq i) = f^{(i)}(x^{(i)}, t; u_t^{(j)}, j \neq i) + \frac{(1-\alpha+\frac{\gamma}{N}) \ln h^{(i)}}{\rho(1-\alpha)} + \frac{\frac{\gamma}{N} \sum_{j \neq i} \ln h^{(j)}}{\rho(1-\alpha)},$$

where the semicolon indicates that we restrict the function to given time dependent paths of the other agents' allocations of human capital. The derivatives of $V^{(i)}$ with respect to the agent's capital stocks $k^{(i)}$ and $h^{(i)}$, with respect to the other agents' human capital stocks $h^{(j)}$ with $j \neq i$, and with respect to time can be expressed in terms of the function $f^{(i)} := f^{(i)}\big(x_t^{(i)}, t; u_t^{(j)}, j \neq i\big)$. Hence, we consider:

$$\begin{aligned}
V_{k^{(i)}}^{(i)} &= \frac{f_x^{(i)} x^{(i)}}{k^{(i)}}, \\
V_{h^{(i)}}^{(i)} &= \frac{1-\alpha+\gamma/N}{1-\alpha} \left(\frac{1}{\rho h^{(i)}} - \frac{f_x^{(i)} x^{(i)}}{h^{(i)}} \right), \\
V_{h^{(j)}}^{(i)} &= \frac{\gamma/N}{1-\alpha} \left(\frac{1}{\rho h^{(j)}} - \frac{f_x^{(i)} x^{(i)}}{h^{(j)}} \right), \\
V_t^{(i)} &= f_t^{(i)},
\end{aligned}$$

where $f_x^{(i)} := \frac{\partial f^{(i)}}{\partial x^{(i)}}$ and $f_t^{(i)} := \frac{\partial f^{(i)}}{\partial t}$. Thus the function $f^{(i)}$ determines decisively the shadow values of the two private production factors and of the other agent's stocks of human capital. The introduction of $f^{(i)}$ allows us to rewrite the first order necessary conditions (7) and (8):

$$c^{(i*)}(x^{(i)}) = \frac{k^{(i)}}{x^{(i)}} \big(\frac{\partial f^{(i)}}{\partial x^{(i)}} \big)^{-1} \tag{11}$$

$$u^{(i*)}(x^{(i)}) = \Bigg(\frac{(1-\alpha)^2 A}{B\big(1-\alpha+\gamma/N\big)\big(\frac{1}{\rho f_x^{(i)}} - x^{(i)}\big)} \Bigg)^{\frac{1}{\alpha}} x^{(i)}. \tag{12}$$

Let $u_{a,t}$ denote the arithmetic mean of all other agents $A^{(j)}$, $j \neq i$, decisions of allocating their stocks of human capital, i.e. we define the optimal average human capital allocation decision at date t as follows:

$$u_{a,t}^* := \frac{1}{N-1} \sum_{j \neq i} u_t^{(j*)}.$$

Using this definition and the above first order necessary conditions we first determine the balanced growth path solution, where the state-like variable $x^{(i)}$ and the controls $q^{(i)}$ and $u^{(i)}$ remain constant over time. In this case the optimal average human capital allocation must also be a constant in the interval $[0,1]$, i.e. $u^*_{a,t} = u^*_a \in [0,1]$ for all $t \geq 0$. Then the HJB-equation is time independent such that $f^{(i)}_t = 0$ holds. We rewrite the HJB equation as follows:

$$f^{(i)} - \frac{B(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N})}{\rho^2(1-\alpha)} + \frac{1+\ln f^{(i)}_x}{\rho} = \frac{B(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N})x^{(i)}}{\rho(1-\alpha)} \left[\frac{(f^{(i)}_x)^{\frac{1}{\alpha}} \varphi^{\frac{1-\alpha}{\alpha}}}{(\frac{1}{\rho} - x^{(i)} f^{(i)}_x)^{\frac{1-\alpha}{\alpha}}} - f^{(i)}_x \right]$$

with

$$\varphi := \frac{1-\alpha}{1-\alpha+\gamma/N} \left(\frac{\alpha^\alpha (1-\alpha)A}{B\left(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N}\right)^\alpha} \right)^{\frac{1}{1-\alpha}}.$$

Note, that we have reduced the HJB equation along the balanced growth path to an implicit differential equation, where φ is a strictly positive constant. We claim that a solution to this equation is given by:

$$f^{(i)}(x^{(i)}) = \frac{B(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N})}{\rho^2(1-\alpha)} + \frac{\ln \rho - 1}{\rho} + \frac{1}{\rho} \ln(x^{(i)} + \varphi). \tag{13}$$

Indeed, we have $f^{(i)}_x = 1/(\rho x^{(i)} + \rho\varphi)$ as well as $1/\rho - x^{(i)} f^{(i)}_x = \varphi f^{(i)}_x$ and hence:

$$0 = f^{(i)} + \frac{1+\ln f^{(i)}_x}{\rho} - \frac{B(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N})}{\rho^2(1-\alpha)} \quad \text{and} \quad 0 = \frac{\left(f^{(i)}_x\right)^{\frac{1}{\alpha}} \varphi^{\frac{1-\alpha}{\alpha}}}{\left(\frac{1}{\rho} - x^{(i)} f^{(i)}_x\right)^{\frac{1-\alpha}{\alpha}}} - f^{(i)}_x.$$

The controls derived from $f^{(i)}$ are given by:

$$u^{(i*)} = \left(\frac{(1-\alpha)^2 A}{B(1-\alpha+\frac{\gamma}{N})\varphi} \right)^{\frac{1}{\alpha}} x^{(i)} \quad \text{and} \quad q^{(i*)} = \rho\left(x^{(i)} + \varphi\right).$$

The insertion of these findings into the dynamics equation (10) for the state-like variable x_t leads us to the following quadratic equation:

$$\dot{x}^{(i)}_t = \mathrm{a}\big(x^{(i)}_t\big)^2 + \big(\mathrm{a}\varphi - \rho\big)x^{(i)}_t - \varphi\rho \quad \text{with} \quad \mathrm{a} := \frac{B(1-\alpha+\gamma-\gamma u_a \frac{N-1}{N})}{\alpha\varphi}. \tag{14}$$

A search for the steady states of $x^{(i)}$ shows that on the positive axis $\dot{x}^{(i)}_t$ only vanishes for the value:

$$x^{(i)}_{ss} = \frac{\rho}{\mathrm{a}} = \frac{\rho(1-\alpha)}{B(1-\alpha+\frac{\gamma}{N})} \left(\frac{\alpha(1-\alpha)A}{B(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N})} \right)^{\frac{1}{1-\alpha}}.$$

Linearizing the right hand side of equation (10) at $x^{(i)} = x^{(i)}_{ss}$ shows that $x^{(i)}_{ss}$ is locally unstable:

$$\dot{x}^{(i)}_t \simeq (\rho + \mathrm{a}\varphi)\left(x^{(i)}_t - x^{(i)}_{ss}\right), \quad \text{with} \quad \rho + \mathrm{a}\varphi > 0.$$

Therefore we infer that $f^{(i)}$ yields the unstable solution branch in the phase diagram. Unfortunately, an analytic expression for the stable solution branch is unknown. The steady state controls implied by $f^{(i)}$ are the following:

$$u^{(i*)}_{ss} = \frac{\rho(1-\alpha)}{B(1-\alpha+\gamma/N)} \quad \text{and} \quad q^{(i*)}_{ss} = \frac{\rho\varphi}{B} \frac{\alpha\rho + B(1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N})}{1-\alpha+\gamma-\gamma u^*_a \frac{N-1}{N}}. \tag{15}$$

We stress that $f^{(i)}$ determines $u^{(i*)}_{ss}$ independently of u^*_a such that the decentralized steady state is unique.

4.2 The symmetric Nash equilibrium

By symmetry, $u^{(i*)} = u^*$ and $c^{(i*)} = c^*$ do not depend on the agent $A^{(i)}$ and in particular the average human capital allocation rule satisfies $u_a = u^{(i*)}$. Hence, the agent's lifetime utility $U^{(i)}$ only depends on his own controls $u^{(i)}$ and $c^{(i)}$ and on the average decision rule u_a^* concerning the optimal allocation of human capital h_a. Thus, u^* and c^* satisfy the Nash condition if

$$U\Big((u_t^{(*)})_{t\geq 0}, (c_t^{(*)})_{t\geq 0}, (u_{a,t}^*)_{t\geq 0}\Big) \geq U\Big((u_t)_{t\geq 0}, (c_t)_{t\geq 0}, (u_{a,t}^*)_{t\geq 0}\Big)$$

holds for all feasible controls $(u_t)_{t\geq 0}$ and $(c_t)_{t\geq 0}$. Note that from now on we drop the superscript (i) in the notation. Furthermore, we restrict our attention of the HJB-equation to the homogeneous form $G^{(t)}(x, f_x(x,t), f_t(x,t))$ with:

$$G^{(t)}(x,p,d) := \frac{B(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N\text{-}1}{N})}{\rho^2(1-\alpha)} + \frac{d - \ln p}{\rho} + \frac{B(1-\alpha+\gamma-\gamma u_{a,t}^* \frac{N\text{-}1}{N})}{\rho(1-\alpha)} xp\left[\frac{\varphi_t^{\frac{1\text{-}\alpha}{\alpha}}}{(\frac{1}{\rho p}-x)^{\frac{1\text{-}\alpha}{\alpha}}} - 1\right],$$

where we have defined

$$\varphi_t := \frac{1-\alpha}{1-\alpha+\gamma/N}\left(\frac{\alpha^\alpha(1-\alpha)A}{B\Big(1-\alpha+\gamma-\gamma u_{a,t}^*\frac{N\text{-}1}{N}\Big)^\alpha}\right)^{\frac{1}{1-\alpha}}.$$

Consequently, $p(x,t) := f_x(x,t)$ solves the partial differential equation:

$$p = G_x^{(t)} + G_p^{(t)} p_x + G_d^{(t)} p_t,$$

where the respective derivatives of the homogeneous form $G^{(t)}$ are given by

$$\begin{aligned}
G_x^{(t)}(x,p,d) &= \frac{B\Big(1-\alpha+\gamma-\gamma u_{a,t}^*\frac{N\text{-}1}{N}\Big)}{\rho(1-\alpha)} p\left[\frac{\varphi_t^{\frac{1\text{-}\alpha}{\alpha}}(\frac{1}{\rho p}+\frac{1-2\alpha}{\alpha}x)}{(\frac{1}{\rho p}-x)^{\frac{1}{\alpha}}} - 1\right],\\
G_p^{(t)}(x,p,d) &= -\frac{1}{\rho p} + \frac{B\Big(1-\alpha+\gamma-\gamma u_{a,t}^*\frac{N\text{-}1}{N}\Big)}{\rho(1-\alpha)} x\left[\frac{\varphi_t^{\frac{1\text{-}\alpha}{\alpha}}(\frac{1}{\alpha\rho p}-x)}{(\frac{1}{\rho p}-x)^{\frac{1}{\alpha}}} - 1\right],\\
G_d^{(t)}(x,p,d) &= \frac{1}{\rho}.
\end{aligned}$$

Note that the dynamics equation (10) for the state-like variable x_t together with the restated first order necessary conditions (11) and (12) imply that $\dot{x}_t = \rho G_p^{(t)}$ holds along the optimal control path. From economic theory we immediately infer that x_t converges monotonically to the steady state x^{ss} such that $\dot{x}_t = \rho G_p^{(t)} \neq 0$ holds off the balanced growth path. Denoting the inverse function of $t \mapsto x_t$ by $x \mapsto t(x)$ we put $\tilde{p}(x) = p(x, t(x))$ for $x > 0$ and $x \neq x^{ss}$. From $G_d^{(t)} = G_p^{(t)} \dot{x}_t^{-1}$ we thus infer

$$\tilde{p}(x) = G_x^{(t)} + G_p^{(t)}\left(\tilde{p}'(x) - \big(t'(x) - \dot{x}_t^{-1}\big)p_t\right) = G_x^{(t)} + G_p^{(t)}\tilde{p}'(x),$$

where $G^{(t)}$ means of course $G^{(t(x))}$ along the solution path. There we know from the Nash condition that $u_{a,t} = u^*(x_t)$ holds which we can finally insert to obtain the ordinary differential equation $\tilde{p} = G_x + G_p\tilde{p}'$ with

$$G_x(x,p) = \frac{B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u(p,x))}{\rho(1-\alpha)}p\left[\left(\varphi(p,x)\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{1}{\rho p}-x\right)^{\frac{-1}{\alpha}}\left(\frac{1}{\rho p}+\frac{1-2\alpha}{\alpha}x\right)-1\right],$$

$$G_p(x,p) = -\frac{1}{\rho p}+\frac{B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u(p,x))}{\rho(1-\alpha)}x\left[\left(\varphi(p,x)\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{1}{\rho p}-x\right)^{\frac{-1}{\alpha}}\left(\frac{1}{\alpha\rho p}-x\right)-1\right].$$

Here the notation $\varphi(p,x)$ reflects that φ_t depends on $u(p,x)$ which is itself derived from formula (12), that is

$$u(p,x) = \left(\frac{(1-\alpha)^2 A}{B(1-\alpha+\gamma/N)\left(\frac{1}{\rho p}-x\right)}\right)^{\frac{1}{\alpha}} x. \tag{16}$$

Hence the optimal average decision rule $u^*_{a,t}$ is replaced by (16) thereby stressing that the agents commit themselves to time dependent control paths $(u_t)_{t\geq 0}$ at the outset of the game. The differential equation is now explicit with

$$\tilde{p}' = \frac{\tilde{p}-G_x}{G_p}.$$

Using the fact that $q = p^{-1}$ and hence $q' = -p^{-2}p'$ holds we can rewrite this equation in terms of the optimal control-like variable such that the following explicit ordinary differential equation has to be solved:

$$q' = q\frac{1-\frac{B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u(q,x))}{(1-\alpha)\rho}\left[\left(\varphi(q,x)\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{q}{\rho}-x\right)^{\frac{-1}{\alpha}}\left(\frac{q}{\rho}+\frac{1-2\alpha}{\alpha}x\right)-1\right]}{\frac{q}{\rho}-\frac{B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u(q,x))x}{(1-\alpha)\rho}\left[\left(\varphi(q,x)\right)^{\frac{1-\alpha}{\alpha}}\left(\frac{q}{\rho}-x\right)^{\frac{-1}{\alpha}}\left(\frac{q}{\alpha\rho}-x\right)-1\right]}. \tag{17}$$

This fraction is indeterminate at x_{ss}:

$$q'(x) = \frac{K\left(x,q(x)\right)}{L\left(x,q(x)\right)} \quad \text{and} \quad K\left(x_{ss},q(x_{ss})\right) = L\left(x_{ss},q(x_{ss})\right) = 0.$$

In order to obtain determinacy at x^{ss} we use L'Hôpital's rule, which gives

$$q'(x^{ss}) = \frac{K_x(x^{ss},q(x^{ss}))+K_q(x^{ss},q(x^{ss}))q'(x^{ss})}{L_x(x^{ss},q(x^{ss}))+L_q(x^{ss},q(x^{ss}))q'(x^{ss})}.$$

This leads us to a quadratic equation in $q'(x^{ss})$, one solution of which we already know from W, namely $q'(x^{ss}) = \rho$. Therefore, there exists exactly one other possible solution of $q'(x^{ss})$ which is given by

$$q'(x^{ss}) = \frac{-K_x\left(x^{ss},q\left(x^{ss}\right)\right)}{\rho L_q(x^{ss},q(x^{ss}))}.$$

As a result the fraction is now determinate with

$$q'(x^{ss}) = \frac{\left(\alpha\rho+B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u^{ss})\right)\left((1-\alpha+\gamma)u^{ss}+(1-\alpha)(1-\alpha+\gamma-\gamma\frac{N-1}{N}u^{ss})(1+\frac{1-\alpha+\gamma}{1-\alpha+\gamma/N})\right)}{(1-\alpha^2)(1-\alpha+\gamma-\gamma\frac{N-1}{N}u^{ss})+\alpha(1-\alpha+\gamma)u^{ss}}. \tag{18}$$

The Appendix states some intermediate results that we obtained when determining this expression. Note that only a simple initial value problem remains to be solved, which is done in next section.

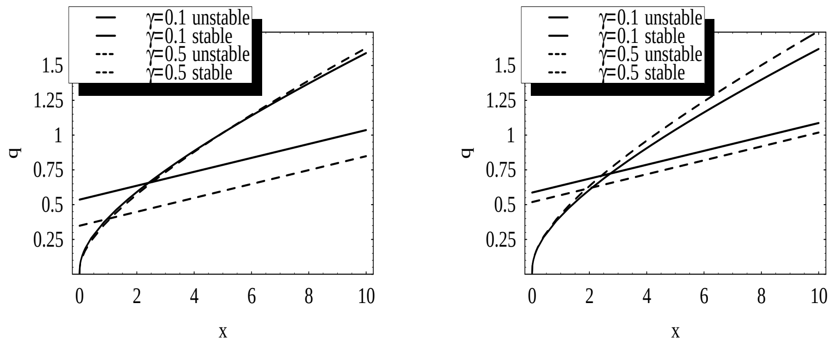


Figure 1: Phase diagrams for $q(x)$ with $\gamma = 0.1$ and $\gamma = 0.5$.
Left: 2 players. Right: 1000 players.

5 Numerical results

In this section we discuss the findings of the preceding section and examine the influence of the number of players and the influence of the external effect γ on the outcome of the differential game which is driven by the human capital allocation u. In order to do this, we consider the following typical calibration of the parameter values

$$A = 1, \quad B = \tfrac{1}{10}, \quad \rho = \tfrac{1}{20}, \quad \text{and} \quad \alpha = \tfrac{1}{3}\ . \tag{19}$$

Figure 1 contains the phase diagrams for the representative players control-like variable $q(x)$ where we have set the degree of the external effect γ equal to 0.1 and to 0.5, respectively. The left diagram displays the duopoly solution and the right diagram refers to the case with 1000 players. The linear lines are derived from the unstable solution (13) to the HJB-equation with the parameter u_a set equal to its steady state value (15). The concave lines that start in the respective origin are the optimal controls q derived from the true value function V, that is the numerical solution of the initial value problem given in equations (17) and (18). Both the stable and the unstable function meet in the respective saddle point (x^{ss}, q^{ss}). Note, that the duopolist's relatively high valuation of human capital leads to lower steady-state values of x in the left diagram than the corresponding steady states in the right diagram where $N = 1000$ holds. On the other hand, if γ increases, we observe a shifting of the steady state to the left, i.e. the N-effect and the γ-effect work in opposite directions. Consequently, in cases with low γ or with a high number of players N, the players' influence on the average productivity is very small such that the steady-state proportion of consumption to the physical capital stock is lower than the two players case or in the case where γ is higher. The pictures show these proportions as the angles between the x-axis and the straight line between the steady state and the origin.

Figure 2 shows the optimal human capital allocation u in the (x, γ) space as a surface. The left part represents the duopoly solution and the right part

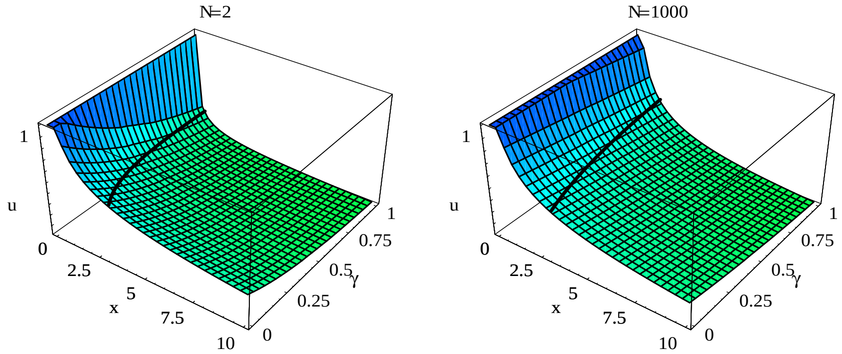
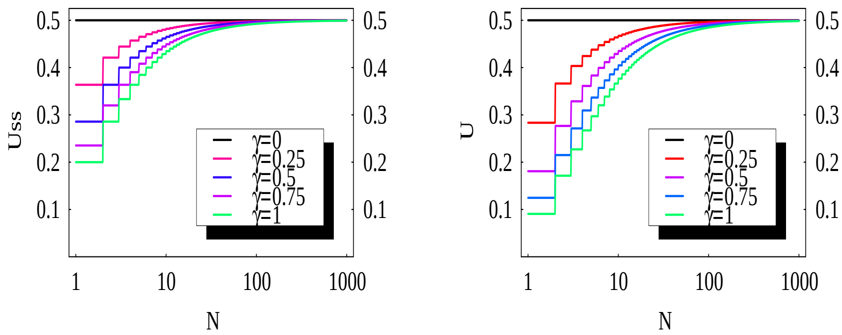


Figure 2: Optimal time share u in (x, γ) space.
Left: 2 players. Right: 1000 players.

the case with $N = 1000$. The black lines correspond to the respective steady-state values u^{ss}. Since the duopolist has more influence on the evolution of the average human capital stock, his valuation of human capital is higher than that of the representative oligopolist in the $N = 1000$ players game. This explains why for positive γ the duopolist's allocation of human capital to the goods sector is always smaller than the corresponding value of u in the right part. For small values of x and γ, they both are about to set u larger than one. Keeping γ fixed, the fraction of time allocated to goods production decreases when x increases. This observation can be rationalized as follows. A high value of x indicates that the economy's endowment of human capital is relatively low. This leads to high marginal returns of human capital in goods production. Arbitrage reasoning implies that the realized marginal productivity of human capital in the schooling sector must also be relatively high[2]. Hence a comparatively high fraction of human capital is attracted by the schooling sector. This explains the relatively low value of u. Since the marginal returns of human capital in goods production and γ are positively related, this reasoning can also explain the negative slope of the surface with respect to γ. Since an increase in the number of players causes the influence of a single player on the average level of human capital to be declining, we observe for the optimal human capital allocation that $u(x)$ decreases in N. In this sense considering different N's means a rescalation of the γ-axis, i.e. the surface is stretched like a rubber blanket if the number N increases (see also Figure 5 in the Appendix). The following two figures examine the interplay of γ and N in detail.

Figure 3 shows the optimal human capital allocation u depending on the number of players N. The left diagram displays the respective steady state values. The case $N = 1$ corresponds to the social planer's solution. Since the social returns are taken into account by the planner, his steady state valuation of human capital is higher than that of an arbitrary oligopolist. This explains

[2] The opportunity costs of schooling are also determined by the shadow values of the representative agent's capital stocks.



Optimal time share u with respect to N.
Figure 3: Left: Steady state u_{ss}. Right: Optimal choice $u(x_{ss}^{ce})$.

why for positive γ the planner's allocation of human capital to the goods sector is smaller than the corresponding value of u in cases where N is bigger than one. A similar argument holds for arbitrary oligopolies with N and N' players, where $N < N'$. The higher marginal returns of the representative players human capital in the N players case lead to a lower steady state value of u compared to the N' players case. Keeping these reflections in mind we now consider optimal allocations along the transition towards the steady state. The right diagram of the figure refers to the optimal human capital allocation if the state-like variable is equal to its competitive equilibrium steady state x_{ss}^{ce}, i.e. in the case where $N = \infty$ holds

$$x_{ss}^{ce} = \frac{\rho}{B}\left(\frac{\alpha(1-\alpha)A}{(1-\alpha+\gamma-\gamma\frac{\rho}{B})B}\right)^{\frac{1}{1-\alpha}}.$$

Since x^{ss} is increasing in N we know that the optimal allocation u of the N players game at this state must be smaller than its respective steady state $u(x_{ss}^{ce}) < u(x^{ss})$. For increasing values of N two effects start to set in. First, the game's steady state of the state-like variable converges against x_{ss}^{ce} from below. Second, as mentioned above the representative player's influence on the average human capital stock is shrinking. Both effects cause the optimal choice of $u(x_{ss}^{ce})$ to increase.

Figure 4 shows the values of u where we consider the state-like variable to be far away from steady state. In the right part of the figure $x = 3x_{ss}^{ce}$ holds, i.e. we consider a relative scarcity of human capital. This causes the productivity of human capital in the goods sector w to be very high, where the representative player's $w^{(i)}$ is given by

$$w^{(i)} = \frac{(1-\alpha+\frac{\gamma}{N})y^{(i)}}{h^{(i)}}.$$

If we neglect the respective shadow values for the moment, it follows that the opportunity costs of schooling are also very high in this case. Then the players optimal allocation of human capital to the goods sector must be relatively small in order to match the marginal productivity of human capital in the two sec-

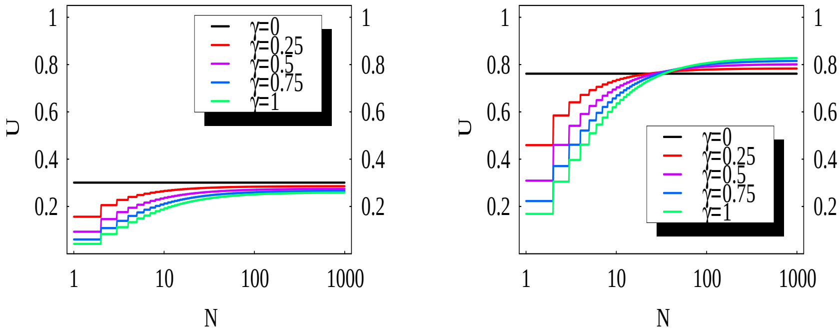


Figure 4: Optimal time share $u(x)$ with respect to N.
Left: For $x = 3x_{ss}^{ce}$. Right: For $x = 1/3x_{ss}^{ce}$.

tors. Clearly, this effect is the more distinct the bigger the parameter γ, i.e. the higher the marginal productivity of his human capital stock in the goods sector. In the right part of the figure we consider $x = 1/3x_{ss}^{ce}$. This corresponds to a relative scarcity in physical capital thereby implying a relatively low marginal productivity of human capital. Note, that for large numbers N the optimal human capital allocation to the goods sector is the bigger the higher the values of γ. This can be rationalized as follows. Let two games be given. In the first game the parameter γ is bigger than the second game's γ. Furthermore, suppose that the number N is relatively large such that we can neglect the representative agent's influence on the economy-wide average human capital stock. If we assume that the average human capital stock in both games grow with the same rate the representative agent knows that the marginal productivity of his human capital stock in the next moment will be higher in the case where γ is big. This means that the shadow value of his present stock of human capital is smaller than in the low γ case. Thereby the shadow values give an incentive for human capital investment in the high γ case. Furthermore, the original assumption that the average stock of human capital in both cases grow at the same rate can not be a symmetric Nash equilibrium, at least in one of the two games considered. We conclude that the average stock of human capital grow faster in the high γ case.

6 Conclusion

In this paper we have derived the open loop solution of a differential game with logarithmic objective functions. The focus on time dependent control paths and on symmetric Nash equilibria has permitted to solve the differential game for an arbitrary number of players. We have shown that the game's solution is completely described by an initial value problem for an ordinary differential equation. Since the allocation of human capital between the two production sectors is crucial for our understanding of the transitional dynamics (cf. Mulligan and Sala-i-Martin, 1993) we have examined the influences of the number of players N and the degree of the external effect γ on the optimal choice of u. We have shown that both parameters have a decreasing influence on the steady-state allocation. Furthermore, the influence of both parameters vanishes for large N. Off the steady state, the optimal allocation is more sensitive to the two parameters. Again the importance of N vanishes. However, we show that the importance of the degree of the external effect γ remains when the number of players increases.

Appendix

Human capital allocation

Figure 5 shows similar plots as Figure 2 in the main text. The only difference here is that we have set N equal to 1, 3, 10, and 100. We observe that the surface indeed the optimal human capital allocation u in the (x, γ) space as a surface. It can be seen that the number N of players causes the surface to rise to the left of the steady state and to part represents the duopoly solution and the right part the case with $N = 1000$. The black lines correspond to the respective steady-state values u^{ss}.

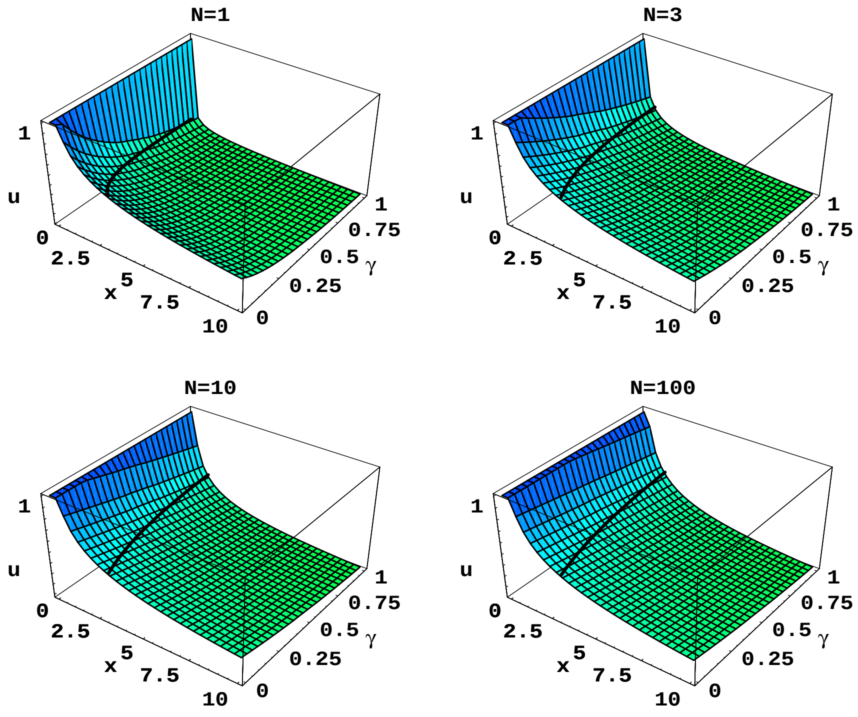


Figure 5: Optimal time share u with respect to (x, γ).

The initial value $q'(x^{ss})$

Note that the derivatives of $\varphi(q,x)$ at $q=q(x^{ss})$ and $x=x^{ss}$ are found to be

$$\varphi_x = \frac{\varphi_a^{ss}\alpha\gamma(N-1)u_x}{(1-\alpha)[N(1-\alpha+\gamma)-\gamma(N-1)u_{ss}]} \quad \text{and} \quad \varphi_q = \frac{\varphi_a^{ss}\alpha\gamma(N-1)u_q}{(1-\alpha)[N(1-\alpha+\gamma)-\gamma(N-1)u_{ss}]}.$$

The respective derivatives of (16) are given by

$$u_x = u_{ss}\frac{B(1-\alpha+\gamma-\gamma\frac{N-1}{N}u_{ss})+\rho}{\alpha\rho\varphi} \quad \text{and} \quad u_q = \frac{u_{ss}}{\rho\varphi_a^{ss}}.$$

Let $K(q(x^{ss}),x^{ss})$ and $L(q(x^{ss}),x^{ss})$ denote the numerator and denominator of the differential equation (17). Since $q'(x^{ss}) = \frac{-K_x(q(x^{ss}),x^{ss})}{\rho L_q(q(x^{ss}),x^{ss})}$ holds for the second root of the quadratic equation we have to look at the following two derivatives:

$$L_q(x^{ss},q(x^{ss})) = \frac{1-\alpha^2}{\alpha} + \frac{(1-\alpha+\gamma)u_{ss}}{1-\alpha+\gamma-\gamma\frac{N-1}{N}u_{ss}}$$
$$K_x(x^{ss},q(x^{ss})) = \frac{-\rho q^{ss}}{\alpha\varphi}\left\{\frac{(1-\alpha+\gamma)(1-\alpha)}{1-\alpha+\gamma/N} + \frac{(1-\alpha)^2(1-\alpha+\gamma-\gamma\frac{N-1}{N}u^{ss})}{(1-\alpha+\gamma/N)u^{ss}}\left(1+\frac{1-\alpha+\gamma}{1-\alpha+\gamma/N}\right)\right\}.$$

This implies the expression given in equation (18).

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