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PATH PROPERTIES OF SIMULATION SCHEMES FOR THE HESTON STOCHASTIC VOLATILITY MODEL

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Path properties of simulation schemes for the Heston stochastic volatility model

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Abstract

The aim of this study is to evaluate some simulation schemes recently suggested for the Heston model by examining their ability in reproducing, on the simulated paths, the autocovariance function of the generated model, when discretely observed. This is done by applying the outcomes of previous research where, based on discrete equi-spaced observations of the log-price, we determined an approximate confidence band for the theoretical autocovariance function of the *mean variance* process.

1 Introduction

It is now well-known that stochastic volatility models allow the volatility to randomly change over time and appear to better resemble some stylized facts in financial markets than simpler models as Black and Scholes (1973). More precisely, such models are able to describe the leptokurtosis which is typical of most financial returns and to explain the classical *smile* form in the graphical representation of the Black-Scholes implied volatility against the strike prices available in the market for the option. These empirical facts are well documented, among others, in Cont (2001).

A renewed attention has been recently devoted to the Heston stochastic volatility model (Heston, 1993) for which a quasi-closed formula is available for computing the price of European *plain vanilla* options, making it possible to calibrate model parameters to market option prices (see Duffie et al., 2000); several numerical techniques have been also introduced to improve further the computation of the quasi-closed formula of the Heston model

and so its calibration to market data (e.g. Kahl, Jackel, 2005, and Lord, Kahl, 2008).

However, the statistical estimation of the parameters involved in the Heston model, as well as in more general stochastic volatility models, is a difficult issue due to the non-observability of the volatility process and it is still a subject of on-going research. Many of the estimation methods recently applied to such models make extensive use of simulations (e.g. Andersen, Lund, 1997, Sorensen, 2003 and Zhang, Shu 2003), and the choice of a good and fast scheme is crucial for doing inference on model parameters. In addition, the Heston model has become a benchmark in the theory of stochastic volatility models, hence new estimators for both spot and integrated volatility are often evaluated on simulated trajectories of the process (see, for istance, Renò, 2006, and Ogawa, Sanfelici, 2008). In order to achieve a reliable comparison of such estimators, simulated paths should of course resemble the theoretical properties of the generated model.

New simulations/dicretization schemes for stochastic volatility models have been proposed by several authors in the last few years, with a special attention to the Heston model (1993). As far as we know, the empirical comparisons of existing simulation methods evaluate each scheme by means of the Root Mean Squared Error (or a similar distance measure) between *plain vanilla* option prices obtained with the quasi-closed formula and those derived with the Monte Carlo method; the smaller this value, the better the scheme. However, this measure only depends on the terminal value of the simulated paths and not at all on the path properties of the trajectores generated by one or the other procedure. We believe instead that properties of the model trajectories may be of great interest for inference purposes on parameters as well as for the pricing of some exotic path-dependent derivatives.

One of the major issues when modelling financial returns is the serial dependence structure for the log-returns and its powers. It is widely accepted that the autocorrelation function of financial log-returns odd powers is not significant while it is very high for even powers. One possible explanation of this feature is the persistence in the volatility (variance) of financial stocks.

Since we are interested in applying the Heston model for financial purposes (pricing, risk management) the aim of this paper is to evaluate the performance of several discretization algorithms in reproducing the autocovariance structure of the simulated model. We focus our attention on the Eulero-Maruyama classical scheme (see Kloeden, Platen, 1992) with the Full truncation fix as suggested in Lord et al. (2008) and to the algorithms proposed recently by Kahl, Jackel (2006) and Andersen (2008). The numerical

results show a better performance, from our viewpoint, of the schemes by Andersen (2008) and Lord et al. (2008).

In order to achieve our scope we use the outcomes of previous research providing confidence bands for the autocovariance function of the *mean variance* process in the Heston model which are based on discrete observations of the price process (see Figà-Talamanca, 2008). The discretization schemes under analysis are then compared on the basis of several outcomes: the graphical representation of the autocovariance functions of the sample paths, superimposing the theoretical counterpart and confidence bands, the number of violations of the confidence bands for each lag and a global RMSE measuring the distance between the theoretical autocovariance and the empirical autocovariance functions and, finally, the computational time.

Sections 2 and 3 sum-up the model specification, the expression for the theoretical autocovariance function of the *mean variance* process for which approximate confidence bands are derived, based on discrete equi-spaced observations of the log-price process. Section 4 is devoted to a brief description of the discretization and simulation algorithms under consideration and Section 5 describes the numerical results and how they are obtained. Concluding remarks are finally reported in Section 6.

2 The Model

The Heston stochastic volatility model (Heston, 1993) describes the dynamics of the price S_t of a given stock through the following bi-dimensional stochastic differential equation (SDE):

$$\frac{dS_t}{S_t} = \mu dt + \sigma_t dZ_t, \tag{1}$$

$$d\sigma_t^2 = \alpha(\beta - \sigma_t^2)dt + c\sigma_t dW_t, \tag{2}$$

where α and β are positive parameters representing respectively the mean-reversion speed and the long-run mean for the instantaneous variance, c is the so-called volatility of volatility and (B, W) is a possibly correlated Brownian motion in $\mathbb{R}^2$. This specification is a very interesting example in financial literature since it provides a closed formula for computing European Option Prices[1]. In the remainder of this paper we consider the following simplified process for the log-price $Y_t = \log S_t$:

[1]This is actually a quasi-closed formula since some integrals involved in the expression are evaluated numerically.

$$dY_t = \sigma_t dZ_t, \qquad Y_0 = 0 \tag{3}$$
$$d\sigma_t^2 = \alpha(\beta - \sigma_t^2)dt + c\sigma_t dW_t, \qquad \sigma_0^2 = \nu. \tag{4}$$

where (Z, W) is a Brownian motion such that $\langle dZ_t, dW_t \rangle = \rho dt$ and ν is a random variable, independent from (Z, W), which has the stationary distribution of the process.

Note that a strong solution exists for the above SDE if $c > 0$, and the origin is unattainable if $2\alpha\beta \geq c^2$; the process also admits a $Gamma(\frac{2\alpha\beta}{c^2}, \frac{2\alpha}{c^2})$ stationary distribution with finite moments of any order given by

$$E[\nu^p] = \left(\frac{c^2}{2\alpha}\right)^p \frac{\Gamma(\frac{2\alpha\beta}{c^2} + p)}{\Gamma(\frac{2\alpha\beta}{c^2})}.$$

As a special case,

$$E[\sigma_0^2] = \beta, \quad Var(\sigma_0^2) = \frac{\beta c^2}{2\alpha}.$$

3 Moments of the discretely observed process

Assume that $Y_1, Y_2, ..., Y_n$ are n equally spaced observations for the process in the first equation of (3) and denote with Δ the fixed observation step. For $i = 1, 2, ..., n$, define the variables $R_i = Y_i - Y_{i-1}$ and $X_i = \frac{R_i}{\sqrt{\Delta}}$. By extending the results of Genon-Catalot et al. (2000), Figà-Talamanca (2008) prove that the process $\{X_i\}_{i=1,2,...,n}$ is stationary and ergodic. Moreover, conditionally on the σ-algebra $\mathcal{F}$ generated by $\left\{\sigma_s^2, s \geq 0\right\}$, the distribution function of $(X_1, X_2, ..., X_n)$ is n-dimensional centered Gaussian with covariance matrix $\Sigma = diag(\overline{V_1}, \overline{V_2}, ...\overline{V_n})$, where

$$\overline{V_i} = \frac{1}{\Delta} \int_{(i-1)\Delta}^{i\Delta} \sigma_s^2 ds, \text{ for } i = 1, 2, ...n, \tag{5}$$

is the *mean variance* in the interval $[(i-1)\Delta, i\Delta[$. If the process in (3) models the dynamics of the log-price of a financial stock, then the quantities R_i, X_i are respectively referred to as the log-returns and the scaled log-returns of the stock; the integral part in (5) is often defined as the *integrated* or *actual volatility* of the stock.

3.1 Second order structure of the mean variance

Under some technical assumptions it is possible to derive closed expressions for the second order moments of the *mean variance* process in terms of model parameters (see Genon-Catalot et al., 2000, Sorensen , 2000, and Barndorff-Nielsen, Shephard, 2001, 2002) which write as

$$E[\overline{V_1}] = \beta,$$

$$E\left[\overline{V_1}^2\right] = \beta^2 + Var(\sigma_0^2)\frac{2(\alpha\Delta - 1 + \exp(-\alpha\Delta))}{\alpha^2\Delta^2}, \tag{6}$$

$$E\left[\overline{V_1 V_{1+h}}\right] = \beta^2 + Var(\sigma_0^2)\exp(-h\alpha\Delta)\frac{\exp(\alpha\Delta)(1-\exp(-\alpha\Delta))^2}{\alpha^2\Delta^2}.$$

Given the above formulas, the theoretical autocovariance function γ_h of the *mean variance* process is

$$\gamma_h = \begin{cases} \frac{Var(\sigma_0^2)}{\Delta^2\alpha^2}\exp(-h\alpha\Delta)\frac{\exp(\alpha\Delta)(1-\exp(-\alpha\Delta))^2}{\alpha^2\Delta^2}, \text{ for } h \geq 1, \\ \\ \frac{2}{\alpha^2\Delta^2}Var(\sigma_0^2)(\alpha\Delta - 1 + \exp(-\alpha\Delta)), \text{ for } h = 0. \end{cases} \tag{7}$$

3.2 Limit results for sample moments of the scaled return

Let us introduce the following sample moments of the scaled log-returns $\{X_i\}_{i=1,2,\ldots,n}$

$$\begin{aligned} \widehat{\beta}^{(n)} &= \frac{1}{n}\sum_1^n X_i^2, \\ M_0^{(n)} &= \frac{1}{3n}\sum_1^n X_i^4 \\ M_h^{(n)} &= \frac{1}{n-h}\sum_1^{n-h} X_i^2 X_{i+h}^2, \text{ for } h \geq 1. \end{aligned} \tag{8}$$

The empirical autocovariance, for $h \geq 0$, is thus defined as

$$\widehat{\gamma}_h^{(n)} = M_h^{(n)} - \widehat{\beta}^2.$$

A simple application of the outcomes in Figà-Talamanca (2008) gives, for $h \geq 1$,

$$\begin{aligned} \widehat{\beta}^{(n)} &\underset{n\to+\infty}{\overset{a.s.}{\to}} \beta, \\ M_0^{(n)} &\underset{n\to+\infty}{\overset{a.s.}{\to}} E\left[\overline{V}_1^2\right], \\ M_h^{(n)} &\underset{n\to+\infty}{\overset{a.s.}{\to}} E\left[\overline{V}_1\overline{V}_{1+h}\right], \end{aligned} \tag{9}$$

and

$$\sqrt{n}\begin{pmatrix} \widehat{\beta}^{(n)} - \beta \\ M_0^{(n)} - E\left[\overline{V}_1^2\right] \\ M_h^{(n)} - E\left[\overline{V}_1\overline{V}_{1+h}\right] \end{pmatrix} \overset{Law}{\longrightarrow} N\left(0, \Sigma(h)\right). \tag{10}$$

The detailed expressions of the entries of the covariance matrix $\Sigma(h)$ is postponed to next subsection.

The following limits are a straightforward consequence of (9, 10) and of the so-called delta-method (see Lehmann, 1998, Theorem 5.2.3 and Corollary 5.4.3.):

1. $\widehat{\gamma}_h^{(n)} \underset{n\to+\infty}{\overset{a.s.}{\to}} \gamma_h$, for $h \geq 0$;

2. if $\hat{\beta}$ is known ($\hat{\beta} \equiv \beta$), then

$$\sqrt{n}\left(\widehat{\gamma}_0^{(n)} - \gamma_0\right) \underset{n\to+\infty}{\overset{law}{\longrightarrow}} N(0, \Sigma_{22}),$$

$$\sqrt{n}\left(\widehat{\gamma}_h^{(n)} - \gamma_h\right) \underset{n\to+\infty}{\overset{law}{\longrightarrow}} N(0, \Sigma_{33}(h)), \quad \text{for} \quad h \geq 1,$$

otherwise

$$\sqrt{n}\left(\widehat{\gamma}_0^{(n)} - \gamma_0\right) \underset{n\to+\infty}{\overset{law}{\longrightarrow}} N(0, \tau_0^*(\beta, E\left[\overline{V_1}^2\right])),$$

$$\sqrt{n}\left(\widehat{\gamma}_h^{(n)} - \gamma_h\right) \underset{n\to+\infty}{\overset{law}{\longrightarrow}} N(0, \tau_h^*(\beta, E\left[\overline{V_1 V_{1+h}}\right])),$$

where $\tau_0^*(u,v) = 4u^2\Sigma_{11} - 4u\Sigma_{12} + \Sigma_{22}(h)$ and $\tau_h^*(u,v) = 4u^2\Sigma_{11} - 4u\Sigma_{13}(h) + \Sigma_{33}(h)$.

For a detailed derivation see Figà-Talamanca (2008).

Hence, for a given confidence level p, and given a large sample $\{x_i\}_{i=1,2,\ldots,n}$ of scaled log-returns, the p-level approximate confidence bands for the sample autocovariance function in h, respectively for $\hat{\beta} \equiv \beta$ or else, are

$$[\gamma_0 - z_{\frac{1-p}{2}}\sqrt{\Sigma_{22}/n}, \gamma_0 + z_{\frac{1-p}{2}}\sqrt{\Sigma_{22}/n}], \text{ and}$$
$$[\gamma_0 - z_{\frac{1-p}{2}}\sqrt{\tau_0^*(\beta, E\left[\overline{V_1}^2\right])/n}, \gamma_0 + z_{\frac{1-p}{2}}\sqrt{\tau_0^*(\beta, E\left[\overline{V_1}^2\right])/n}],$$

and, for $h \geq 1$,

$$[\gamma_h - z_{\frac{1-p}{2}}\sqrt{\Sigma_{33}(h)/n}, \gamma_h + z_{\frac{1-p}{2}}\sqrt{\Sigma_{33}(h)/n}], \text{ and}$$
$$[\gamma_h - z_{\frac{1-p}{2}}\sqrt{\tau_h^*(\beta, E\left[\overline{V_1}\overline{V_{1+h}}\right])/n}, \gamma_h + z_{\frac{1-p}{2}}\sqrt{\tau_h^*(\beta, E\left[\overline{V_1}\overline{V_{1+h}}\right])/n}],$$

with $z_{\frac{1-p}{2}}$ implicitly defined by $\overline{\Phi}(z_{\frac{1-p}{2}}) = \frac{1-p}{2}$, where $\overline{\Phi}$ is the survival distribution function of a standard Gaussian random variable.

3.3 Computing the entries of matrix $\Sigma(h)$

Once parameter values are assigned, the entries of matrix $\Sigma(h)$ can be computed as a function of model parameters. Proposition 2 in Figà-Talamanca (2008) leads to

$$\Sigma_{11} = 3E[\overline{V}_1^2] - E[\overline{V}_1]^2 + 2\sum_{i=1}^{\infty}\left(E[\overline{V}_1\overline{V}_{1+i}] - E[\overline{V}_1]^2\right),$$

$$\Sigma_{22} = \frac{35}{3}E[\overline{V}_1^4] - E[\overline{V}_1^2]^2 + 2\sum_{i=1}^{\infty}\left[E[\overline{V}_1^2\overline{V}_{1+i}^2] - E[\overline{V}_1^2]^2\right],$$

and

$$\begin{aligned}\Sigma_{33}(h) &= 9E[\overline{V}_1^2\overline{V}_{1+h}^2] - E[\overline{V}_1\overline{V}_{1+h}]^2 + 4E[\overline{V}_1\overline{V}_{1+h}^2\overline{V}_{1+2h}] \\ &+ 2\sum_{i=1}^{\infty}(E[\overline{V}_1\overline{V}_{1+h}\overline{V}_{1+i}\overline{V}_{1+h+i}] - E[\overline{V}_1\overline{V}_{1+h}]^2).\end{aligned}$$

Moreover,

$$\Sigma_{12} = 5E[\overline{V}_1^3] - E[\overline{V}_1]E[\overline{V}_1^2] + \sum_{i=1}^{\infty}\left(E[\overline{V}_1\overline{V}_{1+i}^2] - 2E[\overline{V}_1]E[\overline{V}_1^2] + E[\overline{V}_{1+i}\overline{V}_1^2]\right),$$

$$\begin{aligned}\Sigma_{13}(h) &= 3E[\overline{V}_1^2\overline{V}_{1+h}] + 2E[\overline{V}_1\overline{V}_{1+h}^2] - E[\overline{V}_1]E[\overline{V_1V}_{1+h}] \\ &+ \sum_{i=1}^{\infty}(E[\overline{V}_1\overline{V}_{1+i}\overline{V}_{1+h+i}] - 2E[\overline{V}_1]E[\overline{V}_1\overline{V}_{1+h}] + E[\overline{V}_1\overline{V}_{1+i}\overline{V}_{1+h}]),\end{aligned}$$

and

$$\begin{aligned}\Sigma_{23}(h) &= 5E[\overline{V}_1^3\overline{V}_{1+h}] + 4E[\overline{V}_1\overline{V}_{1+h}^3] - 3E[\overline{V}_1^2]E[\overline{V}_1\overline{V}_{1+h}] \\ &+ \sum_{i=1}^{\infty}(E[\overline{V}_1^2\overline{V}_{1+i}\overline{V}_{1+h+i}] - E[\overline{V}_1^2]E[\overline{V}_1\overline{V}_{1+h}]) \\ &+ \sum_{i=1}^{\infty}(E[\overline{V}_1\overline{V}_{1+i}^2\overline{V}_{1+h}] - E[\overline{V}_1^2]E[\overline{V}_1\overline{V}_{1+h}]).\end{aligned}$$

The numerical computation of the above asymptotic variances-covariances involves the third and the fourth order structure of the *mean variance* process. These moments and cross-moments can be calculated as definite third order integrals of $E[\sigma_t^2\sigma_r^2\sigma_s^2]$, for $t < r < s$ and fourth order integrals of $E[\sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2]$, for $t < r < s < w$ (see Appendix A) which can be expressed in terms of model parameters (see Appendix B). Thus, the third and fourth order structure moments of the *mean variance* can also be numerically evaluated once parameters are assigned. Computations, performed with *Mathematica 5.1,* lead to very long formulas which are not reported here.

4 Discretization Schemes

We want to generate a sample $\{(Y_1, \sigma_1^2), (Y_2, \sigma_2^2), ...(Y_n, \sigma_n^2)\}$ of equi-spaced observations, with time-step Δ, for the log-price Y_t and its instantaneous variance σ_t^2 according to the process in (3, 4); denote $Y_i = Y_{i\Delta}$ and $\sigma_i^2 = \sigma_{i\Delta}^2$ for the sake of simplicity. In principle, any discretization of the process may be avoided by applying the Exact Simulation Scheme introduced by Broadie, Kaya (2006). However, this procedure introduces a non-negligible numerical error due to the approximation of an Inverse Fourier transform for which the integrand has a very oscillatory behaviour; in order to generate n observations $\{Y_t\}_{t=1,2,...n}$ of the log-price process, for m scenarios, the Inverse Fourier Transform, the expression of which depends on the ratio of Bessel functions of the first kind,, has to be computed $n \times m$ times and then approximated (for instance with the trapeizoidal rule). Hence, this scheme

is extremely costly in term of computational time as the authors themselves, Smith (2007), Lord et al. (2008) and Andersen (2008) have also remarked and is practically unfeasible for our purposes since n and m should be very large; so we won't apply it in our numerical experiment. The same remark applies for the modification of this scheme given by Smith (2007).

We describe in what follows the Eulero-Maruyama scheme as well as those by Kahl, Jackel (2006) and Andersen (2008). The former is an example of a second order scheme and the latter of a moment-matching algorithm. We also briefly introduce the simulation scheme by Zhu (2008) which is based on a transformation of the variance process.

4.1 The Eulero-Maruyama scheme and the choice of the "fixing method"

The Eulero-Maruyama discretization of the process in (3, 4) gives:

$$\begin{aligned} Y_i &= Y_{i-1} + \sigma_{i-1}\left(\rho\left(W_i - W_{i-1}\right) + \sqrt{1-\rho^2}\left(B_i - B_{i-1}\right)\right), \\ \sigma_i^2 &= \sigma_{i-1}^2 + \alpha(\beta - \sigma_{i-1}^2) + c\sigma_{i-1}\left(W_i - W_{i-1}\right). \end{aligned}$$

for a discretization interval of length Δ.

Note that we have no guarantees that the discretized process σ_t^2 remains non-negative. If at step k the variance σ_t^2 becomes negative, it is not possible to define the process at step $k+1$ since its square root σ_t appears in both equations for Y_t and σ_t^2. By following the approach in Lord et al. (2008) we rewrite the discretization scheme as:

$$\begin{aligned} Y_i &= Y_{i-1} + f_4(\sigma_{i-1})\left(\rho\left(W_i - W_{i-1}\right) + \sqrt{1-\rho^2}\left(B_i - B_{i-1}\right)\right), \\ \sigma_i^2 &= f_1(\sigma_{i-1}^2) + \alpha(\beta - f_2(\sigma_{i-1}^2))\Delta + \varphi(f_3(\sigma_{i-1}))\left(W_i - W_{i-1}\right). \end{aligned}$$

Assuming $f_4 = f_3$, we report here Table 1 of Lord et al. (2008) to describe different fixes.

Lord et al. (2008) conclude that the best choice in term of a trade-off between accuracy and computational time is the Full-Truncation fix. In our numerical examples we take advantage of these results by selecting the Full Truncation fix for the application of the Eulero-Maruyama scheme.

Fix	Paper	$f_1(x)$	$f_2(x)$	$f_3(x)$
Absorption	Unknown	x^+	x^+	x^+
Reflection	Diop (03), Berkaoui et al. (05)	$\|x\|$	$\|x\|$	$\|x\|$
Higham-Mao	Higham, Mao (05)	x	x	$\|x\|$
Partial Truncation	Deelstra, Delbaen (98)	x	x	x^+
Full Truncation	Lord et al. (068)	x	x^+	x^+

Table 1: Several "fixes" for the Eulero-Maruyama discretization scheme (Lord et al. 2008)

4.2 The Kahl-Jackel scheme

Kahl, Jackel (2006) examine in their study many discretization schemes for univariate processes which include the instantaneous variance diffusion in (4) as a special case.

In their study they claim that for the Heston model the best simulation scheme is obtained by applying a second order scheme (the Balanced Milstein Method) to the variance and what they call the IJK scheme to the log-price process. The choice for the balancing functions make their discretization scheme for the instantaneous variance essentially an Implicit Milstein Method. More precisely, if $\delta W_i = W_i - W_{i-1}$,

$$\sigma_i^2 = \frac{\sigma_{i-1}^2 + \alpha\beta\Delta + c\sigma_{i-1}\delta W_i + \frac{1}{4}c^2\delta W_i^2 - \frac{1}{4}c^2\Delta}{(1+\alpha\Delta)},$$

and

$$Y_i = Y_{i-1} + \rho\sigma_{i-1}dW + \frac{1}{2}\sqrt{1-\rho^2}(\sigma_i + \sigma_{i-1})\left(B_i - B_{i-1}\right) + \frac{1}{4}c\rho\left(\delta W_i^2 - \Delta\right).$$

We refer to this scheme as the IJK- IMM algorithm. Note that this scheme guarantees the positivity of the variance process if $4\alpha\beta > c^2$; when this constraint is not fulfilled and the discretized variance process becomes negative we use the Full Truncation Eulero-Maruyama scheme to get the -step ahead variance and insert the positive part of the variance in the the log-price discretization as suggested in Andersen (2008).

4.3 The efficient schemes of Andersen

In the paper by Andersen (2008) the author proposes essentially two schemes, the Truncated Gaussian (TG) and the Quadratic Exponential (QE) algorithms which are both in the family of the moment-matching schemes (the

parameters involved are obtained in order to match the first two moment of the instantaneous variance). The former scheme is based on the observation that the non central chi-squared distribution converges to a Gaussian distribution if the non-centrality parameter is large. The latter arise from the fact that a non central chi-squared random variables with a large non-centrality parameter can be well represented by a power function of a Gaussian variable (see Andersen, 2008, and the references therein); for small values of the non-centrality parameter the author suggests a possible adjustment of the scheme. A switching rule determines whether the adjusment is in order or not.

4.3.1 Truncated-Gaussian (TG)

Define once and for all the function $r(x)$ implicitly as

$$r(x)\phi\left(r(x)\right)+\Phi(r(x))(1+r^2(x))=(1+x)\left(\phi\left(r(x)\right)+r(x)\Phi(r(x))\right),$$

and set

$$\begin{aligned} f_\mu(x) &= \frac{r(x)}{\phi(r(x))+r(x)\Phi(r(x))}, \\ f_\upsilon(x) &= \frac{x^{-1/2}}{\phi(r(x))+r(x)\Phi(r(x))}, \end{aligned}$$

where Φ and φ are respectively the cumulative distribution function and the density of a standard Gaussian random variable.

Using the fact that the distribution of σ_i^2 given σ_{i-1}^2 is, up to a scale factor δ, a non central chi-squared distribution (see Cox. et al, 1985) with non centrality parameter λ and d degrees of freedom with

$$\begin{aligned} \delta &= \frac{c^2(1-e^{-\alpha\Delta})}{4\alpha} \\ \lambda_i &= \frac{e^{-\alpha\Delta}}{\delta}\sigma_{i-1}^2, \text{ and} \\ d &= \frac{4\alpha\beta}{c^2}, \end{aligned} \tag{11}$$

the procedure to obtain σ_i^2 from σ_{i-1}^2 can be summarized as follows:

1. Given σ_{i-1}^2, compute m_{i-1} and S_{i-1}^2 as follows:

$$\begin{aligned} m_{i-1} &= \beta+(\sigma_{i-1}^2-\beta)e^{-\alpha\Delta}, \\ S_{i-1}^2 &= \frac{c^2e^{-\alpha\Delta t}}{\alpha}(1-e^{-\alpha\Delta t})\sigma_{i-1}^2+\frac{\beta c^2}{2\alpha}(1-e^{-\alpha\Delta})^2. \end{aligned}$$

2. Compute $\psi_{i-1} = \frac{S_{i-1}^2}{m_{i-1}^2}$, $\mu_{i-1} = f_\mu(\psi_{i-1})m_{i-1}$ and $\upsilon_{i-1} = f_\upsilon(\psi_{i-1})S_{i-1}$.

3. Generate a Standard Gaussian random number Z_V.

4. Set $\sigma_i^2 = (\mu_{i-1} + \upsilon_{i-1}Z_V)^+$.

4.3.2 Quadratic-Exponential (QE)

This scheme is designed to take into account the behaviour of the distribution of σ_i^2 also when σ_{i-1}^2 approaches 0 making the non centrality parameter in (11) small. It is based on a switching rule to obtain σ_i^2 that splits the scheme in two different algorithms respectively applied for large and small values of σ_{i-1}^2. The scheme can be summarized as follows:

1. Given σ_{i-1}^2, compute m_{i-1} and S_{i-1}^2 as follows:

$$\begin{aligned} m_{i-1} &= \beta + (\sigma_{i-1}^2 - \beta)e^{-\alpha\Delta}, \\ S_{i-1}^2 &= \frac{c^2 e^{-\alpha\Delta}}{\alpha}(1 - e^{-\alpha\Delta})\sigma_{i-1}^2 + \frac{\beta c^2}{2\alpha}(1 - e^{-\alpha\Delta})^2. \end{aligned}$$

   and set $\psi_{i-1} = \frac{S_{i-1}^2}{m_{i-1}^2}$.

2. Generate a uniform random number on the unit interval, U_V and fix a value $\psi_{\max} \in [1, 2]$ to determine the switching rule.

3. If $\psi_{i-1} \leq \psi_{\max}$, compute b_{i-1}, a_{i-1}such that

$$\begin{aligned} b_{i-1}^2 &= 2\psi_{i-1}^{-1} + \sqrt{2\psi_{i-1}^{-1}(2\psi_{i-1}^{-1} - 1)}, \\ a_{i-1} &= \frac{m_{i-1}}{1 + b_{i-1}^2}. \end{aligned}$$

   Set $Z_V = \Phi^{-1}(U_V)$, where Φ is the cumulative distribution function of a standard Gaussian variable, and $\sigma_i^2 = a_{i-1}(b_{i-1} + Z_V)^2$.

4. If $\psi_{i-1} > \psi_{\max}$, compute

$$\begin{aligned} p_{i-1} &= \frac{\psi_{i-1} - 1}{\psi_{i-1} + 1}, \text{ and} \\ \eta_{i-1} &= \frac{1 - p_{i-1}}{m_{i-1}}. \end{aligned}$$

Set

$$H^{-1}(u;p,\eta) = \begin{cases} 0 & if \quad 0 \le u \le p \\ \eta^{-1}\log\frac{1-p}{1-u} & if \quad p < u \le 1 \end{cases}, \text{ and}$$
$$\sigma_i^2 = H^{-1}(U_V; p_{i-1}, \eta_{i-1}).$$

4.3.3 Simulation of the log-price process

In Andersen (2008) the author points out some drawbacks in the application of the Eulero-Maruyama discretization scheme for the log-price process Y_t and suggests to write the log-price process in integral form between time $(i-1)\Delta$ and $i\Delta$, as in Broadie, Kaya (2006), which gives

$$Y_i = Y_{i-1} + \rho\int_{(i-1)\Delta}^{i\Delta}\sigma_s dW_s + \sqrt{1-\rho^2}\int_{(i-1)\Delta}^{i\Delta}\sigma_s dB_s, \tag{12}$$
$$\sigma_i^2 = \sigma_{i-1}^2 + \alpha\beta\Delta - \alpha\int_{(i-1)\Delta}^{i\Delta}\sigma_s^2 ds + c\int_{(i-1)\Delta}^{i\Delta}\sigma_s dW_s. \tag{13}$$

Then the following approximation is considered

$$\int_{(i-1)\Delta}^{i\Delta}\sigma_u^2 du = \Delta\overline{V_i} \simeq \Delta(\gamma_1\sigma_{i-1}^2 + \gamma_2\sigma_i^2),$$

where γ_1 and γ_2 are constant parameters. The Eulero-Maruyama setting is recovered if $\gamma_1 = 1, \gamma_2 = 0$ while a central discretization is obtained if we set $\gamma_1 = \gamma_2 = 0.5$. Since B_t is independent of V_t, then the stochastic integral $\int_{(i-1)\Delta}^{i\Delta}\sigma_u dB_u$, conditional on V_{i-1} and $\int_{(i-1)\Delta}^{i\Delta}\sigma_u^2 du$, is centered Gaussian with variance $\int_{(i-1)\Delta}^{i\Delta}\sigma_u^2 du$. Hence,

$$Y_i = Y_{i-1} + \frac{\rho}{c}(\sigma_i^2 - \sigma_{i-1}^2 - \alpha\beta\Delta) + \frac{\rho\alpha}{c}\Delta(\gamma_1\sigma_{i-1}^2 + \gamma_2\sigma_i^2) + \sqrt{(1-\rho^2)\Delta(\gamma_1\sigma_{i-1}^2 + \gamma_2\sigma_i^2)}Z, \tag{14}$$

where Z is a standard Gaussian variable.

Andersen (2008) compares the methods he proposes with the IJK-IMM scheme and with the FT scheme having in mind, as usual, the ability in obtaining, via the Monte Carlo method, a good estimate of the *plain vanilla* option prices available in closed form. His conclusions are towards a better performance of the QE algorithm with respect to the competitors and we exploit his results not considering the TG algorithm in our numerical simulations.

4.4 From variance to volatility: a look at Zhu scheme

Consider $\sigma_t = \sqrt{\sigma_t^2}$. Applying Ito's Lemma to the variance process in (4) we obtain

$$d\sigma_t = \left(\frac{1}{4\sigma_t}(4\alpha\beta - c^2) - \frac{\alpha}{2}\sigma_t \right) dt + \frac{c}{2} dW_t.$$

This transformation is obviously possible if the variance process stays away of the origin in order to have the twice differentiability of the square-root function needed for the application of Ito's Lemma. The simulation scheme of Zhu (2008) is essentially based on the discretization/simulation of the volatility according to the above process. In Lord et al. (2008) we found enough motivations to refrain us from applying this method in the remainder of our analysis.

5 Numerical comparison of the discretization schemes

In order to check whether a simulation scheme matches the theoretical properties of the discretely observed original process we compute the autocovariance function of each path: if the scheme produces trajectories which are consistent with the original data we expect this autocovariance function to be next to its theoretical counterpart (the autocovariance of the *mean variance* process). In addition, if we simulate m paths, we should not statistically reject consistency, at a confidence level p, if the autocovariance of lag h lies within the $1-p$ approximate confidence band for at least $(1-p) * m$ paths. Our analysis has been thus carried on looking at several outputs:

1. The graphical representation of the autocovariance functions for all simulated trajectories as well as the theoretical autocovariance and its approximate confidence bands, for lags between 0 and H.

2. The number of paths violating the confidence band, computed for each lag $h \leq H$.

3. The "distance" between the theoretical autocovariance and the empirical autocovariance, which is measured by the global Root Mean Squared Error over all paths and for lags between 0 and H:

$$RMSE = \sqrt{\frac{1}{(H+1)m} \sum_{h=0}^{H} \sum_{j=1}^{m} (\widehat{\gamma_h(j)} - \gamma_h)^2},$$

where $\widehat{\gamma_h(j)}$ is the empirical autocovariance of the squared scaled log-returns at lag h for the $j^{th}-$path.

We also measure the computational time required by each method to perform our simulations, though this depends, of course, on the optimization of the codes which is not the main concern of this study.

According to the results in the papers by Andersen (2008), Lord (2006) and Kahl, Jackel (2006), we choose to apply, among the available choices, the Full-Truncation Eulero-Maruyama, the Kahl-Jackel IJK-IMM and the Andersen QE scheme. The algorithms are written as *Matlab* codes.

5.1 Simulation/Discretization program

We simulate $m = 1000$ paths of 3000 observations, the first 500 of which are discarded to accomplish for the stationarity of the process. We used 10 parameter assignments, as detailed in Table 2, corresponding to three sets (S_1, S_2, S_3) for the variance process parameters (α, β, c) with three different values for the correlation ρ, and to a fourth assignment S_4 of the the whole set of parameters which was also considered in Andersen (2008). Once S_1 is assigned, the sets S_2 and S_3 are designed to test respectively how the outcomes of the analysis change with an increase in the mean reversion speed or in the volatility of volatility parameter; the three possible values for the correlation are those considered in Kahl, Jackel (2006). In addition, as Andersen (2008), among others, pointed out, parameter estimates for Heston model often do not fulfill the constraint which guarantees the positivity of the variance process; so, the set S_3 and S_4, where the origin is attainable, are considered in order to verify the robustness of the properties of the simulation algorithms with respect to this constraint.

For each set of parameters, data are generated for an observation step $\Delta \in \left\{\frac{1}{3000}, \frac{1}{250}, \frac{1}{32}, \frac{1}{8}\right\}$ where the first two values corresponds to half-an-hour and daily observations, mostly used in financial applications; the other two values are considered as a benchmark since the results of Andersen (2008), Kahl-Jackel (2006) and Lord et al. (2008) are based on a similar scale of Δ.

5.2 Empirical evidence

Figures 1 and 2 plot the empirical autocovariance function for each of the simulated paths for the parameter set S_1 with correlation $\rho = 0, -0.8$ respectively. Lags are between 0 and 20 and $n = 2500$. From left to right

α	1	1	1	3	3	3	1	1	1	1
β	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.05	0.09
c	0.25	0.25	0.25	0.25	0.25	0.25	0.5	0.5	0.5	1
ρ	0	-0.4	-0.8	0	-0.4	-0.8	0	-0.4	-0.8	-0.3

Table 3: Parameter Assignments

the autocovariance is plotted for the trajectories simulated with the Eulero-Maruyama FT scheme, with the IJK-IMM scheme and the QE algorithm respectively. From top to bottom outcomes are shown for decreasing values of the observation step; the limit autocovariance and the approximate confidence bands at a 5% level are superimposed. Confidence bands are computed for each case with the formulas reported in Section 3.4 for the case of β known. There is no evidence from the figures against one of the discretization schemes since the majority of the paths lie within confidence bands. It seems that the number of violations increases with the observation step and with the absolute correlation value. The results for $n = 500$ are analogous as well as those for the parameter set S_2 .

In Figure 3 we report the graphs for parameter set S_3 with $\rho = -0.4$ for which we observe a significant number of violations especially for the Eulero-Maruyama and the IJK-IMM schemes. Besides, the outcomes for the set of parameters S_4 are summed up in Figure 4.

It is clear from our outcomes that, for all parameter assignments, the number of violations of the confidence bands decreases with an increase in the observation frequency while it increases with the absolute correlation index as well as with the volatility of volatility parameter.

However, it becomes statistically significant only for parameters assignments S_3 and S_4 and for the two higher values of Δ. In these cases the Quadratic Exponential algorithm and the Full-Truncation fix for the Eulero-Maruyama discretization scheme give more robust results with respect to the IJK-IMM scheme of Kahl and Jackel (2006). In particular, all schemes perform similarly for S_4 while for the assignment S_3 the best results are achieved by the simple FT algorithm when $\rho = -0.8$ and by the QE scheme when $\rho = -0.4, \rho = 0$. As an example we report in Table 3 the number of violations in this latter case for $h \leq 10,\ n = 2500$ and $\Delta = \frac{1}{8}, \frac{1}{32}$.

Concerning our third point, in Figure 5 the Root Mean Squared Error is reported according to assignments S_2 (a), S_3 (b) and S_4(c). The RMSE value is reported in blue circles for the Eulero-Maruyama scheme, in green stars

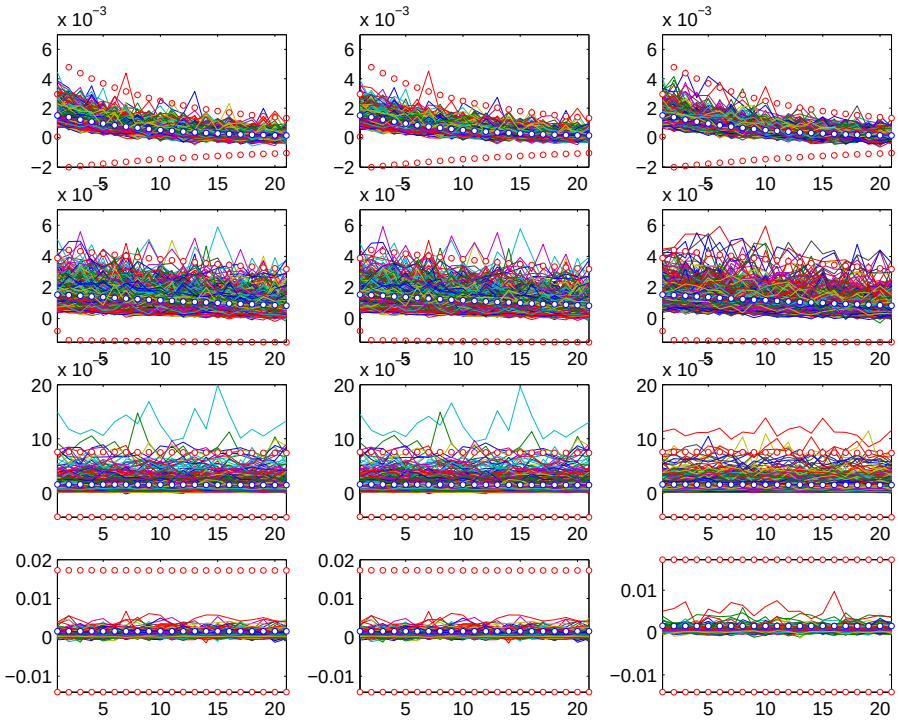

Figure 1: The empirical autocovariance functions of simulated paths (solid lines) and its theoretical value and confidence bands (white circles) for SET 1 of assigned parameters with $\rho = 0$, for $n = 2500$ and for $\Delta = \frac{1}{8}, \frac{1}{32}, \frac{1}{250}, \frac{1}{3000}$ (from top to bottom). From left to right the Full truncated Eulero-Maruyama scheme, the Kahl-Jackel implicit scheme and the Andersen Quadratic-Exponential algorithm are used to perform simulations.

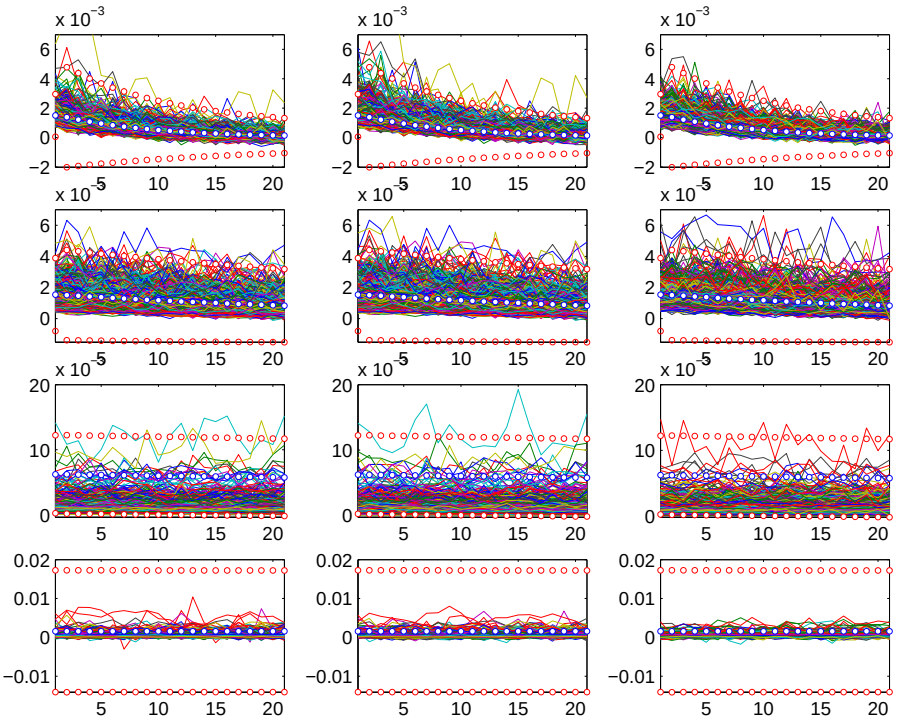

Figure 2: The empirical autocovariance functions of simulated paths (solid lines) and its theoretical value and confidence bands (white circles) for SET 1 of assigned parameters with $\rho = -0.8$, for $n = 2500$ and for $\Delta = \frac{1}{8}, \frac{1}{32}, \frac{1}{250}, \frac{1}{3000}$ (from top to bottom). From left to right the Full truncated Eulero-Maruyama scheme, the Kahl-Jackel implicit scheme and the Andersen Quadratic-Exponential algorithm are used to perform simulations.

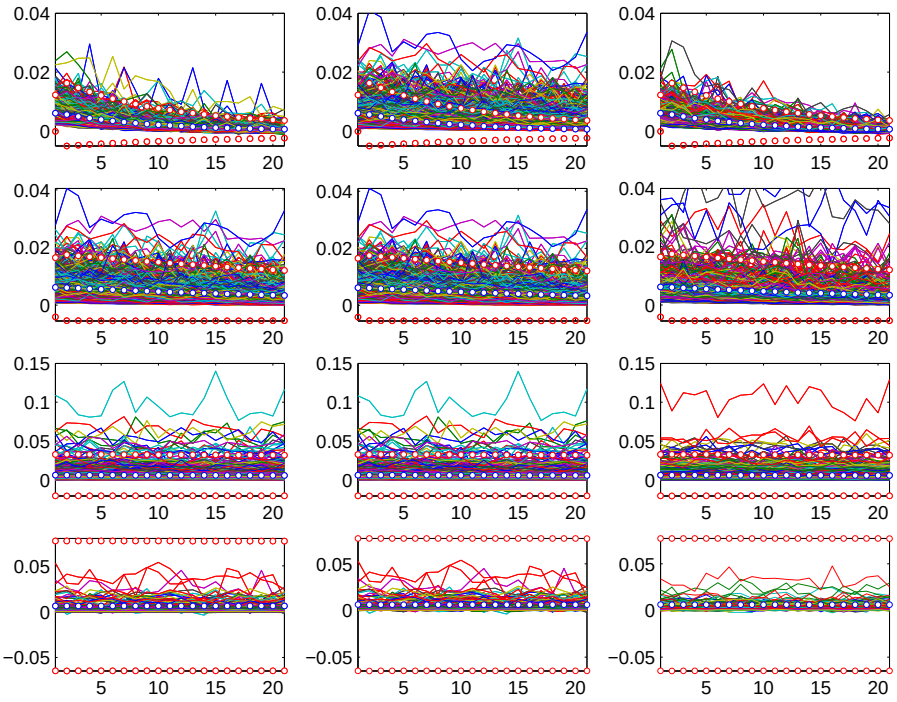

Figure 3: The empirical autocovariance functions of simulated paths (solid lines) and its theoretical value and confidence bands (white circles) for SET 3 of assigned parameters with $\rho = -0.4$, for $n = 2500$ and for $\Delta = \frac{1}{8}, \frac{1}{32}, \frac{1}{250}, \frac{1}{3000}$ (from top to bottom). From left to right the Full truncated Eulero-Maruyama scheme, the Kahl-Jackel implicit scheme and the Andersen Quadratic-Exponential algorithm are used to perform simulations.

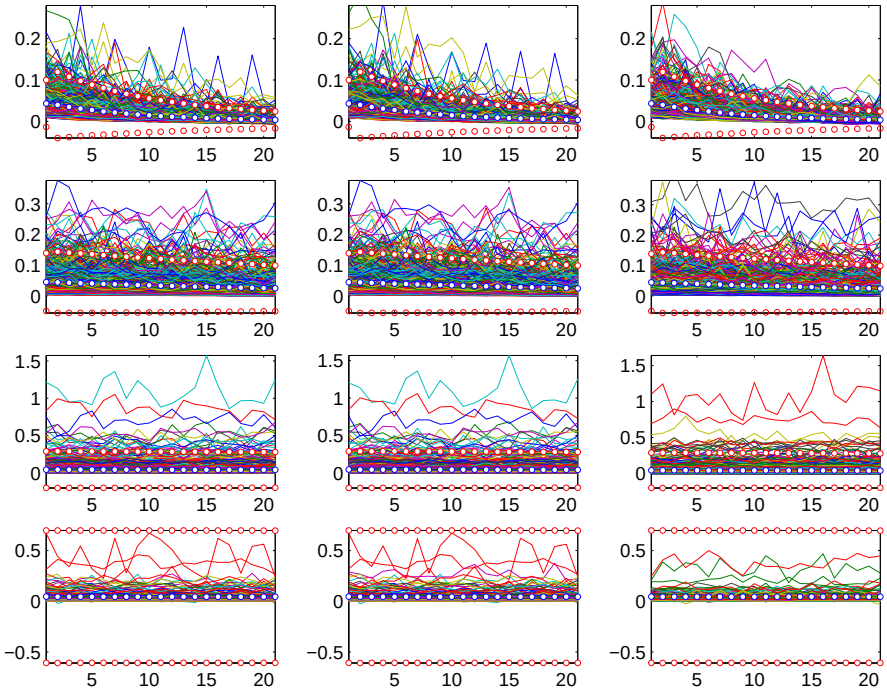

Figure 4: The empirical autocovariance functions of simulated paths (solid lines) and its theoretical value and confidence bands (white circles) for SET 4, $n = 2500$ and $\Delta = \frac{1}{8}, \frac{1}{32}, \frac{1}{250}, \frac{1}{3000}$ (from top to bottom). From left to right the Full truncated Eulero-Maruyama scheme, the Kahl-Jackel implicit scheme and the Andersen Quadratic-Exponential algorithm are used to perform simulations.

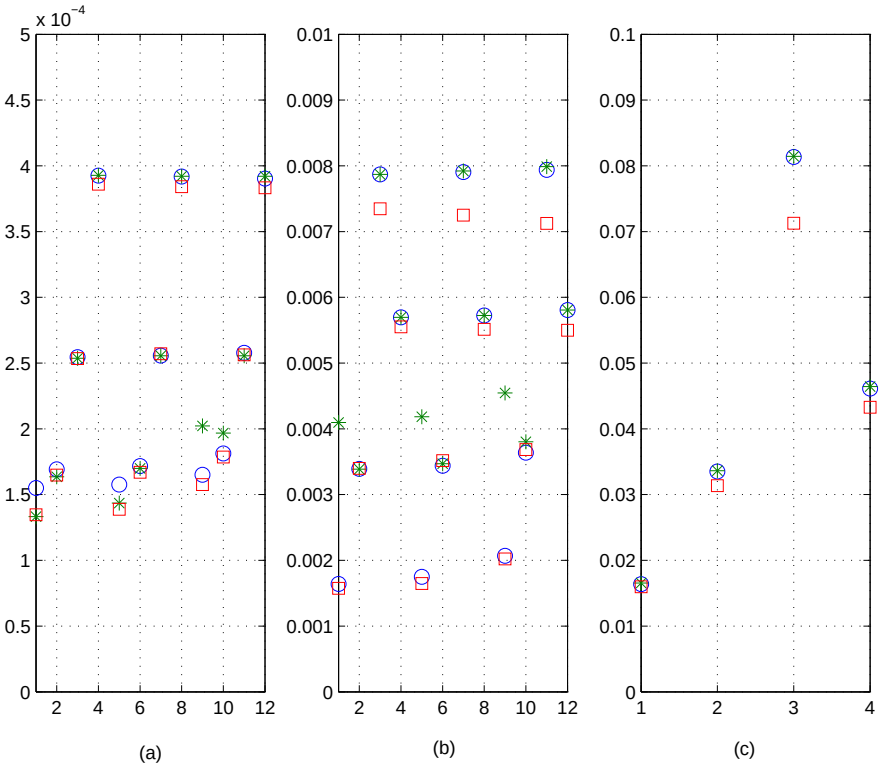

Figure 5: Root Mean Squared Error computed globally for the FT scheme (blue circles), the IJK-IMM scheme (green stars) and the QE algorithm (red squares). RMSE values are given for 12 cases in S_2 (a) and S_3 (b) with decreasing observation step and for all three values of the correlation index; in (c) the RMSE value is plotted for parameter assignment $S_4$with decreasing observation step.

Lag		**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**	**10**
FT	$\Delta = \frac{1}{8}$	**93**	**32**	**40**	**41**	**44**	**46**	**46**	**42**	**31**	**41**	**43**
	$\Delta = \frac{1}{32}$	**71**	**59**	**50**	**44**	**56**	**53**	**52**	**51**	**53**	**54**	**56**
IJK-IMM												
	$\Delta = \frac{1}{8}$	**110**	**68**	**70**	**72**	**93**	**106**	**112**	**131**	**128**	**144**	**155**
	$\Delta = \frac{1}{32}$	**64**	**58**	**47**	**48**	**54**	**53**	**54**	**50**	**53**	**55**	**58**
QE												
	$\Delta = \frac{1}{8}$	**73**	**42**	**34**	**44**	**40**	**42**	**36**	**41**	**39**	**40**	**37**
	$\Delta = \frac{1}{32}$	**49**	**50**	**39**	**49**	**45**	**43**	**44**	**44**	**48**	**42**	**46**

Table 4: Number of paths violating the autocovariance confidence bands for lags between 0 and 10, $n = 2500$ and $\Delta = \frac{1}{8}, \frac{1}{32}$. From top to bottom the Full-Truncation, the IJK-IMM scheme and the QE algorithm for $(\alpha, \beta, c) = (1, 0.05, 0.5)$ and $\rho = 0$.

for the IJK-IMM scheme and in red squares for the QE algorithm. In the first two plots we have 12 points for each scheme: the first four correspond to a decreasing observation step $\Delta = \frac{1}{8}, \frac{1}{32}, \frac{1}{250}$ and $\frac{1}{3000}$ in the $\rho = 0$ case. The second and the third group of four values correspond to $\rho = -0.4$ and $\rho = -0.8$ respectively. In plot (c) the RMSE is reported for S_4 once again according to decreasing values of Δ.

By looking at the minimum values of the RMSE in Figure 5, it seems that for all the assignments the QE algorithm performs the best for $n = 2500$. We do not show any result for $n = 500$ since we found no evidence for a different performance of the considered simulation methods. Nevertheless, it is worth noticing that in this latter case we observed an overall improvement in the performance of the FT scheme for the high negative correlation case.

Computational time does not vary significantly across the different parameter sets considered and the overall fastest scheme is the Eulero-Maruyama FT. For the IJK-IMM scheme the time needed is doubled and for the QE algorithm the computational time is five times more than for the FT. However, let us remark that a proper optimization of the codes, on which we have not focused our attention, may reduce substancially the time required by the QE algorithm.

6 Concluding remarks

In this paper we have given a comparison, based on the autocovariance function of the simulated paths, of some of the available and very recent simulation algorithms for the Heston model. According to the results in the papers by Andersen (2008), Lord et al. (2008) and Kahl, Jackel (2006), we have chosen to apply, among the existing schemes, the Full Truncation Eulero-Maruyama of Lord et al. (2008), the IJK-IMM procedure of Kahl, Jackel (2006) and the Quadratic Exponential algorithm of Andersen (2008).

At first, the serial dependence of the original model, when discretely observed, has been derived and consistent and an asymptotic normal estimators of the *mean variance* autocovariance structure have been introduced which are based on observations of the scaled log-returns. The asymptotic properties of this estimator have been used to obtain an approximate confidence band for the autocovariance structure of the squared scaled log-returns. Then, we have compared the simulations schemes by looking at several outcomes of our study: the graphical representation of the autocovariance functions of the sample paths where the theoretical version and confidence bands are superimposed, the number of violations of the confidence bands for each lag and a global measure for the distance between the theoretical autocovariance and the sample autocovariance functions given by the Root Mean Squared Error (for all lags and all paths). Of course, we have also computed the computational time to perform our simulation study according to each of the considered scheme.

If the parameter contraint that guarantees the positivity of the variance is fulfilled we have found no evidence, from our simulation design, against one of the discretization schemes; for all parameters assignments, observation frequency and time series length we have considered hereby (48 cases) the majority of the paths lie within confidence bands. We have also observed that the number of violations of confidence bands decreases with an increase in the observation frequency while it increases with the absolute correlation index as well as with the volatility of volatility parameter.

Otherwise, when the origin is attainable, the number of violations becomes statistically significant especially for low-frequency data. In these cases the Quadratic Exponential algorithm (Andersen, 2008) and the Full-Truncation fix for the Eulero-Maruyama discretization scheme (Lord et al., 2008) have given, in our analysis, more robust results with respect to the IJK-IMM scheme of Kahl and Jackel (2006). For what concerns the RMSE, it seems that the QE algorithm performs overall the best. Of course, the QE algorithm is the most intensive from a computational viewpoint, though

remaining rather fast and, if computational time is more important than accuracy, the Full Truncation Eulero-Maruyama is the best compromise. However, as already remarked, we have not focused on the optimization of our codes which may improve further the computational time for the QE algorithm.

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Appendix A

The third order structure of the *mean variance* process, for $i > 1$, is given by

$$E[\overline{V_1}^3] = E\left[\frac{1}{\Delta^3}\int_0^\Delta\int_0^\Delta\int_0^\Delta \sigma_t^2\sigma_r^2\sigma_s^2 dtdrds\right] = \frac{3!}{\Delta^3}\int_0^\Delta\int_t^\Delta\int_r^\Delta E[\sigma_t^2\sigma_r^2\sigma_s^2]drdtds, \tag{15}$$

$$E[\overline{V}_1\overline{V}_{1+i}^2] = E\left[\frac{2}{\Delta^3}\int_0^\Delta\int_{i\Delta}^{(i+1)\Delta}\int_r^{(i+1)\Delta}\sigma_t^2\sigma_r^2\sigma_s^2 dtdrds\right] = \frac{2}{\Delta^3}\int_0^\Delta\int_{i\Delta}^{(i+1)\Delta}\int_r^{(i+1)\Delta} E[\sigma_t^2\sigma_r^2\sigma_s^2]dtdrds, \tag{16}$$

$$E[\overline{V}_1^2\overline{V}_{1+i}] = E\left[\frac{2}{\Delta^3}\int_0^\Delta\int_t^\Delta\int_{i\Delta}^{(i+1)\Delta}\sigma_t^2\sigma_r^2\sigma_s^2 dtdrds\right] = \frac{2}{\Delta^3}\int_0^\Delta\int_t^\Delta\int_{i\Delta}^{(i+1)\Delta} E[\sigma_t^2\sigma_r^2\sigma_s^2]dtdrds. \tag{17}$$

The fourth order structure, for $i, h > 1$with $i \neq h$, is given by

$$E[\overline{V}_1^4] = E\left[\frac{1}{\Delta^4}\int_0^\Delta\int_0^\Delta\int_0^\Delta\int_0^\Delta \sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2 dtdrdsdw\right] = \tag{18}$$
$$= \frac{4!}{\Delta^4}\int_0^\Delta\int_t^\Delta\int_r^\Delta\int_s^\Delta E[\sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2]drdtdsdw,$$

$$E[\overline{V}_1^2\overline{V}_{1+h}^2] = E\left[\frac{1}{\Delta^4}\int_0^\Delta\int_0^\Delta\int_{h\Delta}^{(h+1)\Delta}\int_{h\Delta}^{(h+1)\Delta} \sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2 dtdrdsdw\right] = \tag{19}$$
$$= \frac{2}{\Delta^4}\int_0^\Delta\int_t^\Delta\int_{h\Delta}^{(h+1)\Delta}\int_s^{(h+1)\Delta} E[\sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2]dtdrsdw,$$

$$E[\overline{V}_1\overline{V}_{1+h}\overline{V}_{1+i}\overline{V}_{1+h+i}] = \frac{1}{\Delta^4}\int_0^\Delta\int_{(h\wedge i)\Delta}^{(h\wedge i+1)\Delta}\int_{(h\vee i)\Delta}^{(h\vee i+1)\Delta}\int_{(i+h)\Delta}^{(i+h+1)\Delta} E[\sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2]dtdrdsdw, \tag{20}$$

(21)

$$E[\overline{V}_1\overline{V}_{1+h}^2\overline{V}_{1+2h}] = \frac{2}{\Delta^4}\int_0^\Delta\int_{h\Delta}^{(h+1)\Delta}\int_r^{(h+1)\Delta}\int_{2h\Delta}^{(2h+1)\Delta} E[\sigma_t^2\sigma_r^2\sigma_s^2\sigma_w^2]dtdrsdw. \tag{22}$$

Other moments and cross-moments can be computed by using the same technique.

Appendix B

Define $U_t := e^{\alpha t}\sigma_t^2$. By applying Ito's Lemma, we can write:

$$U_r = U_t + \alpha\beta\int_t^r \exp(\alpha s)ds + \int_t^r \exp(\alpha s)\sigma_s dW_s,$$

where
Denote
$A_1 = E[U_t U_r]$,

$A_2 = E[U_t U_r^2]$,
$A_3 = E[U_t U_r^3]$,
$B_1 = E[U_t U_r U_s]$,
$B_2 = E[U_t U_r U_s^2]$
$C_1 = E[U_t U_r U_s U_w]$,
and $f(t,r) = \alpha \int_t^r \exp(\alpha s) ds$.

We aim to compute the above expressions in terms of the raw moments of the process U_t and of model parameters (α, β, c). The computation of A_1 is straightforward

$$A_1 = E[U_t U_r] = E[U_t E[U_r|F_t]] = E[U_t(U_t + \beta f(t,r))] = E[U_t^2] + \beta f(t,r) E[U_t]$$

For the computation of the other quantities we need $E[U_r^2|F_t]$ and $E[U_r^3|F_t]$ hence we need the dynamics of U_r^2 and U_r^3. By applying Ito's Lemma we obtain

$$\begin{aligned}
dU_t^2 &= (2U_t \alpha\beta \exp(\alpha t) + \exp(2\alpha t)\exp(-\alpha t) U_t)\, dt \\
&+ 2U_t \exp(\alpha t)\sqrt{\exp(-\alpha t) U_t} dW_t), \text{ and} \\
dU_t^3 &= 3\left(U_t^2 \alpha\beta \exp(\alpha t) + 3U_t \exp(2\alpha t)\exp(-\alpha t) U_t\right) dt \\
&+ 3U_t^2 \exp(\alpha t)\sqrt{\exp(-\alpha t) U_t} dW_t.
\end{aligned}$$

Hence,

$$\begin{aligned}
A3 &= E[U_t U_r^3] = E[U_t E[U_r^3|F_t]] = \\
&= E\left[U_t\left(U_t^3 + 3(\alpha\beta + c^2)\left(\frac{1}{\alpha} U_t^2 f(t,r) + U_t(2\beta + \frac{c^2}{\alpha})\int_t^r \exp(\alpha u) f(t,u) du\right)\right)\right] \\
&+ E\left[U_t\left(3(\alpha\beta + c^2)\alpha\beta^2 \int_t^r \exp(\alpha u)\int_t^u f(t,x) dx du\right)\right] \\
&+ E\left[U_t\left(3(\alpha\beta + c^2)c^2\beta \int_t^r \exp(\alpha u)\int_t^u \exp(\alpha x) f(t,x) dx du\right)\right] \\
&= E[U_t^4] + 3\frac{(\alpha\beta + c^2)}{\alpha} f(t,r) E[U_t^3] \\
&+ 3(\alpha\beta + c^2)(2\beta + \frac{c^2}{\alpha})\int_t^r \exp(\alpha u) f(t,u) du E[U_t^2] + \\
&+ 6(\alpha\beta + c^2)\alpha\beta^2 \int_t^r \exp(\alpha u)\int_t^u f(t,x) dx du E[U_t] \\
&+ 3(\alpha\beta + c^2)c^2\beta \int_t^r \exp(\alpha u)\int_t^u \exp(\alpha x) f(t,x) dx du] E[U_t],
\end{aligned}$$

$$
\begin{aligned}
B1 &= E[U_t U_r U_s] = E[U_t U_r E[U_s|F_r]] \\
&= E[U_t U_r (U_r + \beta f(r,s))] = \\
&= E[U_t U_r^2] + \beta f(r,s) E[U_t U_r] = \\
&= A2 + \beta f(r,s) A1
\end{aligned}
$$

$$
\begin{aligned}
B2 &= E[U_t U_r U_s^2] = E[U_t U_r E[U_s^2|F_r]] = \\
&= E\left[U_t U_r \left(U_r^2 + U_s(2\beta + \frac{c^2}{\alpha}) f(r,s) + 2\alpha\beta^2 \int_r^s f(r,u)du\right)\right] \\
&+ E\left[U_t U_r c^2 \beta \int_r^s \exp(\alpha u) f(r,u))du\right] \\
&= E[U_t U_r^3] + (2\beta + \frac{c^2}{\alpha}) f(r,s) E[U_t U_r U_s] \\
&+ \left[2\alpha\beta^2 \int_r^s f(r,u)du + c^2\beta \int_r^s \exp(\alpha u) f(r,u))du\right] E[U_t U_r] \\
&= A3 + (2\beta + \frac{c^2}{\alpha}) f(r,s) B1 + \left[2\alpha\beta^2 \int_r^s f(r,u)du + c^2\beta \int_r^s \exp(\alpha u) f(r,u))du\right] A1
\end{aligned}
$$

$$
\begin{aligned}
C1 &= E[U_t U_r U_s U_w] = E[U_t U_r U_s E[U_w|F_s]] = \\
&= E[U_t U_r U_s (U_s + \beta f(s,w))] = \\
&= E[U_t U_r U_s^2] + \beta f(s,w) E[U_t U_r U_s] = \\
&= B2 + \beta f(s,w) B1
\end{aligned}
$$

The quantities $A1$ to $C1$ are thus written as a function the moments of order $p \leq 4$ of the process U_t. Since $V_t = e^{-\alpha t} U_t$, we can obtain $E[\sigma_t^2 \sigma_r^2] = e^{-\alpha(t+r)} A1$, $E[\sigma_t^2 \sigma_r^2 \sigma_s^2] = e^{-\alpha(t+r+s)} B1$ and $E[\sigma_t^2 \sigma_r^2 \sigma_s^2 \sigma_w^2] = e^{-\alpha(t+r+s+w)} C1$. Numerical computations have been performed using *Mathematica 5.1*

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