Natural Resource Abundance and Economic Growth in a Two Country World

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Abstract: We investigate the Ramsey-like dynamics of nonrenewable resource abundance on economic growth and welfare in a two country world. One country is endowed with a nonrenewable-resource Otherwise, countries are identical. The one country result of Rodríguez and Sachs (1999) that the initial stock of the resource influences negatively the GDP growth of the resource-rich country, is shown to not hold in general. The endowment of the nonrenewable resource can have an initial positive effect on the growth rate of the *resource-rich* country provided the elasticity of the initial price of the resource with regard to the initial stock of the resource is greater than minus one. The ratio of the consumption levels of the two countries are shown to be constant over time, and determined by the ratio of initial wealth. An analytical solution of the model allows us to indicate how accumulable and depletable assets affect per country welfare and income growth. For this case we demonstrate that a technological change -that is nonrenewable resource saving- can benefit the resource-rich country's relative welfare.

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1. Introduction

The economics of nonrenewable resources has received increased attention following the empirical observations of Sachs and Warner (1995). They perform cross-country growth regressions and find that economies with large natural resource based exports to GDP ratios in 1971 tended to experience relatively low growth rates in the subsequent period (1971-89). More recent research has led to contradicting results. For example, Stijns (2001) employs different indicators of natural resource abundance than those used by Sachs and Warner and finds no evidence that resource abundance is a detriment for economic growth. Ding and Field (2004) distinguish between natural resource endowments and natural resource dependence and find that, across countries, natural resources do not affect growth. Given this puzzling empirical evidence, the challenge is whether an acceptable theoretical model can help advance our understanding of how resource abundance affects economic growth. To this end, Rodríguez and Sachs (1999) consider a single economy which exports all of the oil extracted, eventually totally depleting the nonrenewable resource stock. However, the economy does not use oil for own production or consumption, implying that the rest of the world must be different than the economy they study, nor can the rest of the world's transition growth impact the resource rich country.

In contrast to Rodríguez and Sachs (1999), we study a two country world economy. In particular, this paper analyzes how initial factor endowments of nonrenewable resources and accumulable resources influence income, growth and welfare in a two country world. We show analytically that, in principle, whether a nonrenewable resource rich country grows faster or slower than another country depends upon economic structure.

Previous work in this area includes Asheim (1986) and Hartwick (1995) who study a two country world and investigate whether constant consumption paths are achievable when the economies invest resource rents in new capital. Chiarella (1980), focuses on international trade aspects of a world economy with two countries and shows that if countries are equally patient then consumption growth across countries is identical. This implies that the ratio between consumption levels (across countries) must be constant over time. Chiarella determines this ratio for the long run (steady state) case when the utility function is logarithmic and discount factors across countries are different. Geldrop and Withagen (1993) generalize Chiarella's (1980) approach, by introducing n number of trading partners. Among others, they concentrate on equilibrium existence, but they do not study cross-country differences on consumption, income or growth.

Our analysis provides insights into how differences in nonrenewable resources and capital endowments affect economic performance. Consequently, we ignore other differences across countries. The economies considered are thus identical, except for the initial endowments of assets (capital) and the nonrenewable resource. As in other models of trade, factor price equalization across countries occurs. The rental rate of capital across countries is equal even in the absence of international borrowing and lending. This result implies that the growth rates of consumption across countries are equal. Thus, as in Chiarella (1980), the ratio of consumption across countries is constant over time. Regardless of *income* growth, relative welfare across countries is shown to remain unchanged. Hence, the question that comes to mind is: shouldn't we instead be concerned about how initial endowments of capital and the nonrenewable resource affect relative welfare instead of how nonrenewable resources affect relative income growth? We show that the ratio of the nonrenewable-resource-rich country's consumption to the nonrenewable-resource-less economy's consumption is constant and is determined by the ratio of their respective value of assets at any point in time. Wealth of the resource-rich country is shown to increases with its initial stock of the resource. This effect is counterbalanced, to some degree, by the negative effect of the initial stock of the depletable resource on the price of the nonrenewable.

We also prove that the initial endowment of the nonrenewable resource has a *positive* effect on the GDP growth rate of the *resource-rich* country as long as the elasticity of the initial price of the resource with regard to the initial stock of the resource is greater than minus one. An analytical solution of the model under a parameter restriction indicates that indeed this elasticity is greater than minus one. Thus, the result of Rodríguez and Sachs (1999) that: the initial stock of the resource influences negatively the GDP growth of an economy that exports a nonrenewable resource, is shown to not hold in general. That is, depending upon structure, a non-renewable resource rich economy can grow faster or slower than an economy without this resource. This result derives mostly from the fact that Rodríguez and Sachs (1999) consider an isolated country, and thus they failed to account for inter-country linkages that influence the rest of the world's transitional dynamics. While it is possible that nonrenewable resources rich economies are growing slower because of rent seeking activities and the like, Rodríguez and Sachs (1999) conclude that owning a large amount of the resource alone is sufficient to generate negative growth, we find this not to be the case. Finally, we show that a technological change that is saving on the nonrenewable resource can benefit the resource-rich country's welfare when compared to the other economy. We also obtain the result that even though the non-renewable resource is an essential input (as in the case of a Cobb-Douglas production function), the two world economy is sustainable forever depending upon the magnitude of the household's rate of time discount relative to the Harrod rate of nonrenewable resource saving technological change.

An overview of the model is as follows. Similar to Chiarella (1980), we model a world economy with two countries that are engaged in international trade. Different from Chiarella (1980), we allow the two countries to be engaged in the production of a final good as in Geldrop and Withagen

(1993). Capital and a nonrenewable resource are used as factors of production and we assume that both countries have access to the same technology to produce the final good. A single country owns the entire stock of the nonrenewable. This country has an extracting sector that depletes the nonrenewable by maximizing discounted profits. The economies trade internationally the final good and the nonrenewable, but international borrowing and lending is not allowed. Each country has a representative consumer that derives satisfaction from consuming the final good and maximizes discounted instant utility, subject to a budget constraint. The rental rate of capital of each country is determined endogenously and equals the marginal physical product of capital in each country. Finally a market clearing condition for the nonrenewable resource endogenously determines its price.

The paper is organized as follows. In section two we introduce the model where a nonrenewable resource and capital are used as inputs in production. In section three we characterize equilibrium, prove the stability of the system and look at the effect of the nonrenewable resource on income growth an relative welfare. In section four we provide an analytical solution under a restriction on parameter values. In section five we provide numerical simulations and conclude in section 6.

2. The model

Consider the environment of a two-country world in which one of the countries is endowed with a nonrenewable-resource (referred to as country one) and the other is a nonrenewable-resource-less country (referred to as country two). The representative consumer of each country seeks to maximize discounted utility of consumption subject to a budget constraint. Across countries consumers have identical preferences and identical discount factors. Whereas country one owns and can deplete the nonrenewable, both economies have an amount of capital that each combines with the nonrenewable resource to produce an identical final good. The final good technology is given by

$$Y_i = F(K_i, R_i) = K_i^{\alpha} \left(e^{\eta t} R_i\right)^{1-\alpha} \tag{1}$$

where K_i and R_i for $i = 1, 2$ denote the amount of capital and the nonrenewable resource employed in the production of output Y_i of country i and η is the growth rate of a resource saving technological progress. The price of the final good is numeraire. Let r_i denote the rental rate of capital in country i. To maximize profits the final good sector of country i sets the marginal physical product of each input factor equal to its rental rate/price as follows:

$$r_i = \alpha \frac{Y_i}{K_i} \qquad\qquad q = (1-\alpha)\frac{Y_i}{R_i}. \tag{2}$$

where q denotes the price of the nonrenewable resource. Since country one exports the nonrenewable to country two, in the absence of trade distortions, the nonrenewable is traded at the same price in both countries. Rearranging these expressions we obtain

$$Y_i = q\frac{R_i}{1-\alpha} = r_i\frac{K_i}{\alpha} \qquad \Rightarrow \qquad K_i = \frac{\alpha}{1-\alpha}\frac{q}{r_i}R_i. \tag{3}$$

Substituting for K_i from (3) into (1) yields

$$Y_i = \left(\frac{\alpha}{1-\alpha}\frac{q}{r_i}\right)^{\alpha} e^{(1-\alpha)\eta t}R_i. \tag{4}$$

Equating this expression to Y_i from (3), provides a relationship between the rental rate of capital r and the price of the nonrenewable resource:

$$r_i = \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{\frac{1}{\alpha}}\left(\frac{e^{\eta t}}{q}\right)^{\frac{1-\alpha}{\alpha}}. \tag{5}$$

Equation (5) suggests that, the Heckscher-Ohlin tendency for factor price equalization is satisfied. Using $r_1 = r_2 = r$ and (3) implies

$$R_i = \frac{1-\alpha}{\alpha}\frac{r}{q}K_i \tag{6}$$

indicating that the ratio of $\frac{R_i}{K_i}$ across countries is the same.[2]

2.1. Extracting sector

Since the final good serves as numéraire, r_1 is the gross interest rate of country one. Define

$$\omega(t) = e^{-\int_0^t (r_1(\tau)-\delta)d\tau} \tag{7}$$

where δ is the constant depreciation rate of capital. Notice that holding one unit of capital yields a gross return $r_1(\tau)$ at instant of time τ, but, since capital depreciates, the net return from holding one unit of capital at time τ equals $r_1(\tau)-\delta$. Presuming perfect capital markets in country one, as in Geldrop and Withagen (1994 p. 1014), $\omega(t)$ is a present-value factor that converts a unit of revenue at time t into an equivalent unit of revenue at time *zero*. We presume that extractions are costless.

[2]If the profit maximization problem of the final good sector were set as a dynamic problem, as in Geldrop and Withagen (1994), the same first order conditions are obtained.

Using $\omega(t)$ to discount profits, the extracting sector finds the optimal path of extractions $R(t)$ that maximizes the present value of profits subject to the constraint that cumulative extractions do not exceed the initial stock of the nonrenewable, i.e.,[3]:

$$\max\{\int_0^\infty q(t)\,R(t)\,\omega(t)\,dt \bigg| \int_0^\infty Rdt \leq S_0\}. \tag{8}$$

The Lagrangian of this isoperimetric problem is given by

$$\mathcal{L} = qRe^{-\int_0^t (r_1(\tau)-\delta)d\tau} - \lambda R \tag{9}$$

and the necessary conditions for a maximum and transversality condition are, respectively, given by

$$q = \lambda e^{\int_0^t (r_1(\tau)-\delta)d\tau}, \qquad \dot{\lambda} = 0, \qquad \lim_{t\to\infty} qRe^{-\int_0^t (r_1(\tau)-\delta)d\tau} = 0. \tag{10}$$

A differential equation for q is obtained by Applying Leibniz's rule to (10) which yields

$$\frac{\dot{q}}{q} = r_1(t) - \delta. \tag{11}$$

This condition indicates that the real price q, of the nonrenewable resource must grow at the real interest rate, or equivalently, the Solow-Stiglitz (Solow, 1974 and Stiglitz, 1974) efficiency condition holds. (11) indicates that the nonrenewable is an asset and for the economy to have incentives to hold the nonrenewable it must be that its price must grow at the real rate of interest. Substituting r_1 from (5) into (11), we obtain an equation describing the motion of the nonrenewable resource price q:

$$\frac{\dot{q}}{q} = r_1 - \delta = \left(\alpha^\alpha (1-\alpha)^{1-\alpha}\right)^{\frac{1}{\alpha}} \left(\frac{e^{\eta t}}{q}\right)^{\frac{1-\alpha}{\alpha}} - \delta. \tag{12}$$

Indeed, this differential equation has an analytical solution given by

$$q(t) = \left(\frac{\Lambda}{\eta+\delta} e^{\frac{1-\alpha}{\alpha}\eta t} + \frac{1}{e^{\frac{1-\alpha}{\alpha}\delta t}} \left(q(0)^{\frac{1-\alpha}{\alpha}} - \frac{\Lambda}{\eta+\delta}\right)\right)^{\frac{\alpha}{1-\alpha}}, \tag{13}$$

[3] If we had integrated the extracting activity into the optimization problem of the consumer of country one the same results would had been obtained.

where $\Lambda = \left(\alpha^{\alpha}(1-\alpha)^{1-\alpha}\right)^{\frac{1}{\alpha}}$ and $q(0)$ remains to be determined. Importantly, notice that the price of the nonrenewable resource can transitionally increase or decline depending on the size of $q(0)$ relative to $\left(\frac{\Lambda}{\eta+\delta}\right)^{\frac{\alpha}{1-\alpha}}$.

Notice from (5), (12) and (13) that the long-run growth rate of q equals

$$\lim_{t\to\infty}\frac{\dot{q}}{q} = \eta \tag{14}$$

Thus, the rate of growth of the resource saving technological change, η, positively influences the long-run growth rate price of the nonrenewable resource. Similar to the standard Ramsey model in which labor augmenting technological change positively influences the long-run labor wage rate, here a resource saving technological change positively influences the long-run price of the nonrenewable resource.

2.2. Consumers' optimization problems

2.2.1. Resource-rich country

The representative consumer of country *one* solves the problem of maximizing discounted utility of consumption subject to an intertemporal budget constraint as follows

$$\max \int_0^{\infty} \frac{c_1(t)^{1-\theta}-1}{1-\theta} e^{-\rho t} dt \tag{15}$$

subject to

$$\dot{K}_1(t) = (r_1-\delta) K_1(t) + \pi(t) - c_1(t) \tag{16}$$

given $K_1(0) = K_{0,1} > 0$,

where $\frac{1}{\theta} > 0$ is the elasticity of intertemporal substitution, $\rho > 0$ is a discount factor, and δ is the constant rate of capital depreciation. π denotes the profits of the extracting sector. Initial and instant t stock of capital for country one are $K_{0,1}$ and $K_1(t)$, respectively. Since extraction is costless $\pi(t) = q(t) R(t)$ holds. Thus, the budget constraint of country one (16) can be rewritten as

$$\dot{K}_1(t) = (r_1-\delta) K_1(t) + q(t) R(t) - c_1(t). \tag{17}$$

Notice that since only country one owns the resource, the rate of extraction consists of what is used for domestic consumption plus an amount that is exported.

2.2.2. Resource less country

The representative consumer of country two solves the problem of maximizing discounted utility of consumption subject to an intertemporal budget constraint, i.e.,

$$\max \int_0^\infty \frac{c_2(t)^{1-\theta} - 1}{1-\theta} e^{-\rho t} dt \tag{18}$$

subject to

$$\dot{K}_2(t) = (r_2 - \delta) K_2(t) - c_2(t) \tag{19}$$

given $K_2(0) = K_{0,2} > 0$.

Where $K_{0,2}$ and $K_2(t)$ are, respectively, the initial and instant t capital stocks of country two.

The Euler condition of the consumer's problem of country i is given by

$$\frac{\dot{c}_i}{c_i} = \frac{r_i - \delta - \rho}{\theta} \quad \text{for } i = 1, 2 \tag{20}$$

with transversality condition

$$\lim_{t\to\infty} \frac{K_i}{e^{\rho t} c_i^{\theta}} = 0 \quad \text{for } i = 1, 2 \tag{21}$$

Notice that the implication of r_1 equaling r_2, implies that c_1 and c_2 grow at identical rates. Using $r_1 = r_2 = r$, and $r - \delta = \frac{\dot{q}}{q}$, (20) yields

$$\frac{\dot{c}_i}{c_i} = \frac{1}{\theta}\left(\frac{\dot{q}}{q} - \rho\right) \quad \text{for } i = 1, 2. \tag{22}$$

The solution to this differential equation is

$$c_i(t) = \left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\theta}} \frac{c_i(0)}{e^{\frac{\rho}{\theta} t}} \quad \text{for } i = 1, 2. \tag{23}$$

Since $q(t)$ can transitionally decline or increase, $c_i(t)$ can follow a similar pattern, albeit influenced by the discount factor and the elasticity of intertemporal substitution. Using (23) the transversality condition of country i can be rewritten as

$$\lim_{t\to\infty} \frac{K_i}{q(t)} = 0 \quad \text{for } i = 1, 2. \tag{24}$$

3. Equilibrium characterization

An equilibrium for this economy are paths of quantities $c_i(t)$, $Y_i(t)$, $K_i,(t)$ $R_i(t)$, $R(t)$ and prices $q(t)$, $r_i(t)$ for $i = 1, 2$ such that $c_i(t)$, and $K_i(t)$ solve the consumer's optimization problem of country i, $Y_i(t)$, $K_i(t)$, $R_i(t)$ solve the optimization problem of the final good sector of country i, $R(t)$ solves the maximization problem of the extracting sector of country one, and the *nonrenewable resource* market clears, i.e.,

$$R_1(t) + R_2(t) = R(t). \tag{25}$$

Proposition 1. The ratio $\frac{c_1(t)}{c_2(t)}$ is constant for all t and equals the present value of the ratio of the assets of country one and country two $\frac{q(t)S(t)+K_1(t)}{K_2(t)}$ at time t.

Proof. See proof in Appendix A■

Let $\mu \equiv \frac{c_1(t)}{c_2(t)}$ where μ is a constant. Since $\mu = \frac{c_1(t)}{c_2(t)} = \frac{q(t)S(t)+K_1(t)}{K_2(t)}$ for all t then,

$$\mu = \frac{c_1(t)}{c_2(t)} = \frac{q(0)S_0 + K_{0,1}}{K_{0,2}} \tag{26}$$

Remark: This result implies that relative consumption is determined from time zero. This result suggests that consumption is influenced forever by each country's initial wealth, i.e., by the value of each country's assets in time $t = 0$. Since S_0, $K_{0,1}$ and $K_{0,2}$ are given, then μ can be determined if $q(0)$ were known.

We have shown thus far that:

Corollary 1.

i) Factor price equalization across countries result. The rental rate of capital across countries is equal even in the absence of international borrowing and lending.

ii) The Solow-Stiglitz efficiency criterion that the price of the nonrenewable resource grows at the real interest rate is satisfied.

iii) The Harrod rate of growth positively influences the long-run growth rate of price of the nonrenewable resource.

iv) The ratio of consumption levels of the resource-rich to the resource less country is constant over time.

v) The ratio $\frac{c_1(t)}{c_2(t)}$ equals the ratio of the value of the assets of each country $\left(\frac{q(t)S(t)+K_1(t)}{K_2(t)}\right)$.

3.1. The reduced system

To study the long-run stability properties of the model, it is useful to normalize the variables of the model, and in order to decrease the dimensionality of the system.

Proposition 2. An equilibrium, if it exists, converges to a balanced growth path with growth rates

$$g_q = \eta, \qquad g_{K_i} = g_{c_i} = \frac{\eta - \rho}{\theta} \equiv g_K, \qquad g_{R_i} = g_R = g_S = \frac{(1-\theta)\,\eta - \rho}{\theta} \tag{27}$$

for $i = 1, 2$ and r is constant. Where g_v denotes the long-run growth rate of variable v.

Proof See appendix A$\blacksquare$

Remark: Notice the important result suggested by (27) ,namely, even though $S(0)$ is an essential non-renewable resource, these economies are sustainable in the long run ($g_{c_i} \geqq 0$) if the growth rate of technological change is equal to or greater than the consumers' discount factor.

Let $K = K_1 + K_2$. Variables are now normalized by their corresponding growth rate as follows

$$\begin{aligned} \hat{K} &= \frac{K}{e^{g_K\,t}}, \qquad \hat{K}_i = \frac{K_i}{e^{g_K\,t}}, \qquad \hat{c}_i = \frac{c_1}{e^{g_K\,t}}, \\ \hat{q} &= \frac{q}{e^{\eta\,t}}, \qquad \hat{S} = \frac{S}{e^{g_S\,t}}, \qquad \hat{R} = \frac{R}{e^{g_S\,t}}, \qquad \hat{R}_i = \frac{R_i}{e^{g_S\,t}} \end{aligned} \tag{28}$$

In this way the normalized variables ($\widehat{\ }$) will remain constant along the growth path. To study the stability of the model we find it useful to reduce the system of equations to the smallest possible number of differential equations. Let

$$T \equiv \frac{\hat{S}}{\hat{K}}, \qquad g \equiv \frac{\hat{c}_2}{\hat{S}}, \qquad \hat{c}_1 = \mu \hat{c}_2 \tag{29}$$

where T is a state variable and g is a control like variable, and $\mu > 0$ is a constant. Since all the variables have been normalized, T and g will remain constant in the long-run. Recall that since c_1 and c_2 grow (forever) at the same rate, then $\hat{c}_1(t) = \mu\hat{c}_2(t)$ for all t. Taking the time derivative of T and using $\dot{\hat{S}} = -\hat{R} - g_S\hat{S}$, (6) and (25) so that $\hat{R} = \frac{r}{\hat{q}}\frac{1-\alpha}{\alpha}\hat{K}$ we obtain:

$$\dot{T} = -\frac{r}{\hat{q}}\frac{1-\alpha}{\alpha} - T\left(\frac{r}{\alpha} - \delta - \eta - (1+\mu)\, gT\right) \tag{30}$$

Note that $\hat{c}_2 = g\hat{S}$ and $\frac{\hat{c}_2}{\hat{K}} = gT$. Similarly, taking the log time derivative of g yields:

$$\dot{g} = \left(\frac{r-\delta-\rho}{\theta} - \eta + \frac{r}{\hat{q}}\frac{1-\alpha}{\alpha}\frac{1}{T}\right) g \tag{31}$$

and

$$\dot{\hat{q}} = (r-\delta-\eta)\,\hat{q} \tag{32}$$

with

$$r = \left(\frac{\alpha^{\alpha}(1-\alpha)^{1-\alpha}}{\hat{q}^{1-\alpha}}\right)^{\frac{1}{\alpha}} \tag{33}$$

Thus, the first order conditions of the model can be reduced to a system of three differential equations, (32), (30) and (31) in three variables T, g and $\hat{q}$.

3.1.1. *Stability*

Here we investigate the stability properties of the of the lung run equilibrium of the reduced system. The equilibrium is locally unique if the Jacobian of the reduced system has two eigenvalues with positive real parts and one with a negative real part. The reason is because the initial condition $T(0)$ is given but $q(0)$, and $g(0)$ are free. Let

$$\begin{pmatrix} \dot{\hat{q}} \\ \dot{g} \\ \dot{T} \end{pmatrix} = \begin{pmatrix} \bar{q}(\hat{q}, g, T) \\ \bar{g}(\hat{q}, g, T) \\ \bar{T}(\hat{q}, g, T) \end{pmatrix} \tag{34}$$

By setting $\frac{\dot{T}}{T}$, $\frac{\dot{g}}{g}$ and $\frac{\dot{\hat{q}}}{\hat{q}}$ equal to zero and employing (33) we obtain the long run or steady state values for T, g $\hat{q}$ and r respectively given by

$$\hat{q}^* = \left(\frac{\alpha^\alpha (1-\alpha)^{1-\alpha}}{(\delta+\eta)^\alpha} \right)^{\frac{1}{1-\alpha}} \qquad g^* = \left(\frac{\eta + \delta(1-\alpha)}{\alpha} - g_K \right) \frac{1}{(1+\mu)\, T^*} \tag{35}$$

$$T^* = \frac{r^*}{\hat{q}^*} \frac{1-\alpha}{\alpha} \frac{1}{\eta - g_K} \qquad \text{and} \qquad r^* = \eta + \delta \tag{36}$$

variables with the superscript $*$ denote steady state values. The Jacobian matrix of (34) evaluated at the steady state equals

$$J^* = \begin{pmatrix} -\frac{1-\alpha}{\alpha} r^* & 0 & 0 \\ \bar{g}_q^* & 0 & \bar{g}_T^* \\ \bar{T}_q^* & \bar{T}_g^* & \bar{T}_T^* \end{pmatrix} \tag{37}$$

where $\bar{g}_q^*$ denotes the partial derivative of $\bar{g}$ with respect to q evaluated at the steady state, and similarly for the other elements of J^*. The eigenvalues (ξ_i) of J^* are given by $\xi_1 = -\frac{1-\alpha}{\alpha} r^* = -\frac{1-\alpha}{\alpha} (\eta + \delta)$ together with the two solutions to the quadratic equation in ξ

$$\xi \left(\bar{T}_T - \xi \right) + \bar{T}_g \bar{g}_T = 0 \tag{38}$$

which equal

$$\xi_2 = -g_S > 0 \qquad \xi_3 = (1+\mu)\, g^* T^* > 0 \tag{39}$$

This leads to the next proposition.

Proposition 3. The equilibrium is locally unique.

Proof. Notice that $\xi_1 = -\frac{1-\alpha}{\alpha} (\eta + \delta)$ is negative. In Appendix A we show that ξ_2 and ξ_3 are the solutions to (38) which are positive. Since there is a single negative eigenvalue the equilibrium is locally unique and stable■

Remark: This result facilitates an empirical analysis as it eliminates the need to search for other equilibria.

3.2. Growth

To provide insights on how the nonrenewable resource affects GDP growth, we proceed by examining the share of country i GDP in aggregate (global) GDP. If the share of country $i's$ GDP in world GDP changes along the growth path, then one of the countries transitionally grows slower than the other.

Let $s_i(t)$ denote the share of GDP of country i on global GDP as follows

$$s_i(t) = \frac{GDP_i(t)}{GDP^w(t)}, \tag{40}$$

where $GDP^w(t)$ equals $GDP_1(t) + GDP_2(t)$. In particular[4] $s_2(t) = \alpha \frac{K_2(t)}{K(t)}$ where as before $K = K_1 + K_2$.

Proposition 4. At time *zero,* the share of GDP of country i for $i = 1, 2$ on global GDP^w does not depend on the level of the resource and

$$s_1(0) = \frac{K_{0,1} + (1-\alpha) K_{0,2}}{K_0}, \qquad s_2(0) = \alpha \frac{K_{0,2}}{K_0} \tag{41}$$

where K_0 denotes the sum of the initial stocks of capital of country one and two $(K_{0,1} + K_{0,2})$.

Proof . Country GDP is, respectively, given by

$$GDP_1 = rK_1 + qR \qquad GDP_2 = rK_2. \tag{42}$$

From equation (6) and the market clearing condition (25) yields

$$R = \frac{r}{q} \frac{1-\alpha}{\alpha} (K_1 + K_2) \tag{43}$$

Substituting this result into the definition of GDP of country *one* and computing GDP^w yields the proposition.■

Remark: A reason for the share $s_i(0)$ of the i-th country to depend on the stock of the resource S, but rather on capital, is because the productivity of $R(0)$ is influenced by capital but not by the stock of the resource.

Let κ_i be the share of capital of country i in total (global) capital as follows

[4] Note that $GDP^w = rK + qR = rK + \frac{1-\alpha}{\alpha} rK$

$$\kappa_i = \frac{K_i}{K} = \frac{\hat{K}_i}{\hat{K}} \tag{44}$$

Since K_i and K growth at the same rate in the long run, the share of capital of country i in the long run is given by

$$\kappa_i^* = \frac{\hat{K}_i^*}{\hat{K}^*} \tag{45}$$

where at time zero $\kappa_{0,i} = \frac{K_{0,i}}{K_0}$ are given to the economy. For the case of country two

$$\hat{K}_2^* = \kappa_2^* \hat{K}^*. \tag{46}$$

Recall that we showed that the ratio of consumption across countries equals

$$\frac{c_1(t)}{c_2(t)} = \mu = \frac{q(t)\,S(t) + K_1(t)}{K_2(t)} \tag{47}$$

where μ is a constant for all t. This implies that

$$\frac{q(0)\,S(0) + K_1(0)}{K_2(0)} = \frac{\hat{q}^* \hat{S}^* + \hat{K}_1^*}{\hat{K}_2^*} \tag{48}$$

where, as before, variables with the superscript $*$ denote state values and $\hat{}$ denote normalized variables. Condition (48) can be rewritten as

$$\frac{q(0)\,S(0) + K_1(0)}{K_2(0)} = \frac{\hat{q}^* \hat{S}^*}{\kappa_2^* \hat{K}^*} + \frac{1-\kappa_2^*}{\kappa_2^*} \tag{49}$$

Using $T^* = \frac{\hat{S}^*}{\hat{K}^*}$ we obtain

$$\frac{q(0)\,S(0) + (1-\kappa_2(0))\,K(0)}{\kappa_2(0)\,K(0)} = \frac{\hat{q}^*}{\kappa_2^*} T^* + \frac{1-\kappa_2^*}{\kappa_2^*} \tag{50}$$

Solving for κ_2^* we obtain

$$\kappa_2^* = \frac{\kappa_2(0)\,K(0)\,(1 + q^* T^*)}{K(0) + q(0)\,S(0)} \tag{51}$$

Since the share of GDP of country two in total GDP equals $s_2(t) = \alpha \frac{K_2(t)}{K(t)}$, in the long run s_2 equals

$$s_2^* = \alpha\kappa_2^* = \alpha\frac{\kappa_2(0)\,K(0)\,(1+q^*T^*)}{K(0)+q(0)\,S(0)}. \tag{52}$$

Thus, if

$$s_2(0) = \alpha\frac{K_{0,2}}{K_0} < s_2^* = \alpha\frac{\kappa_2(0)\,K_0\,(1+q^*T^*)}{K_0+q(0)\,S_0}. \tag{53}$$

then we have the result that that the resource-less country is transitionally growing faster than the resource-rich economy. A result that would seem consistent with the results of Sachs and Warner (1995).

But to what degree is s_2^* affected by the natural resource and how? Our problem relies on the fact that in general it is not possible to solve for the value of the resource at time zero $q(0)$. If $q(0)$ is a relation of S_0, perhaps also of K_0, lets presume that such solution for $q(0)$ exists and denote this solution as

$$Q_0(S_0, K_0) \tag{54}$$

The effect of S_0 on s_2^* is given by

$$-\alpha\frac{\kappa_2(0)\,K(0)\,(1+q^*T^*)}{(K(0)+q(0)\,S(0))^2}q(0)\left(1+\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)}\right) \tag{55}$$

Proposition 5. If $s_2(t)$ is a monotonous function of time and if $\left(1+\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)}\right) > 0$, then the *nonrenewable resource enhances GDP growth of the resource-rich* economy.

Proof. Inspection of (55) shows that if $\left(1+\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)}\right) > 0$ holds the effect of S_0 on s_2^* is negative. Since

$$s_1^* = 1 - s_2^* \tag{56}$$

the effect of S_0 on s_1^* is positive■

Remark: We have thus established conditions whereby either empirical results of Sachs and Warner (1995) or the contrary results of Stijns (2001) are obtainable.

Unfortunately we can not claim that $\left(1+\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)}\right) > 0$ since we expect $\frac{\partial Q_0}{\partial S_0} < 0$. That is, as the stock of the resource increases we expect the price $q(0)$ to decline. While in general we are not able to prove that $\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)} > -1$, our numerical solutions confirm that $\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)} > -1$. But, our specific analytical solution indicates that $\frac{\partial Q_0}{\partial S_0}\frac{S(0)}{q(0)} = -\alpha$.

We next proceed to an analytical solution for the special case where $\alpha = \theta$..

4. Analytical solution

So far integration techniques have provide a general analytical solution. We now proceed with an analytical solution for the case where the parameter values are restricted, to $(\alpha = \theta)$. The first step is to determine the solution for this case reveals about evolution of the two economies.

Proposition 6. If $\alpha = \theta$ the price of the nonrenewable resource and aggregate consumption $(C \equiv c_1 + c_2)$ at time zero equal

$$q(0) = \frac{(1-\alpha)\,\alpha^{\alpha}}{(\rho - (1-\alpha)\,\eta)^{\alpha}} \left(\frac{K_0}{S_0}\right)^{\alpha} \qquad C(0) = K_0 \left(\delta \frac{1-\alpha}{\alpha} + \frac{\rho}{\alpha}\right) \quad (57)$$

and the rate of extraction and aggregate capital $(K \equiv K_1 + K_2)$ for all t equal

$$R(t) = \frac{\rho - (1-\alpha)\,\eta}{\alpha} S_0 e^{\frac{(1-\alpha)\eta - \rho}{\alpha} t} \qquad K(t) = K_0 \left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\alpha}} e^{-\frac{\rho}{\theta} t}. \quad (58)$$

Proof See appendix B■

Remark: Notice that

$$\frac{dQ_0}{dS_0} \frac{S_0}{q(0)} = -\alpha \quad (59)$$

Proposition 5 implies that with a sufficiently large initial endowment of the nonrenewable resource, the resource-rich economy can experience larger GDP growth rates than the resource-less economy. This result contradicts the results found by Rodríguez and Sachs (1999). This contradiction comes from the fact that Rodríguez and Sachs (1999) consider an isolated country, and thus they fail to take into account the inter-country linkages and the effects of these linkages on the world's transition dynamics.

From (57) we observe that aggregate consumption at time zero $(C(0))$ depends on the aggregate level of capital K_0, Consumption at time zero is not influenced by the initial level of the nonrenewable resource. In Appendix B, we show that for the transversality condition to hold, it must be that $\rho - (1-\alpha)\,\eta > 0$, which also guarantees $q(0) > 0$. Finally, the restriction $\alpha = \theta$ also implies that the rate of depletion of R declines at the negative constant rate of $\frac{(1-\alpha)\eta - \rho}{\alpha}$.

Proposition 7. If $\alpha = \theta$, the constant $\mu \left(\equiv \frac{c_1(t)}{c_2(t)}\right)$ equals

$$\mu = \frac{\frac{(1-\alpha)\alpha^{\alpha}}{(\rho-(1-\alpha)\eta)^{\alpha}} \left(K_{0,1} + K_{0,2}\right)^{\alpha} S_0^{1-\alpha} + K_{0,1}}{K_{0,2}} \tag{60}$$

Proof. Substituting $q(0)$ into $\frac{q(0)S_0+K_{0,1}}{K_{0,2}}$ the result is obtained.

Remark. From simple calculation one can obtain the effect of $K_{0,2}$ on μ. The effect of country two's initial capital endowment on μ is negative. Thus, the capital stock of country two negatively influences the consumption of the resource rich country *relative* to the resource less economy. Note that ratio $\frac{c_1}{c_2}$ is influenced by the stock of the resource at time zero. The effect of S_0 in μ equals

$$\frac{\partial \mu}{\partial S_0} = \frac{(1-\alpha)(1-\alpha)\alpha^{\alpha}}{(\rho-(1-\alpha)\eta)^{\alpha}} \frac{1}{K_{0,2}} \left(\frac{K_{0,1}+K_{0,2}}{S_0}\right)^{\alpha} \tag{61}$$

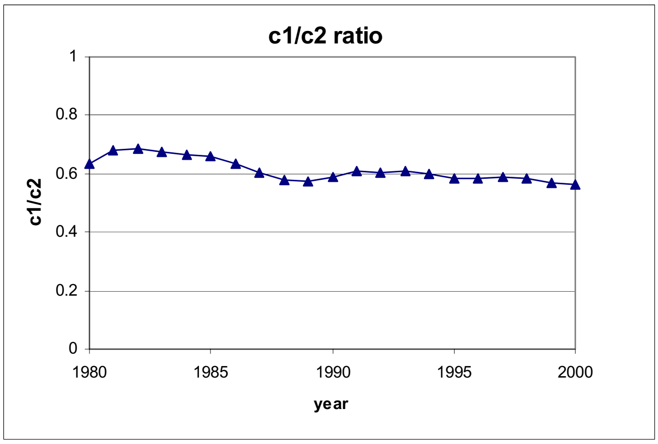
However, this effect is important to the extent by which the share of the resource is large. In particular, if $(1-\alpha)$ is small, the consumer of country one's benefit from owning the resource is relatively small. Our conclusion comes from observing that $(1-\alpha)$ is the exponent of the stock of resource $\left(S_0^{1-\alpha}\right)$. Thus, a relatively small $(1-\alpha)$ severely reduces the effect of the resource.

Interestingly, the consumption of country one (the resource-rich economy) relative to the consumption of country two (the resource-less economy) increases with the size of the growth rate of the resource saving technological change η.

5. Simulations

At this point our main concern is that the analytical solution's result presented in section 4, indicating that larger amounts of the resource positively affect the growth rate of the resource rich economy, may only be coincidental. To verify if this is the case we simulate the model under parameter values that are more consistent with other features of the data. First we want to verify if indeed the ratio $\frac{c_1}{c_2}$ is constant over time. Later we find initial endowments of the nonrenewable resource stock and capital that are consistent with $\frac{c_1}{c_2} = \frac{q(0)S_0+K_{0,1}}{K_{0,2}}$ as the model indicates. To this end we use the consumption levels in purchasing power parity obtained from the World Bank World Development Indicators (WDI, 2004) of all countries for which data is available for the period 1980-2000. To obtain consumption levels in purchasing power parity we have computed the share of consumption in total *GDP* for each country in the sample, and multiplied this share by *GDP* in purchasing power parity. When it was impossible to obtain the share of consumption on *GDP* from the WDI (2004) we have obtained consumption share data from the International Monetary Fund International Financial Statistics Yearbook (2002). Combining both sources of data allows us to include most of the non-oil-exporting and oil-exporting countries, including Saudi Arabia, Kuwait and Iran. To follow the model's setting as much as we can, we have divided the world into two regions, a nonrenewable-resource-rich region (countries whose oil exports are more than 25% of total merchandise exports) and a resource-poor region otherwise. Our sample includes 74 resource-poor[5] and 19 resource-rich countries. We have added the consumption level in purchasing power parity of the countries of each region and have divided this number by its respective population, and obtained what we called per-capita consumption of a resource rich (c_1) and resource poor (c_2) economy . Figure 1 shows the ratio $\mu = \frac{c_1}{c_2}$ from 1980-2000:

Figure 1

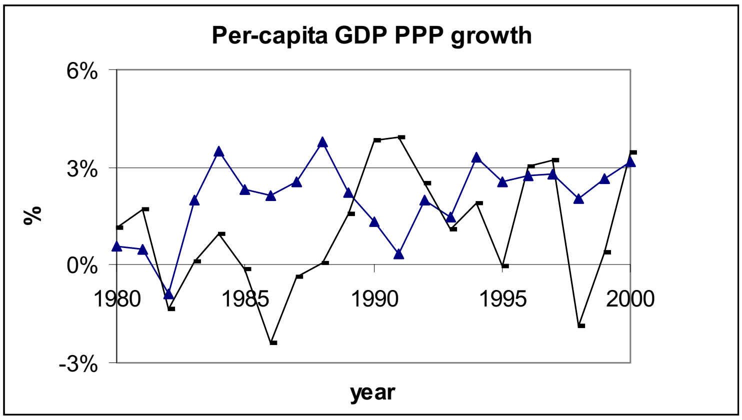


Most likely the larger ratio $\frac{c_1}{c_2}$ observed at the begging of the 80's relates to the oil crises experienced during that period. Interestingly, the ratio of the per-capita consumption levels remains within the 56-68% interval. In particular, during the 1990's this ratio only fluctuated between the 56-61% interval . For illustrative purposes Figure 2 shows the growth experience of the two regions. While indeed the oil-rich countries in average grew slower during the 1980-2000 period (the yearly average of the oil-rich economies was 1.1% versus 2% for the oil-poor economies). Since our objective is to verify if, in the absence of any distortion, owning larger amounts of the resource is sufficient to generate smaller or negative growth rates, as Rodriguez and Sachs indicate, Table 2 does not

[5] We have excluded South Africa due to its large mining activity.

speak against or in favor of any argument.

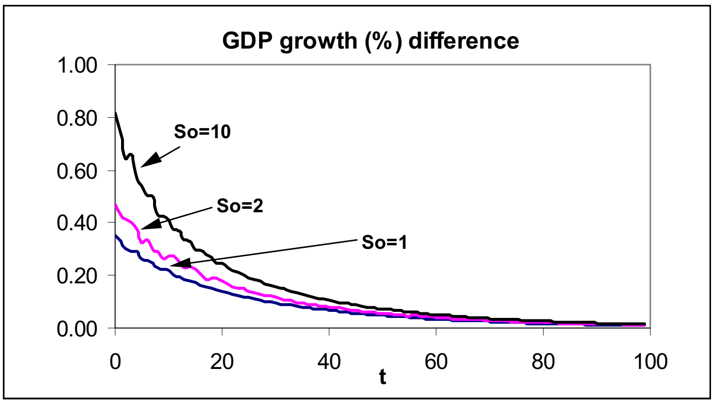
Figure 2



▲ nonrenewable-resource-poor region, − nonrenewable-resource-rich region

To perform our simulations we assume the following parameter values $\alpha = 0.82$, $\theta = 2.5$ (a relatively large number, but we want to depart from the value of α) $\delta = 0.04$, $\rho = 0.03$ and $\eta = 0.1$ (the value $\eta = 0.1$ generates a long run growth of 2.8% $\left(g_K = \frac{\eta - \rho}{\theta}\right)$). We then choose values for $S_0\,(=1)$, $K_{0,1}\,(=10)$, and $K_{0,2}\,(=90)$ that generate a ratio of $\frac{c_1}{c_2} = 0.62$. Our simulations consist of increasing the endowment of the nonrenewable resource as to see if indeed this can generate lower growth rates for the resource-rich region that the ones of the resource-poor region. In Figure 3 we have plotted the GDP growth difference between the resource rich and the resource poor country $\left(\frac{\dot{GDP_1}}{GDP_1} * 100 - \frac{\dot{GDP_2}}{GDP_2} * 100\right)$ for three different levels of the stock of the nonrenewable resource (1, 2 and 10). From Figure 3 we observe that increases in the endowment of the nonrenewable resource positively increase the difference between the growth rates, indicating that as the initial stock of the resource increases the more the resource-rich economy grows. We have performed many other simulations, that we do not report here, but in all the cases larger endowments of the resource do not affect negatively the GDP growth of the resource-rich country as our analytical solution indicates.

Figure 2 $\left(\frac{\dot{GDP_1}}{GDP_1} * 100 - \frac{\dot{GDP_2}}{GDP_2} * 100 \right)$



In Table 1 we also report how the price of the nonrenewable resource at time zero changes as a result of a change in the resource endowment.

Table 1

	S=1	S=2	S=10
q(0)	45.8833	23.3479	4.8472
μ	0.6200	0.6300	0.6497

From Table 1 we see that with an initial stock of the resource equal to one the price of the resource equals 45.88. If we double the stock of the resource to $S_0 = 2$ the price of the resource declines and equals 23.35. Note, however, that this decline is less than proportional to the increase of the stock of the resource, indicating that the elasticity of the price of the resource is negative and greater than minus one. This result is thus, consistent with Proposition 5 which indicates that if the elasticity of the price of the resource at time zero with regard to the stock of the resource is grater than minus one this is sufficient for the stock of the resource to positively influence the relative income growth rate of the resource-rich economy. In Table 1 we have also included the value of $\mu = \frac{c_1}{c_2}$ for the different levels of the initial stock of the resource considered. Our simulations indicates that the effect of the increase of the resource in relative welfare $(\frac{c_1}{c_2})$ is small but positive. This reflects the fact that larger endowments of the nonrenewable negatively affect the price of the resource and thus its end effect on μ is small.

6. Conclusions

This paper examines the impact of nonrenewable resources on economic growth and relative welfare in a two country world economy. We introduce a nonrenewable - resource - rich economy and a nonrenewable-less country in an otherwise standard neoclassical growth model. As in other models of trade, factor price equalization across countries is shown to result. In particular, the rental rate of capital across countries is equal even in the absence of international borrowing and lending. This result implies that the growth rates of consumption across countries are equal and therefore, the ratio of consumption levels is constant. We show that the ratio of the consumption of the resource-rich country to the consumption of the resource-less economy equals the ratio of the value of assets at any point in time. While the resource-rich country wealth increases with the initial stock of the resource, this effect is counterbalanced, to some degree, by the negative effect of the initial stock of the resource on the price of the nonrenewable.

We show that the initial endowment of the nonrenewable resource has a positive effect on the GDP growth rate of the *resource-rich* country as long as the elasticity of the initial price of the resource with regard to the initial stock of the resource is greater than minus one. An analytical solution of the model under a parameter restriction indicates that indeed this elasticity is greater than minus one. Thus, we find the result of Rodríguez and Sachs (1999), indicating that the initial stock of the resource influences negatively the GDP growth of an economy that exports a nonrenewable resource, does not to hold in general. This contradiction comes from the fact that Rodríguez and Sachs (1999) consider an isolated country, and thus they miss to account for any iteration of the economy they analyze and the rest of the world transitional dynamics.

Finally, we show that a technological change -that is saving on the nonrenewable resource- can benefit the resource-rich country welfare when compared to the resource-less economy.

References

Asheim, Geir B. (1986). Hartwick's Rule in open economies. The Canadian Journal of Economics, 19, 395-402.

Chiarella, Carl (1980). Trade between resource-poor and resource-rich economies as a differential game. Exhaustible Resources, Optimality, and Trade. Ed. M.C. Kemp and N.V. Long. North-Holland Publishing Company, 219-246.

Dasgupta, P., and G. Heal (1974). The Optimal Depletion of Exhaustible Resources. Review of Economic Studies (Symposium on the economics of exhaustible resources), 41, 3-28.

Ding and Field (2004) Natural resource abundance and economic growth. Department of Resource Economics, University of Massachusetts Amherst, Working paper no. 2004-7.

Geldrop, Jan H. van and Cees A. A. M. Withagen (1993). General equilibrium and international trade with exhaustible resources. Journal of International Economics, 34, 341-357.

Geldrop, Jan H. van and Cees A. A. M. Withagen (1994). General equilibrium in an economy with exhaustible resource and an unbounded horizon. Journal of Economic Dynamics and Control, 18, 1011-1035.

Hartwick, John M. (1995). Constant consumption paths in open economies with exhaustible resources. Review of International Economics, 3, 275-283.

International Monetary Fund (2002). International Financial Statistics Yearbook, Washington DC, USA.

Rodríguez, Francisco and Jeffrey D. Sachs (1999). Why do resource-abundant economies grow more slowly? Journal of Economic Growth, 4, 277-303.

Sachs, Jeffrey D. and Andrew M. Warner (1995). Natural resource abundance and economic growth. NBER Working Paper 5398.

Solow, Robert M. (1974). Intergenerational Equity and Exhaustible Resources. Review of Economic Studies (Symposium on the economics of exhaustible resources), 41, 29-45.

Stiglitz, Joseph E. (1974). Growth with Exhaustible Natural Resources: Efficient and Optimal Growth Paths. Review of Economic Studies (Symposium on the economics of exhaustible resources), 41, 123-37.

Stijns, Jean-Phillip (2001). Natural resource abundance and economic growth revisited Unpublished manuscript. Department of Economics, University of California Berkeley.

World Bank (2004). World Development Indicators CD.

Appendix A

Proof of Proposition 1.
Using (23) , and $r-\delta=\frac{\dot{q}(t)}{q(t)}$ the budget constraint of country two can be rewritten as

$$\frac{\dot{K}_2(t)}{q(t)}-\frac{\dot{q}(t)}{q(t)}\frac{K_2(t)}{q(t)}=-\left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\theta}}\frac{c_2(0)}{e^{\frac{\rho}{\theta}t}}\frac{1}{q(t)}. \tag{62}$$

Notice that the left hand side of expression (62) equals $\frac{d\frac{K_2}{q}}{dt}$, integrating (62) we obtain

$$\lim_{t\to\infty}\frac{K_2(t)}{q(t)}-\frac{K_2(\tau)}{q(\tau)}=-\frac{c_2(0)}{q(0)^{\frac{1}{\theta}}}\int_{\tau}^{\infty}\frac{q(t)^{\frac{1-\theta}{\theta}}}{e^{\frac{\rho}{\theta}t}}. \tag{63}$$

Employing the transversality condition (24) implies

$$K_2(\tau)=q(\tau)\frac{c_2(0)}{q(0)^{\frac{1}{\theta}}}\int_{\tau}^{\infty}\frac{q(t)^{\frac{1-\theta}{\theta}}}{e^{\frac{\rho}{\theta}t}}. \tag{64}$$

Since the ratio $\frac{c_1}{c_2}$ is constant for all t, let $c_1(t)=\mu c_2(t)$ where μ is a constant. (6) and (25) imply

$$R=\frac{r}{q}\frac{1-\alpha}{\alpha}(K_1+K_2). \tag{65}$$

The budget constraint of the consumer of country one can be rewritten as

$$\dot{K}_1-(r-\delta)K_1+\mu c_2=q\frac{r}{q}\frac{1-\alpha}{\alpha}(K_1+K_2). \tag{66}$$

Using $\frac{\dot{q}}{q}=r-\delta$, $\dot{S}=-R$, setting $c_2(t)=\left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\theta}}\frac{c_2(0)}{e^{\frac{\rho}{\theta}t}}$, dividing by q and integrating we get:

$$\begin{aligned}\int_{\tau}^{\infty}\left(\frac{\dot{K}_1(t)}{q(t)}-\frac{\dot{q}(t)}{q(t)}\frac{K_1(t)}{q(t)}+\frac{\mu}{q(t)}\left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\theta}}\frac{c_2(0)}{e^{\frac{\rho}{\theta}t}}\right)dt &= \int_{\tau}^{\infty}R(t)\,dt \\ &= S(\tau)-\lim_{t\to\infty}S(t).\end{aligned} \tag{67}$$

Since depletion is costless $\lim_{t\to\infty}S(t)=0$. Therefore, (67) can be written as

$$\int_{\tau}^{\infty}\frac{d\frac{K_1}{q}}{dt}dt+\mu\frac{c_2(0)}{q(0)^{\frac{1}{\theta}}}\int_{\tau}^{\infty}\frac{q(t)^{\frac{1-\theta}{\theta}}}{e^{\frac{\rho}{\theta}t}}dt=S(\tau) \tag{68}$$

or

$$\lim_{t\to\infty}\frac{K_1(t)}{q(t)}-\frac{K_1(\tau)}{q(\tau)}+\mu\frac{c_2(0)}{q(0)^{\frac{1}{\theta}}}\int_{\tau}^{\infty}\frac{q(t)^{\frac{1-\theta}{\theta}}}{e^{\frac{\rho}{\theta}t}}dt=S(\tau). \tag{69}$$

Using the transversality condition (equation 24) and rearranging we obtain

$$\mu q(\tau) \frac{c_2(0)}{q(0)^{\frac{1}{\theta}}} \int_{\tau}^{\infty} \frac{q(t)^{\frac{1-\theta}{\theta}}}{e^{\frac{\rho}{\theta}t}} dt = q(\tau) S(\tau) + K_1(\tau). \tag{70}$$

From (64) we yield

$$q(\tau) \frac{c_2(0)}{q(0)^{\frac{1}{\theta}}} \int_{\tau}^{\infty} \frac{q(t)^{\frac{1-\theta}{\theta}}}{e^{\frac{\rho}{\theta}t}} = K_2(\tau), \tag{71}$$

and therefore,

$$\mu = \frac{q(\tau) S(\tau) + K_1(\tau)}{K_2(\tau)}. \tag{72}$$

■

Proof of Proposition 2

First notice from (5), (12) and (13) that the long-run growth rate of q and the long-run value of r equal

$$\lim_{t\to\infty} \frac{\dot{q}}{q} = \eta = g_q, \qquad \lim_{t\to\infty} r = \eta + \delta \equiv r^* \tag{73}$$

hence, r is constant in the long run. Thus, the per country growth rate of consumption is constant in the long run and given by

$$\lim_{t\to\infty} \frac{\dot{c}_i}{c_i} = \lim_{t\to\infty} \frac{r - \delta - \rho}{\theta} = \frac{\eta - \rho}{\theta} = g_{c_i} \tag{74}$$

The first order conditions of the final good sectors in each country (equation (6)) and the market clearing condition for R (equation (25)) imply

$$R = \frac{1-\alpha}{\alpha} \frac{r}{q} (K_1 + K_2) \qquad \Rightarrow \qquad \frac{\alpha}{1-\alpha} q \frac{R}{K} = r \tag{75}$$

where $K = K_1 + K_2$. Since r is constant in the long run, (75) indicates that in the long run the growth rate of K and R must satisfy the following relation

$$\lim_{t\to\infty} \frac{\dot{q}}{q} + \lim_{t\to\infty} \frac{\dot{R}}{R} - \lim_{t\to\infty} \frac{\dot{K}}{K} = 0 \qquad \Rightarrow \qquad \lim_{t\to\infty} \frac{\dot{K}}{K} - \lim_{t\to\infty} \frac{\dot{R}}{R} = \eta \tag{76}$$

Next notice that $\dot{S} = -R$, thus the growth rate of S equals the ratio $\frac{\dot{S}}{S} = -\frac{R}{S}$. Since extraction is costless, as time goes to infinity both R[6] and S approach zero. Using L'Hôpital's rule we have that

$$\lim_{t\to\infty}\frac{\dot{S}}{S} = \lim_{t\to\infty} -\frac{R}{S} = \lim_{t\to\infty} -\frac{\dot{R}}{\dot{S}} = \lim_{t\to\infty}\frac{\dot{R}}{R}. \tag{77}$$

Implying that as $t \to \infty$ both R and S grow at the same rate, (this result was previously derived by Dasgupta and Heal, 1974). What remains to show is that $g_{c_i} = g_{K_i}$. Observe that (75) implies that $q\frac{R}{K}$ is constant in the long run, since in the long run S and R grow at the same rate, then, it is also the case that $\chi \equiv \frac{qS}{K}$ is constant in the long run. Taking the log time derivative of χ, using $\dot{S} = -R$, and $qR = \frac{1-\alpha}{\alpha}Kr$ we obtain

$$\frac{\dot{\chi}}{\chi} = \dot{q}\frac{S}{K}\frac{1}{\chi} + \frac{q\dot{S}}{K}\frac{1}{\chi} - \frac{qS}{K}\frac{\dot{K}}{K}\frac{1}{\chi}. \tag{78}$$

Taking the limit we obtain

$$\lim_{t\to\infty}\frac{\dot{\chi}}{\chi} = 0 = \eta - \lim_{t\to\infty}\frac{1-\alpha}{\alpha}\frac{r}{\chi} - \lim_{t\to\infty}\left(r - \delta + r\frac{1-\alpha}{\alpha} - \frac{C}{K}\right) \tag{79}$$

where $C = c_1 + c_2$. Since both c_1 and c_2 grow at the same rate, then C also grows at the same rate as c_i. Notice that since χ is constant in the long run $\left(\frac{\dot{\chi}}{\chi} = 0\right)$, thus, for (79) to hold it must be that $\frac{C}{K}$ is constant in the long run and therefore, C and K must asymptotically grow at the same rate. Therefore, $g_{c_i} = g_K$. Employing (76) we obtain

$$\lim_{t\to\infty}\frac{\dot{R}}{R} = \frac{\eta(1-\theta)-\rho}{\theta} = g_R$$

For the transversality condition (10) to hold it must also be the case that $g_R = g_S < 0$. Finally, since $K = K_1 + K_2$ then it is straightforward to show that

$$g_{K_i} = g_K = g_{c_i}\blacksquare$$

Proof of Proposition 3. The Jacobian matrix J^* equals

$$J^* = \begin{pmatrix} -\frac{1-\alpha}{\alpha}r^* & 0 & 0 \\ \hat{g}_q & 0 & -\frac{g^*}{T^{*2}}\frac{r^*}{q^*}\frac{1-\alpha}{\alpha} \\ \hat{T}_q & (1+\mu)T^{*2} & (1+\mu)g^*T^* - g_S \end{pmatrix} \tag{80}$$

[6] For the tranversality condition (10) to hold it must be that $\lim_{t\to\infty} R = 0$

The eigenvalues of J^* are the values of ξ that solve the characteristic equation

$$\left(-\frac{1-\alpha}{\alpha}r^* - \xi\right)\left(-\xi\left(\left(1+\mu\right)g^*T^* - g_S - \xi\right) + \frac{r^*}{q^*}\frac{1-\alpha}{\alpha}\left(1+\mu\right)g^*\right) = 0 \tag{81}$$

One of the solution to (81) is given by

$$\xi_1 = -\frac{1-\alpha}{\alpha}r^* < 0. \tag{82}$$

The other two are values of ξ that solve the quadratic equation in ξ given by

$$\xi^2 - \xi\left(\left(1+\mu\right)g^*T^* - g_S\right) + \frac{r^*}{q^*}\frac{1-\alpha}{\alpha}\left(1+\mu\right)g^* = 0 \tag{83}$$

or

$$\xi^2 + b\xi + d = 0 \tag{84}$$

where $b = -\left(\left(1+\mu\right)g^*T^* - g_S\right)$ and $d = \frac{r^*}{q^*}\frac{1-\alpha}{\alpha}\left(1+\mu\right)g^*$. Using g^* from (35) we obtain

$$\left(\frac{\eta+\delta\left(1-\alpha\right)}{\alpha} - g_K\right) = \left(1+\mu\right)T^*g^*. \tag{85}$$

Substituting T^* from (36) into (85) we get

$$\frac{r^*}{\hat{q}^*}\frac{1-\alpha}{\alpha}\left(1+\mu\right)g^* = \left(\eta - g_K\right)\left(\frac{\eta+\delta\left(1-\alpha\right)}{\alpha} - g_K\right). \tag{86}$$

Substituting (85) into (86) yields

$$\frac{r^*}{\hat{q}^*}\frac{1-\alpha}{\alpha}\left(1+\mu\right)g^* = \left(\eta - g_K\right)\left(1+\mu\right)T^*g^* \Rightarrow \frac{r^*}{\hat{q}^*}\frac{1-\alpha}{\alpha} = \left(\eta - g_K\right)T^*. \tag{87}$$

From (27) it is the case that $\left(\eta - g_K\right) = -g_S$. Using (87) notice that d can be rewritten as

$$d = -g_S\left(1+\mu\right)g^*T^* \tag{88}$$

Thus the other two eigenvalues of J^* equal

$$\frac{-b \pm \sqrt{D}}{2} \tag{89}$$

where $D = ((1+\mu)\, g^* T^* - g_S)^2 + 4 g_S (1+\mu)\, g^* T^*$. Further manipulation of D yields

$$\begin{aligned} D &= (1+\mu)^2 (g^* T^*)^2 + 2 g_S (1+\mu)\, g^* T^* + g_S^2 \\ &= ((1+\mu)\, g^* T^* + g_S)^2 \end{aligned} \tag{90}$$

substituting into (89)

$$\frac{-b \pm ((1+\mu)\, g^* T^* + g_S)}{2} \tag{91}$$

the remaining eigenvalues of J^* equal

$$\xi_2 = -g_S \qquad\qquad \xi_3 = (1+\mu)\, g^* T^*$$

which are positive■

Appendix B

Proof of Proposition 6. Here we show that $q(0) = \frac{(1-\alpha)\alpha^\alpha}{(\rho-(1-\alpha)\eta)^\alpha} \left(\frac{K_0}{S_0}\right)^\alpha$ when $\alpha = \theta$. Since $R = \frac{1-\alpha}{\alpha} \frac{r}{q} (K_1 + K_2)$. To determine R we need to find a solution for aggregate capital $K = K_1 + K_2$. Since $\dot{K} = \dot{K}_1 + \dot{K}_2$ using both budget constant and denoting $C = c_1 + c_2$, $\dot{K}$ equals

$$\dot{K} = (r - \delta) K + \frac{1-\alpha}{\alpha} r K - C, \tag{92}$$

which is a first order differential equation with a variable coefficient. Using $\frac{\dot{q}}{q} = r - \delta$, its general solution is given by

$$K(t) = K_0 e^{\int_0^t \frac{r(\xi)}{\alpha} - \delta d\xi} - \int_0^t C(\tau)\, e^{\int_\tau^t \frac{r(\xi)}{\alpha} - \delta d\xi} d\tau \tag{93}$$

$$= K_0 \left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\alpha}} e^{\delta\left(\frac{1-\alpha}{\alpha}\right) t} - \int_0^t C(\tau) \left(\frac{q(t)}{q(\tau)}\right)^{\frac{1}{\alpha}} e^{\delta\left(\frac{1-\alpha}{\alpha}\right)(t-\tau)} d\tau. \tag{94}$$

Using $\alpha = \theta$ and $C(t) = \left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\theta}} \frac{C(0)}{e^{\frac{\rho}{\theta}t}}$ from (23) we get

$$\begin{aligned} K(t) &= K_0 \left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\alpha}} e^{\delta\left(\frac{1-\alpha}{\alpha}\right)t} - \frac{C(0)}{q(0)^{\frac{1}{\theta}}} \int_0^t q(t)^{\frac{1}{\alpha}} \frac{q(\tau)^{\frac{1}{\theta}}}{q(\tau)^{\frac{1}{\alpha}}} \frac{e^{\delta\left(\frac{1-\alpha}{\alpha}\right)(t-\tau)}}{e^{\frac{\rho}{\theta}\tau}} d\tau \\ &= K_0 \left(\frac{q(t)}{q(0)}\right)^{\frac{1}{\alpha}} e^{\delta\left(\frac{1-\alpha}{\alpha}\right)t} + \frac{C(0)}{q(0)^{\frac{1}{\theta}}} \frac{q(t)^{\frac{1}{\alpha}}\left(e^{-\frac{\rho}{\theta}t} - e^{\delta\left(\frac{1-\alpha}{\alpha}\right)t}\right)}{\left(\delta\left(\frac{1-\alpha}{\alpha}\right) + \frac{\rho}{\theta}\right)}. \end{aligned} \tag{95}$$

Using $r = \left(\frac{\alpha^\alpha (1-\alpha)^{1-\alpha} e^{(1-\alpha)\eta t}}{q^{1-\alpha}}\right)^{\frac{1}{\alpha}}$ from (5), we obtain that the ratio $\frac{r}{q}$ equals

$$\frac{r}{q} = \frac{\alpha(1-\alpha)^{\frac{1-\alpha}{\alpha}} e^{\left(\frac{1-\alpha}{\alpha}\right)\eta t}}{q^{\frac{1}{\alpha}}}. \tag{96}$$

Using $R = \frac{1-\alpha}{\alpha}\frac{r}{q}K = \frac{(1-\alpha)^{\frac{1}{\alpha}} e^{\left(\frac{1-\alpha}{\alpha}\right)\eta t}}{q^{\frac{1}{\alpha}}}K$, $\int_0^\infty R dt = S_0$ and (95) we obtain

$$\begin{aligned} \int_0^\infty R(t)\,dt &= (1-\alpha)^{\frac{1}{\alpha}} \int_0^\infty \left(\frac{K_0}{q(0)^{\frac{1}{\alpha}}} e^{\frac{1-\alpha}{\alpha}(\eta+\delta)t} + \frac{C(0)}{q(0)^{\frac{1}{\theta}}} \frac{\left(e^{\left(\frac{1-\alpha}{\alpha}\eta - \frac{\rho}{\theta}\right)t} - e^{\frac{1-\alpha}{\alpha}(\eta+\delta)t}\right)}{\frac{1-\alpha}{\alpha}\delta + \frac{\rho}{\theta}} \right) dt \\ &= \frac{(1-\alpha)^{\frac{1}{\alpha}}}{q(0)^{\frac{1}{\alpha}}} \lim_{t\to\infty} \left(\frac{\alpha\left(e^{\frac{1-\alpha}{\alpha}(\eta+\delta)t} - 1\right)}{(\eta+\delta)(1-\alpha)} \left(K_0 - \frac{C(0)}{\frac{1-\alpha}{\alpha}\delta + \frac{\rho}{\theta}}\right) + \frac{C(0)\left(e^{\left(\frac{1-\alpha}{\alpha}\eta - \frac{\rho}{\theta}\right)t} - 1\right)}{\left(\frac{1-\alpha}{\alpha}\eta - \frac{\rho}{\theta}\right)\left(\frac{1-\alpha}{\alpha}\delta + \frac{\rho}{\theta}\right)} \right). \end{aligned} \tag{97}$$

For (97) to converge to the constant S_0 it must be that

$$C(0) = K_0\left(\delta\frac{1-\alpha}{\alpha} + \frac{\rho}{\theta}\right). \tag{98}$$

For the transversality condition to hold $\left(\frac{1-\alpha}{\alpha}\eta - \frac{\rho}{\theta}\right) < 0$ must be satisfied, this implies that (97) converges. Solving for $q(0)$ we get,

$$q(0) = \frac{(1-\alpha)\alpha^\alpha}{(\rho - (1-\alpha)\eta)^\alpha}\left(\frac{K_0}{S_0}\right)^\alpha \tag{99}$$

also

$$
\begin{aligned}
K(t) &= K_0 \left(\frac{q(t)}{q(0)} \right)^{\frac{1}{\alpha}} e^{\delta\left(\frac{1-\alpha}{\alpha}\right)t} - \frac{C(0)}{q(0)^{\frac{1}{\theta}}} \int_0^t q(t)^{\frac{1}{\alpha}} \frac{q(\tau)^{\frac{1}{\theta}}}{q(\tau)^{\frac{1}{\alpha}}} \frac{e^{\delta\left(\frac{1-\alpha}{\alpha}\right)(t-\tau)}}{e^{\frac{\rho}{\theta}\tau}} d\tau \\
&= K_0 \left(\frac{q(t)}{q(0)} \right)^{\frac{1}{\alpha}} e^{-\frac{\rho}{\theta}t} = q(t)^{\frac{1}{\alpha}} \frac{(\rho - (1-\alpha)\eta)}{e^{\frac{\rho}{\theta}t}(1-\alpha)^{\frac{1}{\alpha}}\alpha} S_0. \qquad (100)
\end{aligned}
$$

Substituting (100) into $R(t) = \frac{1-\alpha}{\alpha}\frac{r(t)}{q(t)}K(t)$ and the solution for $q(0)$ we get

$$
R = \frac{1-\alpha}{\alpha}\frac{r}{q}K = \frac{\rho - (1-\alpha)\eta}{\alpha} S_0 e^{\frac{(1-\alpha)\eta - \rho}{\alpha}t}. \qquad (101)
$$

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