BANACH FAMILIES AND THE IMPLICIT FUNCTION THEOREM

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ABSTRACT. We generalise the classical implicit function theorem (IFT) for a family of Banach spaces, with the resulting implicit function having derivatives that are locally Lipschitz to very strong operator norms.

Notation. For Banach spaces X and Y, $\mathbb{L}(X,Y)$ is the Banach space of continuous linear maps from X to Y; so $\mathbb{L}(X,X)$ is the Banach algebra of *operators* on X. $X \times Y$ and[1] $X \cap Y$ have by default the maximum norm.

Definition 1. Fix a *pointed set* A, i.e. a pair (A, α_0) with $\alpha_0 \in A$. Fix also a class of subsets $\mathcal{A}$ of A with $\alpha_0 \in S\ \forall S \in \mathcal{A}$.

A *Banach family* $(B, \mathcal{A})$, or shortly B, is a Banach space $(B, \|\cdot\|)$ endowed with a collection of pseudo-norms $(\|\cdot\|_\alpha)_{\alpha\in A}$, s.t. $\|\cdot\|_{\alpha_0} = \|\cdot\|$, where we allow pseudo-norms to take infinite values.[2] $B_\alpha = \{x \in B \mid \|x\|_\alpha < \infty\}$.[3,4]

For 2 Banach families X and Y, and a linear map φ from X to Y, let $\|\varphi\|_\alpha = \mathsf{sup}\{\|\varphi(x)\|_\alpha \mid \|x\|_\alpha \le 1\}$, and, for $S \in \mathcal{A}$, $\|\varphi\|_S = \mathsf{sup}_{\alpha\in S}\|\varphi\|_\alpha$; and let $\mathbb{L}_{\mathcal{A}}(X,Y) = \{\varphi \in \mathbb{L}(X,Y) \mid \forall S \in \mathcal{A}, \|\varphi\|_S < \infty\}$, endowed with the family of norms $(\|\cdot\|_S)_{S\in\mathcal{A}}$.

Remark 1. Nothing prevents to endow the same Banach space with 2 different Banach family structures; those should however be distinguished notationally then. E.g., $\mathbb{R}^n$ will denote denote this space with the constant family of pseudo-norms, while $(\mathbb{R}^n, \{0\})$ will be used when, for $\alpha \neq \alpha_0$, $\mathbb{R}^n_\alpha = \{0\}$.

Remark 2. The main intent is to be able to view B also as some sort of B_α-manifold; i.e., to speak of α-neighbourhoods of points in B.

Remark 3. As is clear from the definition, the purpose of constructing Banach families is to get operator norms, later used to formulate the IFT. One could, conceivably, use a classical IFT for each of the norms $\alpha \in A$ and get as final conclusion:

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[1]This will be used only when X and Y inject continuously and linearly into a common Hausdorff topological vector space (e.g, the equivalence classes of measurable functions, with convergence in measure on compact subsets), with the injections coinciding on $X \cap Y$ and mapping it to the intersection of the images, ensuring $X \cap Y$ is a Banach space.

[2]Else it would basically amount to a locally convex space.

[3]If $(B_\alpha, \|\cdot\|_\alpha)$ is a Banach space it suffices for $\|\cdot\|_\alpha \ge \varepsilon_\alpha\|\cdot\|$ that $\|x_n\|_\alpha \to 0$ and $\|x_n - x\| \to 0$ implies $x = 0$ (closed graph theorem).

[4]All $\|\cdot\|_\alpha$ used in [4] are l.s.c., and s.t. B_α is a Banach space under $\mathsf{max}\{\|\cdot\|, \|\cdot\|_\alpha\}$.

$\forall\alpha\exists$ neighbourhood of a point x_0 on which there exists an "α-smooth" implicit function, however the intersection of those neighbourhoods can very well be reduced to $\{x_0\}$! Clearly, such a statement would be not be very useful.

Banach families allow to formulate a unified IFT, thm. 1. Prop. 3 can be used to prove by decomposition the required smoothness property of the underlying map.

The explicit operator norms for the Banach families used in [4, def. 7] are obtained in that paper in sect. 5.5.3.

Remark 4. If $\#A = 1$ or $X_\alpha = \{0\}$ for $\alpha \neq \alpha_0$, $\mathbb{L}_\mathcal{A}(X,Y) = \mathbb{L}(X,Y)$, so the definitions reduce to the usual ones.

Lemma 1. *For $f_i \in \mathbb{L}_\mathcal{A}(Y,Z)$ and $g_i \in \mathbb{L}_\mathcal{A}(X,Y)$, $f_i \circ g_i \in \mathbb{L}_\mathcal{A}(X,Z)$ and, $\forall S \in \mathcal{A}$, $\|f_1 \circ g_1 - f_2 \circ g_2\|_S \leq \|f_1\|_S\|g_1 - g_2\|_S + \|g_2\|_S\|f_1 - f_2\|_S$.*

Proof. Establish first $\|f \circ g\|_S \leq \|f\|_S\|g\|_S$ (the particular case where $g_2 = 0$); use then the triangle inequality. ■

Definition 2. A map $g\colon E \to F$ between topological vector spaces is $\mathcal{C}$-*differentiable* [6] at $x \in E$, for a class $\mathcal{C}$ of subsets of E, iff there exists $g' \in \mathbb{L}(E,F)$, its $\mathcal{C}$-*derivative* at x, s.t. $y \mapsto \frac{1}{\varepsilon}[g(x+\varepsilon y) - g(x)] - g'(y) \xrightarrow[\varepsilon\to 0]{} 0$ uniformly on all sets in $\mathcal{C}$.

We use F-differentiable (Fréchet), resp. H- (Hadamard), sH- (strongly —), G- (Gâteaux), when $\mathcal{C}$ is the class of bounded, resp. compact, weakly compact, finite subsets (cf. e.g. [3, 6, 7, 2, 1] for background).

Definition 3. For Banach families $(X,\mathcal{A})$ and $(Y,\mathcal{A})$, and O open in X, $F\colon O \to Y$ is S^1 (or: $S^1_\mathcal{A}$) if F is Gâteaux-differentiable at each $x \in O$, with *derivative* $F'_x \in \mathbb{L}_\mathcal{A}(X,Y)$, and if, $\forall\|\cdot\|_S$ $(S \in \mathcal{A})$, $x \mapsto F'_x$ is locally Lipschitz on O.

Remark 5. The local Lipschitz constant, $\ell^S_F(x) \stackrel{\text{def}}{=} \inf_{\varepsilon>0} \sup_{\|y_i - x\| \leq \varepsilon} \frac{\|\varphi_{y_1} - \varphi_{y_2}\|_S}{\|y_1 - y_2\|}$, is by definition u.s.c. (upper semi-continuous).

Remark 6. We will *need* the "locally Lipschitz" aspect above only at 1 point, however crucial, in prop. 1, where we will use this 'equi-Lipschitz' aspect to get δ independent of $\alpha \in S$ ('equi' refers to the comparability of norms for different α); all the rest would go through one or other way with just e.g. 'continuous for each α' instead.

Remark 7. When F and F' have a priori values in some larger linear spaces, suffices, if O is connected (thus: by piece-wise linear paths), to prove there is 1 point mapped by F to Y and 1 by F' to $\mathbb{L}_\mathcal{A}(X,Y)$: continuity of F' will then imply it is everywhere in $\mathbb{L}_\mathcal{A}(X,Y)$, next differentiability (prop. 1.i) will imply $F(x) \in Y\, \forall x$.

Lemma 2. *If $f\colon X \to Y$ is locally Lipschitz, where X, Y are metric spaces, then each compact subset of X has a neighbourhood on which f is Lipschitz.*

Proof. Fix $C \subseteq X$ compact. Let O_n denote a finite family of open subsets of X that cover C and such that f is Lipschitz on each O_n, say with constant L. Consider the (Lipschitz) function $g(x) = \max_n d(x, \complement O_n)$ on X; since $g > 0$ on the compact set C, $\exists\varepsilon > 0$ s.t. $g(x) > \varepsilon$ on the open neighbourhood $O = \{x \mid d(x,C) \leq \varepsilon\}$ of C: $x, y \in O$ and $d(x,y) \leq \varepsilon \Rightarrow \exists n\colon (x,y) \in O_n \times O_n$. Thus on O we have that $d(x,y) \leq \varepsilon \Rightarrow d(f(x), f(y)) \leq L d(x,y)$. Consider then the locally Lipschitz function $F(x,y) = \frac{d(f(x),f(y))}{d(x,y)}$ on $\{x, y \in O \times O \mid d(x,y] \geq \varepsilon$: by continuity, it is bounded, say by $L' \geq L$, on the compact set $C \times C$. So, again by continuity and compactness of $C \times C$, $\exists\varepsilon' > 0, \varepsilon' \leq \varepsilon\colon F(x,y) \leq L' + 1$ on the ε'-neighbourhood of $C \times C$. ■

Next proposition shows that S^1 implies C^1, and much more.

Proposition 1. *For Banach families $(X,\mathcal{A})$ and $(Y,\mathcal{A})$, let $F\colon O \to Y$ be S^1 with O open in X. Then:*

(i) *$\forall S \in \mathcal{A}$ and each compact subset C of O there exists a neighbourhood of C, $V \subseteq O$, and $\exists K, \varepsilon > 0$ s.t., for any $x \in V$ and $\alpha \in S$, $\|\delta x\| \leq \varepsilon \Rightarrow \|F(x+\delta x) - F(x) - F'_x(\delta x)\|_\alpha \leq K\|\delta x\|\|\delta x\|_\alpha$.*

(ii) *Let, for $x \in O$, V_α be the $\|\cdot\|$-connected component of 0 in $(O-x)\cap X_\alpha$. V_α and its complement in $(O-x)\cap X_\alpha$ have disjoint closures in $(O-x, \|\cdot\|)$ and V_α is connected via piecewise-linear paths, so, in any vector topology.*

(iii) *$\delta x \mapsto F(x+\delta x) - F(x)$ is $\mathcal{B}_\alpha$-differentiable from V_α to Y_α with $F'_{x+\delta x}$ as derivative at δx, $\mathcal{B}_\alpha$ being the class of subsets bounded both in X and in X_α.*

(iv) *$\forall S \in \mathcal{A}$, each point of O has a $\|\cdot\|$-neighbourhood U and $\exists K$ s.t., $\forall \alpha \in S$, $\forall x, y \in U$, $\|F(x) - F(y)\|_\alpha \leq K\|x-y\|_\alpha$.*

(v) *F is (Fréchet) C^1.*

Remark 8. Clearly e.g. point iii becomes much stronger and simpler when $\forall \alpha\ \|\cdot\|_\alpha \geq \varepsilon_\alpha \|\cdot\|$. But one verifies immediately that if one defines a new family $\|\cdot\|'_\alpha \stackrel{\text{def}}{=} \mathsf{max}\{\|\cdot\|_\alpha, \|\cdot\|\}$ then $\forall S\ \|\cdot\|'_S \leq \|\cdot\|_S$, so that all conclusions available from the S^1 property with the new family are already so with the original family, plus some more.

Proof. i: By lemma 2, there is a neighbourhood $W \subseteq O$ of the compact set C s.t. $F' \colon X \to \mathbb{L}_{\mathcal{A}}(X, Y)$ is Lipschitz on W w.r.t. $\|\cdot\|_S$, say with constant K.

Each point of C has a convex neighbourhood $U \subseteq W$. Assume $x \in U, x+\delta x \in U$. For $t \in [0,1]$ let $f(t) = F(x+t\delta x) - F(x) - tF'_x(\delta x)$: $f'_t = (F'_{x+t\delta x} - F'_x)(\delta x)$, so $\|f'_{t_1} - f'_{t_2}\|_\alpha \leq \|F'_{x+t_1\delta x} - F'_{x+t_2\delta x}\|_S \|\delta x\|_\alpha \leq K|t_1 - t_2|\|\delta x\|\|\delta x\|_\alpha$: f' has Lipschitz constant $L = K\|\delta x\|\|\delta x\|_\alpha$. Since $f(0) = f'(0) = 0$, this implies first $\|f'_t\|_\alpha \leq L$, next, by integration, $\|f(1)\|_\alpha \leq K\|\delta x\|\|\delta x\|_\alpha$.

Use now those neighbourhoods as in the proof of lemma 2 to construct an open covering O_n of C, and then to find $\varepsilon > 0$ s.t. $d(x, C) \leq \varepsilon$, $d(x+\delta x, C) \leq \varepsilon$ and $\|\delta x\| \leq \varepsilon$ imply $\|F(x+\delta x) - F(x) - F'_x(\delta x)\|_\alpha \leq K\|\delta x\|\|\delta x\|_\alpha$. Halving this ε yields then the statement, since $d(x, C) \leq \frac{\varepsilon}{2}$ and $\|\delta x\| \leq \frac{\varepsilon}{2}$ imply $d(x+\delta x, C) \leq \varepsilon$.

ii: Since $F'_x \in \mathbb{L}(X_\alpha, Y_\alpha)$, (i) implies first that $F(x+\delta x) - F(x) \in Y_\alpha$ for $\delta x \in X_\alpha$, $\|\delta x\|$ sufficiently small. Re-applying this at each point of $W_x = \{\delta x \in (O-x)\cap X_\alpha \mid F(x+\delta x) - F(x) \in Y_\alpha\}$ shows W_x is a $\|\cdot\|$-open neighbourhood of 0 in X_α.

$V' = \{z \in V_\alpha \mid \exists m, \exists x_i \text{ with } i = 1 \ldots 2m+1\colon\ x_1 = x,\ x_{2m+1} = x+z,\ x_{2i\pm1} - x_{2i} \in W_{x_{2i}} \text{ for } i = 1 \ldots m\}$ is trivially open and closed in V_α, so $V' = V_\alpha$. Hence the second statement. For the first, let else z belong to both closures: a $\|\cdot\|$-ball around z is contained in $O-x$ and intersects V_α and its complement, say in z_1 and z_2. Then the segment from z_1 to z_2 lies in the ball, hence in $O-x$, and also in X_α: z_2 is connected to V_α, hence $\in V_\alpha$: contradiction.

iii: Since $F(x_{2i\pm1}) - F(x_{2i}) \in Y^\alpha$, $F(x+z) - F(x) \in Y^\alpha\ \forall z \in V_\alpha$. Use then (i) at each $x+z$ for $z \in V_\alpha$.

iv: Intersect in (i) the $\frac{\varepsilon}{2}$-neighbourhood of the compact set $\{x\}$ with V; restrict still more if needed to ensure that $\|F'_x\|_S$ is bounded on U.

v: Take $\alpha = \alpha_0$ in i. ∎

Corollary 1. *If $F\colon O \to Y$ is S^1 and O is either connected, with X_α dense in X, or is convex, then $z \mapsto F(x+z) - F(x)$ is, $\forall x \in O$, $\mathcal{B}_\alpha$-differentiable from $(O-x)\cap X_\alpha \subseteq X_\alpha$ to Y_α.*

Lemma 3. *$f\colon \prod_1^n X_i \to Y$ (X_i, Y metric spaces) is locally Lipschitz if it is so for arguments that differ in a single coordinate.*

Definition 4. For Banach families $(X_i, \mathcal{A})$ $(i = 1 \ldots n)$ the product $(X, \mathcal{A})$ is defined by $X = \prod X_i$, $\|x\|_\alpha = \mathsf{max}_i \|x_i\|_{i,\alpha}\ \forall \alpha \in A$.

Lemma 4. *For S^1 maps $f_i\colon (Y, \mathcal{A}) \to (X_i, \mathcal{A})$, $f = \prod f_i\colon (Y, \mathcal{A}) \to \prod(X_i, \mathcal{A})$ is S^1.*

Proof. Let $X = \prod_i X_i$. One checks immediately that $f' \stackrel{\text{def}}{=} (f_1', \ldots, f_n') \in \mathbb{L}_{\mathcal{A}}(Y, X)$, with $\|f'\|_S = \mathsf{max}_i \|f_i'\|_S$ — which implies $\ell_f^S = \mathsf{max}_i \, \ell_{f_i}^S$. Finally, since $f' \in \mathbb{L}_{\mathcal{A}}(Y, X)$, the differentiability of f follows straight from that of the f_i. ∎

Remark 9. The analogue of this lemma for $g_i \colon (Y_i, \mathcal{A}) \to (X_i, \mathcal{A})$ and $g = \prod g_i \colon \prod(Y_i, \mathcal{A}) \to \prod(X_i, \mathcal{A})$ will be an immediate corollary (of prop. 3), using $f_i = g_i \circ \mathsf{proj}_i$ with $\mathsf{proj}_i \colon \prod(Y_j, \mathcal{A}) \to (Y_i, \mathcal{A})$ (and a corollary of prop. 2 (with $n = 1$) for proj_i).

Lemma 5. *A map $f \colon \prod(X_i, \mathcal{A}) \to (Y, \mathcal{A})$ is S^1 iff it is separately so and each partial derivative is locally Lipschitz for arguments that differ in a single coordinate.*

Proof. The condition is clearly necessary. Assume it holds; then the vector of partial derivatives satisfies the Lipschitz condition (lemma 3); suffices thus to prove it equals the Gâteaux differential. This follows from the same argument as in the proof of prop. 1.i, using a path where 1 coordinate is changed at a time. ∎

Definition 5. A map φ between linear spaces is *affine* if $\varphi(\alpha x + (1 - \alpha) y) = \alpha \varphi(x) + (1 - \alpha)\varphi(y)$ for any x, y in the domain and any scalar α.

A map from a product of linear spaces to a linear space is *multi-affine* if it is affine in each coordinate, for any fixed values of the other coordinates.

$\|\varphi\| \stackrel{\text{def}}{=} \mathsf{sup}_{\|x_i\| \le 1 \forall i} \|\varphi(x_1, \ldots, x_n)\|$ for a multi-affine map between normed spaces.

Lemma 6. *Let φ be a multi-affine map between normed spaces. If $\|\varphi(x)\| \le K$ in the ε-ball around $\bar{x}$, then $\forall x \ \|\varphi(x)\| \le K\big(\frac{\|x - \bar{x}\|}{\varepsilon}\big)^n$, so $\|\varphi\| < \infty$, and φ has local Lipschitz constant $\le n \, \mathsf{max}\{1, \|x\|^{n-1}\} \|\varphi\|$.*

Proof. For the bound, one may assume by translation $x_0 = 0$. Enlarge the ε-ball to radius R one coordinate at a time: each time, by affinity, the bound on the norm is multiplied by at most $\frac{R}{\varepsilon}$. Hence the result.

For the Lipschitz aspect, deduce first that, for $R \ge 1$, $\|x_1\| \le \varepsilon, \|x_i\| \le R \Rightarrow \|\varphi(x_1, \ldots, x_n) - \varphi(0, x_2, \ldots, x_n)\| \le \varepsilon R^{n-1} \|\varphi\|$ (multiplying x_1 by $\frac{R}{\varepsilon}$). So $\|\varphi(x) - \varphi(y_1, x_2, \ldots, x_n)\| = \|\varphi(x_1 - y_1, x_2, \ldots, x_n) - \varphi(0, x_2, \ldots, x_n)\| \le R^{n-1} \|x_1 - y_1\| \|\varphi\|$. Applying this 1 coordinate at a time allows to pass in n steps from an arbitrary vector x to an arbitrary vector y, provided both are in the R-ball. Thus $\|\varphi(x) - \varphi(y)\| \le R^{n-1} \|\varphi\| \sum_i \|x_i - y_i\|$, so $\le n\big(\mathsf{max}\{1, \|x\|, \|y\|\}\big)^{n-1} \|\varphi\| \|x - y\|$. ∎

Next proposition, especially in the multi-linear case where $\varphi^i = \varphi$, might suggest to look at tensor products, at least for $\# A = 2$ ("Banach pairs").

Proposition 2. *For Banach families X_i $(i = 1 \ldots n)$ and Y, let $\varphi \colon \prod X_i \to Y$ be multi-affine. Let $\varphi^i \stackrel{\text{def}}{=} \delta x_i \mapsto \varphi(\delta x_i, x_{-i}) - \varphi(0, x_{-i})$ (the i^{th} partial derivative), and let $\|\varphi^i\|_\alpha$ be its norm as a multi-affine map, equivalently from $X_{i,\alpha} \times \prod_{j \ne i} X_j$ to Y_α or from $\prod_{j \ne i} X_j$ to $\mathbb{L}(X_{i,\alpha}, Y_\alpha)$. For $S \in \mathcal{A}$, let $|||\varphi|||_S = \mathsf{max}_i \, \mathsf{sup}_{\alpha \in S} \|\varphi^i\|_\alpha$. Then:*

(i) *For $O \ne \emptyset$ open in $\prod X_i$, φ is S^1 on O iff $|||\varphi|||_S < \infty \ \forall S \in \mathcal{A}$.*
(ii) *$\ell_\varphi^S \le n(n-1) \, \mathsf{max}\{1, \|x\|^{n-2}\} |||\varphi|||_S$.*

Proof. Assume φ is S^1 somewhere. Its derivative there can only be $\varphi' = \sum_i \varphi^i \colon \delta x \mapsto \sum_i \varphi^i(\delta x_i)$. Since $\|\varphi_x'\|_\alpha \ge \mathsf{max}_i \, \mathsf{sup}_{\|\delta x_i\|_\alpha \le 1} \|\varphi_{x_{-i}}^i(\delta x_i)\|_\alpha$, $\|\varphi_x'\|_S \ge \mathsf{max}_i \, \mathsf{sup}_{\alpha \in S} \mathsf{sup}_{\|\delta x_i\|_\alpha \le 1} \|\varphi_{x_{-i}}^i(\delta x_i)\|_\alpha$. Now $\|\varphi_x'\|_S$ is locally bounded where φ is S^1; thus $\exists \bar{x}, K, \varepsilon$: $\forall \alpha \in S, \forall i, \|x_{-i} - \bar{x}_{-i}\| \le \varepsilon, \|\delta x_i\|_\alpha \le 1 \Rightarrow \|\varphi_{x_{-i}}^i(\delta x_i)\|_\alpha \le K$. Hence, by lemma 6, viewing φ^i as a multi-affine function of x_{-i} to $\mathbb{L}(X_{i,\alpha}, Y_\alpha)$: $\forall x$, $\|\delta x_i\|_\alpha \le 1 \Rightarrow \|\varphi_{x_{-i}}^i(\delta x_i)\|_\alpha \le K\big(\frac{\|x - \bar{x}\|}{\varepsilon}\big)^{n-1}$; so $\mathsf{sup}_{\alpha \in S} \|\varphi^i\|_\alpha \le K\big(\frac{\|\bar{x}\| + 1}{\varepsilon}\big)^{n-1}$. Thus $|||\varphi|||_S < \infty$.

Assume now $|||\varphi|||_S < \infty\ \forall S \in \mathcal{A}$. Since the norm of $\varphi^i\colon \prod_{j\neq i} X_j \to \mathbb{L}(X_{i,\alpha}, Y_\alpha)$ is $\|\varphi^i\|_\alpha$, the norm of $\varphi^i\colon \prod_{j\neq i} X_j \to \mathbb{L}_{\mathcal{A}}(X_i, Y)$, when the latter space is endowed with $\|\cdot\|_S$, is $\leq |||\varphi|||_S$. Thus, by lemma 6, φ^i has local Lipschitz constant $\leq (n-1)\max\{1, \|x_{-i}\|^{n-2}\}|||\varphi|||_S$ w.r.t. the $\|\cdot\|_S$ norm on its values: since $\|\varphi'\| \leq \sum_i \|\varphi^i\|$, $\ell^S_\varphi \leq n(n-1)\max\{1, \|x\|^{n-2}\}|||\varphi|||_S$. So the local Lipschitz requirement is satisfied. Hence the result by lemma 5. ■

Remark 10. $|||\varphi||| = 0$ implies (lemma 6, using $\alpha = \alpha_0$) that all partial derivatives are identically 0, so φ is constant. In particular, $|||\varphi|||$ is a norm on multilinear maps.

Remark 11. A *polynomial map* $\varphi\colon \prod X_i \to Y$ is the composition of the diagonal maps $X_i \to \prod_{j\in I_j} X_{i,j}$, where the $X_{i,j}$ are copies of X_i, with a multiaffine map from $\prod_{i,j} X_{i.j}$ to Y (and the latter can always be chosen symmetric in each I_j). Prop. 2 allows thus to prove the S^1 property of adequate polynomial maps, using prop. 3 for the composition and lemma 4 for the diagonal maps.

But is there a direct and natural extension of prop. 2 to polynomial maps?

Proposition 3. *For $O \subseteq X$ and $U \subseteq Y$ both open, if $g\colon O \to U$ and $f\colon U \to Z$ are S^1, $f \circ g$ is so.*

Proof. Lemma 1 implies $f'_{g_x} \circ g'_x \in \mathbb{L}_{\mathcal{A}}(X, Z)\ \forall x$. By lemma 1.v it is the Fréchet derivative, using the stability of C^1 maps under composition. Lemma 1 again yields then the local Lipschitz aspect of $x \mapsto f'_{g_x} \circ g'_x$, using the continuity of g (lemma 1.v) and the fact that locally Lipschitz functions are locally bounded. ■

Corollary 2. *S^1maps from $(X,\mathcal{A})$ to $(Y,\mathcal{A})$ form a vector space, and an algebra if Y is a Banach algebra s.t., $\forall S$, $\sup_{\alpha\in S}\sup_{\|y\|\leq 1, \|z\|_\alpha\leq 1}\max\{\|yz\|_\alpha, \|zy\|_\alpha\} < \infty$ (so Y_α is a 2-sided ideal). They are also a module on the algebra of S^1 maps from $(X,\mathcal{A})$ to $(\mathbb{R},\{0\})$, where $\mathbb{R}$ stands for the base-field.*

Proof. By prop. 2 the sum ($n = 1$, triangle inequality) [resp., product ($n = 2$)] and the product with given scalars ($n = 1$) on $(Y,\mathcal{A})$ are S^1. Apply then prop. 3. For the last sentence, argue similarly, this time for the product with scalars viewed as a bilinear map, since the algebra-property of this set of maps is now established. ■

The core of next proposition, consisting essentially of the beginning of the proof, would not need the Lipschitz assumption (and not obtain the Lipschitz aspect in the conclusion). Instead, it would suffice to assume a form of 'equal continuity' of the partial derivatives: that $\forall\varepsilon > 0 \exists\delta > 0\colon \|(\frac{\partial F}{\partial x})_{x,y} - (\frac{\partial F}{\partial x})_{x_0,y_0}\|_\alpha \leq \varepsilon\ \forall\alpha$ if $\|x - x_0, y - y_0\| < \delta$. However, the only practical way we found to ensure such an 'equal continuity' of maps with values in a family of different spaces was using computations of Lipschitz constants; further, continuity of ϖ'_y on Y is very important. Thus we use such assumptions, and draw the corresponding additional conclusions.

Theorem 1 (IFT)**.** *Given 2 Banach families $(X,\mathcal{A})$ and $(Y,\mathcal{A})$, let $F\colon X\times Y \to X$ vanish at (x_0, y_0), and be S^1 in a neighbourhood V of (x_0, y_0). If $(\frac{\partial F}{\partial x})_{x_0,y_0}$ is invertible in $\mathbb{L}_{\mathcal{A}}(X,X)$, then $\forall S \in \mathcal{A}$, $\exists\delta,\delta' > 0$ and an $S^1_{\{S\}}$ map $\varpi\colon \{y \mid \|y - y_0\| < \delta\} \to X$ s.t. $x = \varpi(y)$ is the unique solution of $F(x,y) = 0$ with $\|x - x_0\| \leq \delta'$, and s.t. $\forall y$, $(\frac{\partial F}{\partial x})_{\varpi(y),y}$ has an inverse in $\mathbb{L}_{\{S\}}(X,X)$ which is a Lipschitz function of y.*

Proof. Reduce V to ensure it is open, bounded, and that ℓ^S_F is bounded on V, say by L; so $\|(\frac{\partial F}{\partial y})_{x,y}\|_S$ is also bounded, say by D. The theorem with C^1, when $X_\alpha = Y_\alpha = \{0\}\ \forall\alpha$, is classical [e.g. 5, theorems 25, 26, vol. 1]. Use it first, with prop. 1.v, to obtain a C^1 ϖ, in a δ_0-neighbourhood of $y_0 \in Y$, s.t. the graph of ϖ above this δ_0-neighbourhood belongs to V, and a $\delta' > 0$ s.t. uniqueness holds for $\|x - x_0\| \leq \delta'$ and $\|y - y_0\| \leq \delta_0$. In particular, ϖ is Gâteaux-differentiable.

Let $M = (\frac{\partial F}{\partial x})_{x_0,y_0}$, $Y = M + Z$, then $Y^{-1} = M^{-1}(\mathbf{1} + ZM^{-1})^{-1}$; so, since $(\mathbf{1}-X)^{-1}$ exists and $= \sum_{n\geq 0} X^n$, with norm ≤ 2 when $\|X\| \leq \frac{1}{2}$, if $\|ZM^{-1}\|_S \leq \frac{1}{2}$ then $\|Y^{-1}\|_S \leq 2\|M^{-1}\|_S$; i.e., with $C = 2\|(\frac{\partial F}{\partial x})^{-1}_{x_0,y_0}\|_S$, and using lemma 1, if $\|Z\|_S \leq \frac{1}{C}$, then $\|Y^{-1}\|_S \leq C$. Thus, if $\|(\frac{\partial F}{\partial x})_{x,y} - (\frac{\partial F}{\partial x})_{x_0,y_0}\|_S < \frac{1}{C}$, then $(\frac{\partial F}{\partial x})_{x,y}$ is invertible and $\|(\frac{\partial F}{\partial x})^{-1}_{x,y}\|_S \leq C$. This condition holds if $\|x - x_0, y - y_0\| \leq \frac{1}{LC}$. By continuity of ϖ at y_0, $\exists \delta < \delta_0\colon \|y - y_0\| \leq \delta \Rightarrow \|\varpi(y) - x_0, y - y_0\| < \frac{1}{LC}$. So $\|y - y_0\| \leq \delta \Rightarrow (\frac{\partial F}{\partial x})_{\varpi_y,y}$ is invertible in $\mathbb{L}_S(X, X)$ and $\|(\frac{\partial F}{\partial x})^{-1}_{\varpi_y,y}\|_S \leq C$.

Since $\varpi'_y = \frac{d\varpi}{dy} = -\big(\frac{\partial F}{\partial x}\big)^{-1} \frac{\partial F}{\partial y}$ at $x = \varpi_y$, $\|\varpi'(y)\|_S \leq CD$, by lemma 1.

And since $A^{-1} - B^{-1} = A^{-1}(B - A)B^{-1}$, lemma 1 again yields $\|A^{-1} - B^{-1}\| \leq \|A^{-1}\|\|B^{-1}\|\|A-B\|$. Thus $\|(\frac{\partial F}{\partial x})^{-1}_{\varpi_{y_1},y_1} - (\frac{\partial F}{\partial x})^{-1}_{\varpi_{y_2},y_2}\|_S \leq C^2L\|\varpi_{y_1} - \varpi_{y_2}, y_1 - y_2\|$.

So, $\|\varpi'_{y_1} - \varpi'_{y_2}\|_S = \|(\frac{\partial F}{\partial x})^{-1}_{\varpi(y_1),y_1}(\frac{\partial F}{\partial y})_{\varpi(y_1),y_1} - (\frac{\partial F}{\partial x})^{-1}_{\varpi(y_2),y_2}(\frac{\partial F}{\partial y})_{\varpi(y_2),y_2}\|_S \leq C\|(\frac{\partial F}{\partial y})_{\varpi(y_1),y_1} - (\frac{\partial F}{\partial y})_{\varpi(y_2),y_2}\|_S + D\|(\frac{\partial F}{\partial x})^{-1}_{\varpi(y_1),y_1} - (\frac{\partial F}{\partial x})^{-1}_{\varpi(y_2),y_2}\|_S$, by lemma 1; $\leq (CL + D(C^2L))(1 + CD)\|y_1 - y_2\|$: $\ell^S_\varpi \leq LC(1 + CD)^2$. Thus ϖ is S^1. ∎

References

[1] Dudley, R. M. (1994): "The Order of the Remainder in Derivatives of Composition and Inverse Operators for p-Variation Norms," *The Annals of Statistics*, 22(1), 1–20.

[2] Durdil, J. (1973): "On Hadamard differentiability," *Commentationes Mathematicae Universitatis Carolinae*, 14(3), 457–470.

[3] Hadamard, J. S. (1923): "La notion de différentielle dans l'enseignement," *Scripta Universitatis atque Bibliothecae Hierosolomitanarum*, I(4), Mathematica et Physica, curavit A. Einstein. Leipzig: Kreising.

[4] Mertens, J.-F., and A. Rubinchik (2011): "Regularity and stability of equilibria in an overlapping generations growth model," Discussion paper, Working paper. `http://econ.haifa.ac.il/~arubinchik/papers/Reg_Stab_OLG.pdf`.

[5] Schwartz, L. (1957-59): *Théorie des distributions*, vol. I and II. Hermann, Paris.

[6] Sebastião e Silva, J. (1956): "Le calcul différentiel et intégral dans les espaces localement convexes, réels ou complexes. I, II," *Atti della Accademia Nazionale dei Lincei. Classe di Scienze Fisiche, Matematiche e Naturali. Rendiconti Lincei. Serie VIII. Matematica e Applicazioni*, 20 (6) pp. 743–750, 21(1–2) pp. 40–46.

[7] Sova, M. (1966): "Conditions of differentiability in linear topological spaces," *Czech. Math. J.*, 16, 339–362.