COVER SHEET

Dependence Structures in Financial Time Series: a Chaos-theoretic Approach

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Summary

Of much interest in financial econometrics is the recovery of joint distributional behaviour of collections of contemporaneous financial time series, e.g., two related commodity price series, or two asset returns series. An approach to model their joint behaviour is to use copulas. Essentially, copulas are selected on the basis of a measure of correlation between the two series and are made to match their marginal properties. Of course, generalisations exist for more than two series. A possible limitation of this approach is that only linear correlations between series might be captured. We consider incorporating more general dependence structures, through the use of the correlation integral (as in the BDS test), as a means to refine the choice of candidate copulas in an empirical situation.

Some keywords. Archimedean copula; copula; correlation integral; dependence; Poisson convergence.

1. Introduction

The year 1959 was world-famous for the transit of the first living being – a Russian dog named Laika – in outer space orbit around the Earth. Perhaps less conspicuously, but nevertheless famous in the statistical world, in and about that same year was the announcement of some important foundation results concerning dependence of random variables. This knowledge – including other results accumulated in the thirty-odd years prior to that – laid the foundation for the theory of copulas. In the bivariate case, one can envisage a copula as being the remainder factor after "dividing out" the marginal densities from the associated joint density: it is, in a sense, an algebraic term which "couples" two distributions together. In this paper, we briefly review notions relating to dependence structures, and their bearing on copulas. We point out how the choice of the right copula might be ambiguous, and proceed to illustrate how this choice might be improved if one adopts an approach from dynamical systems. In a financial setting, the importance of this work is to help to understand how, for instance, returns from two stocks, or even returns on indices from two markets, depend on each other. This is particularly of interest when one wishes to determine how shocks in one market might transfer to another, or how high-frequency events "drive" each other. See Guégan and Ladoucette (2004) for an empirical study.

2. Dependence Structures

Rényi (1959) established a framework, by means of an axiom set, for a measure of dependence, $R(X,Y)$, between continuous random variables, X and Y, which we recite here as a preliminary to development of later observations.

(a) $R(X,Y)$ is defined for any X and Y.

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(b) $R(X,Y) = R(Y,X)$.
(c) $0 \le R(X,Y) \le 1$
(d) $R(X,Y) = 0$ if and only if *X* and *Y* are independent.
(e) $R(X,Y) = 1$ if and only if each of *X* and *Y* is almost surely a strictly monotone function of the other.
(f) If m_1 and m_2 are strictly monotone functions almost surely on the range of *X* and the range of *Y*, respectively, then $R\{m_1(X), m_2(X)\} = R(X,Y)$.
(g) If the joint distribution of *X* and *Y* is bivariate Normal, with correlation coefficient ρ, then $R(X,Y)$ is a strictly increasing function of $|\rho|$.
(h) If (X,Y) and (X_n, Y_n) $(n = 1,2,\ldots)$ are pairs of random variables with joint distribution functions *H* and H_n, respectively, and if the sequence $\{H_n\}$ converges weakly to *H*, then $\lim_{n\to\infty} R(X_n, Y_n) = R(X,Y)$.

As discussed in Schweizer and Wolff (1981), this set of axioms may be too strong. Indeed, Rényi showed that only the maximal correlation coefficient, given by the expression $\sup_{m_1,m_2} \text{corr}\{m_1(X), m_2(Y)\}$, satisfies all axioms (where the supremum is taken over all Borel-measurable functions m_1, m_2; moreover, the maximal correlation coefficient has several drawbacks, such as attaining the value of unity "too often" (Hall, 1969). What is important to note is that Rényi's axioms admit a U-statistic-like structure for dependence measures (Lee, 1990), which connects with discussion later in the present paper.

3. Copulas

Further statistical contributions, contemporaneously with the space race, came from Kruskal (1958) and Sklar (1959), who accessed the notion of dependence through definition of a copula. Consider, in the general multivariate case, a random vector $[X_1, \cdots, X_n]'$ which has a joint cumulative distribution function given by $F(x_1, \cdots, x_n) = P(X_1 \le x_1 \,\&\, \cdots \,\&\, X_n \le x_n)$. Assume that each component random variable X_j $(j = 1, \ldots, n)$ has a continuous marginal cumulative distribution function given by $F(x_j) = P(X_j \le x_j)$. Sklar showed that the joint distribution function *F* can be written in terms of its marginals in the form

$$F(x_1, \cdots, x_n) = C\{F_1(X_1), \cdots, F_n(X_n)\}$$

for a unique function *C*, called the copula of *F* (or the copula of *X*), so called because it "couples" the marginal distributions together with the joint distribution. It is illuminating to note that, by differentiating the above expression, one obtains

$$f(x_1, \cdots, x_n) = f_1(X_1), \ldots f_n(X_n) c\{F_1(X_1), \cdots, F_n(X_n)\},$$

where *f* is the probability density function corresponding to cumulative distribution function *F*, f_j is the probability density function corresponding to cumulative distribution function F_j $(j = 1, \ldots, n)$, and *c* is the derivative of the function *C*. Now $X_1, \cdots, X_n$ are independent if and only if $c\{F_1(X_1), \cdots, F_n(X_n)\} = 1$, which gives an immediate interpretation of *c* (or *C*) as a measure of departure from independence. Note also that the function *c* (or *C*), having arguments being marginal cumulative distribution functions, is generically defined in terms of uniform marginals on the unit interval, since $F_j(X_j)$ is a standard uniform random variable for any random variable X_j.

Either side of the contributions from the 1950s was the pioneering work in bivariate distributional modelling by Pretorius (1930) and, 40 years later, Mardia's (1970) observation that very little progress had been made to that date in being able to identify and fit an appropriate bivariate distribution given marginal behaviour (and other statistics). Archimedean copulas (Schweizer and Sklar, 1983, Ch 3; Genest and MacKay, 1986a, 1986b), to an extent, simplify some issues in fitting bivariate copulas, in that the functional form of the copula is restricted to take a particular form, given by

$$C_\varphi(x_1, x_2) = \varphi^{-1}\{\varphi(x_1) + \varphi(x_2)\} \quad (0 < x_1, x_2 < 1),$$

where φ is a convex decreasing function with $\varphi(1) = 0$, and we take $\varphi^{-1}(x) = 0$ whenever $x \geq \varphi(0)$. These conditions are necessary and sufficient for $C_\varphi(x_1, x_2)$ to be a cumulative distribution function (Schweizer and Sklar, 1983, Ch 5), and are equivalent to $1 - \varphi^{-1}(x)$ being a unimodal cumulative distribution on $[0, \infty)$ with mode at 0. Noting the inverse transformations

$$U_1 = \varphi(X_1)/\{\varphi(X_1) + \varphi(X_2)\}, \quad U_2 = C(X_1, X_2),$$

it can be shown that U_1 is distributed uniformly on $(0,1)$, independently of U_2 which is distributed as $u_2 - \varphi(u_2)/\varphi'(u_2)$ on $(0,1)$. Genest and Rivest (1993) show that, in the special Archimedean case for U_2, it is possible to solve the resulting differential equation involving φ to obtain

$$\varphi(u_2) = \exp\left\{\int_{u_0}^{u_2} \frac{\varphi'(t)}{\varphi(t)}\,dt\right\},$$

where $0 < u_2 < 1$ is an arbitrarily chosen constant. In some sense, it captures the "projection" of almost any dependence function *C* within the class of Archimedean copulas. They then proceed to estimate the copula in a non-parametric (though unsmoothed) fashion.

An example of a dependence function is Kendall's tau, which is a generalised form of correlation; specifically, it is the difference between the probability of concordance and the probability of discordance, namely

$$\tau(X_1, X_2) = P\{(X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) > 0\} - P\{(X_1 - \tilde{X}_1)(X_2 - \tilde{X}_2) < 0\},$$

where $\tilde{X}_j$ is an independent copy of the random variable X_j $(j = 1,2)$. Kendal's tau can be expressed as a double integral of the copula function *C* (Nelsen, 1999) but, for the special case of an Archimedean copula, it is given by

$$\tau = 1 + 4\int_0^1 \frac{\varphi(t)}{\varphi'(t)}\,dt\,.$$

4. Copula Selection: Statement of the Problem

Joe (1997) presents a catalogue of copulas from which one may choose one or several candidates for fitting a particular set of data. Clearly, in data analysis, given only a sample from a bivariate or multivariate population, it may not – and is usually not – possible to select the “correct” copula: for instance, when restricting attention only to Archimedean copulas, one might still have several candidates (from several families) for a given estimate of Kendall’s tau. Therefore, an important issue remains over how to choose the “best” copula.

Particularly when one moves to the time series case, computing simple correlations from data and comparing with the resulting correlations arising from the copula may not be adequate, in that such a calculation captures only linear dependence. Thus, when two copulas render a similar theoretical correlation, and which is also close to the sample correlation from the data, choosing the copula which is the closer of the two may not suffice, particularly when there is no significance in the comparison.

We propose to use a dependence measure the correlation integral, as discussed in the next section.

5. Correlation Integral

The correlation integral is used in dynamical systems theory to estimate the correlation dimension of a dynamical system, based on the observation of a trajectory. We do not consider aspects of dynamical systems here, and a full account is given in a very elegant introduction by Ruelle (1989).

Consider a time series $X_1, \cdots, X_n$ of length *n*. A *p*-dimensional embedding of the time series is defined to be

$$V_j = \left[X_j, X_{j+1} \cdots, X_{j+p-1}\right]' \quad (j = 1, \ldots, n-p+1).$$

Then the correlation integral is given by

$$K^*(n,p,r) = \binom{n-p+1}{2}^{-1} \sum_{i<j} I\left(0 < \left\|V_i - V_j\right\| \le r\right),$$

where *I* is an indicator function. The limit

$$\lim_{n\to\infty, r\downarrow 0} \frac{\log K^*(n,p,r)}{\log r}$$

is called the correlation dimension of the dynamical system from which the original time series arose (assuming that the limit exists, and that the correct embedding dimension *p* has been selected).

Consider the unscaled correlation integral

$$K(n,p,r) = \sum_{i<j} I\left(0 < \left\|V_i - V_j\right\| \le r\right)$$

and let $I_{ij} = I\left(0 < \left\|V_i - V_j\right\| \le r\right)$. Then *K* is simply the sum of correlated Bernoulli random variables, each having mean $\pi_{ij} = E\left(I_{ij}\right) = P\left(0 < \left\|V_i - V_j\right\| \le r\right)$. Given a random sample (or, for that matter, a time series) $X_1, \cdots, X_n$, the probabilities π_{ij} can clearly be estimated consistently.

Analogously to a binomial random variable with parameters m and π converging in law to a Poisson random variable with parameter λ as $m \to \infty$, $\pi \to 0^+$ and $m\pi \to \lambda$, one may also prove Poisson convergence of correlated Bernoulli random variables – as in the function K – using the method of Chen (1975) and Stein (1972).

Wolff (1995) establishes a Poisson law for the unscaled correlation integral under certain assumptions, chiefly M-dependence. In that case, it is shown that

$$E(I_{ij}) \sim r^p v_p g_{ij}(0),$$

where v_p is the volume of the unit ball in p dimensions, and g_{ij} is the probability density function of the difference of embedding vectors $V_i - V_j$, which features in the mean of the attained Poisson law. Corresponding to the ordinary convergence of the binomial to the Poisson, Wolff obtains the rate of convergence for the random variable $K(n, p, r)$: $n \to \infty$, $r \to 0^+$ and $n^2 r^p \to \delta$, where δ is a constant depending on the underlying density of the observations evaluated at the origin. By estimating δ from a sample, one can find the optimal value of r at which to evaluate $K(n, p, r)$, namely

$$r = \delta n^{-2/p},$$

where optimality is in terms of closeness to the Poisson distribution for the unscaled correlation integral, and thus giving best calibration against that distribution.

Finally, note that the correlation integral is a U-statistic, and is admissible under Rényi's (1959) scheme as a dependence measure.

6. An Algorithm for Copula Selection

We propose the following algorithm for copula selection, noting that, from the equation in densities

$$f(x_1, \cdots, x_n) = f_1(X_1), \ldots f_n(X_n) c\{F_1(X_1), \cdots, F_n(X_n)\},$$

all information about dependence is contained in the function c, and so we may concentrate purely on the copula in order to assess the extent to which dependence has been captured in the modelling process.

Our algorithm is as follows, and assumes that time series data are available.

(i) Compute sample values of the marginal densities underlying the observed data, after Wolff's (1995) calculations, and thus determine the optimal value of r at which to evaluate the correlation integral, as well as fitting the Poisson parameter.

(ii) Compute the sequence $K_{\text{data}}(n, p, r)$ for the data, indexed by p.

(iii) For a proposed copula which is claimed to fit the data, generate a very large number simulated observations using stochastic simulation (such as by an appropriate Monte Carlo method, such as given by Hammersley and Handscombe, 1964).

(iv) Compute a corresponding sequence $K_{\text{cop}}(n, p, r)$ from the simulated observations.

(v) Treating $K_{\text{cop}}(n, p, r)$ as a constant – by virtue of the large volume of simulated observations – measure its closeness to $K_{\text{data}}(n, p, r)$, calibrating appropriately from the fitted underlying Poisson distribution for it.

(vi) Repeat for all candidate copulas, and choose the one which gives the closest calibrated match between $K_{\text{data}}(n,p,r)$ and $K_{\text{cop}}(n,p,r)$.

The efficacy of the method will be demonstrated in the conference presentation, and in forthcoming developments based on this paper.

7. Some Final Remarks

The only ambiguity in the preceding algorithm lies in choosing an appropriate embedding for the data. For the bivariate time series case, the natural embedding is given by $[X_{1,t}, X_{2,t}]'$, where indexing refers to the group and time epoch. This can be compared naturally with the marginal (with respect to time) dependence structure from the copula, simply *via* $[U_1, U_2]'$.

In order to capture time dependence, one may construct such embedding vectors as $[X_{1,t}, X_{1,t-1}, X_{2,t}, X_{2,t-1}]'$ in order to capture within-group correlation and cross-correlation, thus leading to a 4-variate copula problem. (This appeals to the dynamical system notion of delays within embeddings: see, again, Ruelle, 1989).

There is clearly a trade-off here between expanding the temporal range of the time series embedding and the rapidly diminishing loss of accuracy in calibrating the correlation integral through lack of data, namely, having exponentially growing “emptiness” in the multivariate space chosen for the embedding, the so-called curse of dimensionality. While the optimal situation is clearly to have the smallest sized embedding possible, a detailed understanding of the curse of dimensionality for this problem might shed light on the limits of how sophisticated a time-dependent copula model might be constructed.

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