University of Zurich

Department of Economics

Working Paper Series

ISSN 1664-7041 (print)
ISSN1664-705X(online)

Working Paper No. 32

Free-riding on Liquidity

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September 2011

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September 30, 2011

Abstract

Do financial market participants free-ride on liquidity? To address this question, we construct a dynamic general equilibrium model where agents face idiosyncratic preference and technology shocks. A secondary financial market allows agents to adjust their portfolio of liquid and illiquid assets in response to these shocks. The opportunity to do so reduces the demand for the liquid asset and, hence, its value. The optimal policy response is to restrict (but not eliminate) access to the secondary financial market. The reason for this result is that the portfolio choice exhibits a pecuniary externality: An agent does not take into account that by holding more of the liquid asset, he not only acquires additional insurance but also marginally increases the value of the liquid asset which improves insurance to other market participants.

1 Introduction

This paper addresses a basic, yet unresolved, question: Do financial market participants free-ride on liquidity? This question has emerged as one of the key issues in the policy discussion regrading the unprecedented freeze of money markets during the financial crisis of 2007 and 2008. One of the criticism was that many financial intermediaries became highly dependent on the availability of short-term instruments to finance their operations. In this paper, we show that free-riding on liquidity is indeed a problem even in the absence of aggregate shocks to the economy. Moreover, it can be so strong that the optimal policy response is to restrict access to facilities that provide short-term financing.

We derive this result in a dynamic general equilibrium model with two nominal assets: a liquid asset and an illiquid asset.[1] By liquid (illiquid), we mean that

[1] Our basic framework is the divisible money model developed in Lagos and Wright (2005).

the asset can be used (cannot be used) as a medium of exchange in goods market trades. Agents face idiosyncratic preference and technology shocks which generate an ex-post inefficiency in that some agents have "idle" liquidity holdings, while others are liquidity constrained in the goods market. This inefficiency generates an endogenous role for a secondary financial market where agents can trade the liquid for the illiquid asset before trading in the goods market. We show that restricting (but not eliminating) access to this secondary financial market can be welfare improving.

The basic mechanism for this result is as follows. In the absence of the secondary financial market, the demand for the liquid asset is high, since agents need to self-insure against the idiosyncratic preference and technology shocks. The aggregate demand for the liquid asset depends on the distribution of the liquidity risk and the cost of self-insurance. With the secondary financial market, they attempt to allocate a larger fraction of their wealth to the higher yielding illiquid asset. This reduces the demand for the liquid asset, and so its equilibrium price falls. This can reduce aggregate activity, and the effect can be so strong that it dominates the benefits provided by the secondary financial market in reallocating liquidity.

In a sense made precise in the present paper, the secondary financial market allows market participants to free-ride on the liquidity holdings of other participants. An agent does not take into account that by holding more of the liquid asset he not only acquires additional insurance against his own idiosyncratic liquidity risks, but also marginally increases the value of the liquid asset which improves insurance to other market participants. This pecuniary externality can be corrected by restricting, but not eliminating, access to the secondary financial market.

1.1 Liquidity

There are many competing definitions of liquidity. In our model, liquidity measures the ease of converting an asset into consumption. An asset that can be directly exchanged for consumption goods is called liquid. In our model, the liquid asset is fiat money and the illiquid asset is a one-period government bond. The former can be used as a medium of exchange to acquire consumption goods, while the latter cannot be used. Nevertheless, government bonds obtain a liq-

The main departure from this framework is that we add government bonds and a secondary bond market, and that we assume that all markets are perfectly competitive.

uidity premium, since they can be exchanged for fiat money in the secondary financial market, and the money can then subsequently be used to acquire consumption goods.

The fact that government bonds cannot be used as a medium of exchange to acquire consumption goods is the consequence of certain assumptions that we impose on our environment as explained in the main discourse of this paper. Note, though, that government bonds are only essential if they are illiquid.[2] That is, if agents can use government bonds as a medium of exchange in the goods market, the resulting allocation would be identical to the one obtained in an economy without government bonds. The intuition for this result is provided in Kocherlakota (2003, p. 184): "If bonds are as liquid as money, then people will only hold money if nominal interest rates are zero. But then the bonds can just be replaced by money: there is no difference between the two instruments at all." An interesting implication of this result is that "any essentiality of nominal bonds can be traced directly to their (relative) illiquidity (Kocherlakota 2003, p. 184)."[3]

1.2 Literature

During the financial crisis of 2007 and 2008, liquidity (markets that provide short-term financing) dried up dramatically. This phenomenon sparked a renewed interest by policy makers. However, liquidity shortages and liquidity crises have been studied intensively in the literature. Most of these papers share elements of the seminal contribution by Diamond and Dybvig (1983).

Our paper differs from this literature in several dimensions. First, our model is *not* a model of crisis: there are neither aggregate shocks nor multiple equilibria. Liquidity shortages and free-riding on liquidity occur in "normal" times; i.e., in the unique steady state equilibrium. Second, we propose a novel policy response by showing that restricting access to secondary financial markets can be optimal. Third, to our knowledge our paper is the only attempt to study free-riding on liquidity in a dynamic general equilibrium model. The infinite horizon allows us to determine endogenously the value of money, which is an

[2] By essential, we mean that illiquid government bonds improve the real allocation in the model.

[3] Alternatively, if nominal bonds are liquid and pay a strictly positive nominal interest rate, then the real value of fiat money must be zero. But since interest on nominal bonds is paid out in units of fiat money, the real value of the interest on bonds is zero as well. It then follows that nominal bonds can never be sold at a discount, so a zero nominal interest rate is the only equilibrium.

intrinsically useless object and, so, can have no value in a deterministic model with a finite time horizon.

Real assets and finite time horizon models[4] Diamond and Dybvig (1983) study an economy where depositors are subject to idiosyncratic liquidity shocks which are private information. They show that deposit contracts are subject to a coordination problem which may lead to banking panics. Given their assumptions of first-come, first-served and costly liquidation, if depositors believe that other depositors will withdraw money from the bank, they may decide to also make early withdrawals which then leads to a liquidity crisis. Diamond and Dybvig (1983) show that a pre-commitment policy, in the form of "suspension of convertibility", leads to a Pareto-superior Nash equilibrium.

Building on Diamond and Dybvig (1983), Bhattacharya and Gale (1987) develop a model of an inter-bank market. The inter-bank market creates a free-riding problem: banks tend to underinvest in liquid assets since they know they can meet their liquidity needs on the inter-bank market. They show that an optimal insurance contract, in the form of central bank reserve requirements, can be welfare improving.

Diamond and Rajan (2005) study whether bank failures can cause a shortage of liquidity in an economy with real contracts. They show that if too many projects are delayed, the bank may run short of liquidity. This may lead the bank to call loans, forcing a costly restructuring of investments, which may lead to runs on other banks, even if depositors are optimistic.[5]

Nominal assets and finite time horizon models Holmström and Tirole (1998) study the role of government provision of liquidity in a model with idiosyncratic and aggregate liquidity shocks. Without aggregate uncertainty, there is no need for government intervention. The optimal allocation can be achieved by financial intermediaries that offer credit lines. With aggregate uncertainty, the private sector cannot attain the constrained-efficient allocation, because each firm needs liquidity exactly when all the other firms need it too. In this case, government provision of liquidity is Pareto-improving.

[4] The following literature review is in no way complete. The literature that followed the seminal contribution by Diamond and Dybvig (1983) is simply too large for a comprehensive review.

[5] Diamond and Rajan (2006) extend Diamond and Rajan (2005) to nominal contracts. While this extension is useful to understand the relation between variables like prices, money, and real production, it does not affect the main findings of Diamond and Rajan (2005).

Allen and Gale (1998) argue that bank runs are related to business cycles rather than random events. During an economic downturn, the value of bank assets decreases, which makes it difficult for banks to meet their obligations. Anticipating this, depositors withdraw money, which worsens the banks liquidity problem and accelerates the crisis. They study the optimal central bank policy during panics. In some cases, a laissez-faire central bank which does not respond to crisis is efficient. In other cases, central bank intervention is optimal.

In Cao and Illing (2010a), banks choose how much they invest in lower yielding liquid projects and higher yielding illiquid projects. Because of an inter-bank market, banks have an incentive to free-ride on the liquidity holdings of other banks, which results in excessive investments in illiquid projects. In the following period, impatient investors run the bank if they learn that the bank's liquidity is not sufficient to cover their claims. Borrowing from this setup, Cao and Illing (2010b) study the optimal policy response in this framework. They show that ex-ante liquidity requirements combined with an ex-post lender of last resort policy attains the highest payoffs for investors. This is because the former policy response prevents banks from free-riding, whereas the latter prevents bank runs in the presence of aggregate shocks.

Infinite time horizon models with nominal assets An exception to the above finite time horizon models is Rojas-Breu (2010). In Rojas-Breu (2010), some agents use credit cards and some fiat money to acquire consumption goods. She shows that restricting the use of credit cards can be welfare improving. The intuition for this result is that marginally increasing the fraction of agents that use credit cards can have a general equilibrium effect on the price level, which makes the agents that have no credit card worse off. This effect can be so strong that overall welfare decreases. In contrast to our model, in her model restricting the use of credit cards is a local optimum only, since it would be optimal to endow all agents with credit cards.

2 The Model

Time is discrete, and in each period there are three perfectly competitive markets which open sequentially.[6] The first market is a secondary bond market

[6] Our basic framework is the divisible money model developed in Lagos and Wright (2005). This model is useful, because it allows us to introduce heterogeneous preferences while still keeping the distribution of asset holdings analytically tractable. The main departure from

where agents trade money for bonds. The second market is a goods market where agents produce or consume market-2 goods. The third market is the settlement market where all agents consume and produce market-3 goods. All goods are nonstorable, which means that they cannot be carried from one market to the next.

There is a $[0, 1]$ continuum of infinitely lived agents. At the beginning of each period, agents receive two idiosyncratic iid shocks: a preference shock and an entry shock. The preference shock determines whether an agent can produce or consume in the goods market. With probability $1 - n$ an agent can consume but not produce, and with probability n he can produce but not consume. Consumers in market 2 are called buyers, and producers are called sellers. The entry shock determines whether agents can participate in the secondary bond market. With probability π they can, and with probability $1 - \pi$ they cannot. Agents who participate in the bond market are called *active*, while agents who do not are called *passive*.

In the goods market, buyers get utility $u(q)$ from consuming q units of the market-2 goods, where $u'(q), -u''(q) > 0$, $u'(0) = \infty$, and $u'(\infty) = 0$. Sellers incur the utility cost $c(q) = q$ from producing q units of market-2 goods.[7]

As in Lagos and Wright (2005), for tractability, we impose assumptions that yield a degenerate distribution of portfolios at the beginning of the secondary bond market. That is, we assume that in the last market all agents can produce and consume market-3 goods. The production technology is linear such that h units of time produce h units of goods. The utility of consuming x units of goods is $U(x)$, where $U'(x) > 0$, $U''(x) \geq 0$, $U'(0) = \infty$, and $U'(\infty) = 0$.

Agents discount between, but not within, periods. The discount factor between two consecutive periods is $\beta = 1/(1 + r)$, where $r > 0$ represents the real interest rate.

2.1 First-best allocation

The planner treats all agents symmetrically. His optimization problem is

$$\mathcal{W} = \max_{h,x,q,q_s} [(1 - n) u(q) - nq_s] + U(x) - h \tag{1}$$

Lagos and Wright (2005) is that we add government bonds and a secondary bond market as in Berentsen and Waller (2011).

[7] We assume a linear utility cost for ease of exposition. It is a simple generalization to allow for a more general convex disutility cost.

subject to $h \geq x$ and $nq_s \geq (1-n)\, q$. The first inequality is the feasibility constraint for market-3 goods, and the second inequality is the one for market-2 goods. The first-best allocation satisfies $U'(x^*) = 1$, $u'(q^*) = 1$, $h^* = x^*$, and $q_s^* = n^{-1}(1-n)\, q^*$. These are the quantities chosen by a social planner who dictates consumption and production.

2.2 Information frictions, money and bonds[8]

There are two perfectly divisible financial assets: money and one-period, nominal discount bonds. Both are intrinsically useless, since they are neither arguments of any utility function nor are they arguments of any production function. Both assets are issued by the central bank in the last market. Bonds are payable to the bearer and default free. One bond pays off one unit of currency in the last market of the following period.

At the beginning of a period, after the idiosyncratic shocks are revealed, agents can trade bonds and money in the secondary bond market. The central bank acts as the intermediary for all bond trades, by recording purchases/sales of bonds, and redistributing money receipts.[9] Since bonds are intangible objects, they are incapable of being used as a medium of exchange in the goods market, hence they are illiquid.[10] Since agents are anonymous and cannot commit, a buyer's promise in market 2 to deliver bonds to a seller in market 3 is not credible.

To motivate a role for fiat money, search models of money typically impose three assumptions on the exchange process (Shi 2008): a double coincidence problem, anonymity, and costly communication. First, our preference structure creates a single-coincidence problem in the goods market, since buyers do not have a good desired by sellers. Second, agents in the goods market are

[8] The description of the environment in this subsection follows very closely Berentsen and Waller (2011). However, the question investigated in Berentsen and Waller (2011) are not related to the questions studied in this paper.

[9] The central bank is assumed to have a record-keeping technology over bond trades in the secondary bond market, and bonds are book-keeping entries – no physical object exists. This implies that agents are not anonymous to the central bank. Nevertheless, despite having a record-keeping technology over bond trades, the central bank has no record-keeping technology over goods trades.

[10] The beneficial role of illiquid bonds has been studied in Kocherlakota (2003). Sun (2007), Shi (2008), and Berentsen and Waller (2011) also find that it is optimal that bonds are illiquid. All these papers, including Kocherlakota (2003), assume unrestricted access to secondary financial markets. One of our contributions to this literature is to show that it is not only optimal that bonds are illiquid but that one has to go one step further. It can be optimal to reduce their liquidity even further by restricting participation in the secondary financial market where these bonds are traded for money.

anonymous, which rules out trade credit between individual buyers and sellers. Third, there is no public communication of individual trading outcomes (public memory), which, in turn, eliminates the use of social punishments in support of gift-giving equilibria. The combination of these frictions implies that sellers require immediate compensation from buyers. In short, there must be immediate settlement with some durable asset, and money is the only such durable asset. These are the micro-founded frictions that make money essential for trade in market 2. Araujo (2004), Kocherlakota (1998), Wallace (2001), and Aliprantis, Camera and Puzzello (2007) provide a more detailed discussion of the characteristics that make money essential. In contrast, in the last market all agents can produce for their own consumption or use money balances acquired earlier. In this market, money is not essential for trade.[11]

2.3 Money supply process

Denote M_t as the per capita money stock and B_t as the per capita stock of newly issued bonds at the end of period t. Then M_{t-1} (B_{t-1}) is the beginning-of-period money (bond) stock in period t. Let ρ_t denote the price of bonds in market 3. Then, the change in the money stock in period t is given by

$$M_t - M_{t-1} = \tau_t M_{t-1} + B_{t-1} - \rho_t B_t. \tag{2}$$

The change in the money supply at time t is given by three components: a lump-sum money transfer ($T = \tau_t M_{t-1}$); the money created to redeem B_{t-1} units of bonds; and the money withdrawal from selling B_t units of bonds at the price ρ_t. We assume there are positive initial stocks of money M_0 and bonds B_0 with $\frac{B_0}{M_0} > \frac{n}{1-n}$. For $\tau_t < 0$, the government must be able to extract money via lump-sum taxes from the economy.

3 Agent's decisions

For notational simplicity, the time subscript t is omitted when understood. Next-period variables are indexed by $+1$, and previous-period variables are indexed by -1. In what follows, we look at a representative period t and work backwards, from the settlement to the bond market.

[11] One can think of agents as being able to barter perfectly in this market. Obviously in such an environment, money is not needed.

3.1 Settlement market

In the settlement market, agents can consume and produce market-3 goods. Furthermore, they receive money for maturing bonds, buy newly issued bonds, adjust their money balances by trading money for goods, and receive the lump-sum money transfer T. An agent entering the settlement market with m units of money and b units of bonds has the indirect utility function $V_3(m, b)$. An agent's decision problem in the settlement market is

$$V_3(m, b) = \max_{x,h,m_{+1},b_{+1}} \left[U(x) - h + \beta V_1(m_{+1}, b_{+1})\right], \tag{3}$$

subject to

$$x + \phi m_{+1} + \phi\rho b_{+1} = h + \phi m + \phi b + \phi T. \tag{4}$$

The first-order conditions with respect to m_{+1}, b_{+1} and x are $U'(x) = 1$, and

$$\frac{\beta \partial V_1}{\partial m_{+1}} = \rho^{-1} \frac{\beta \partial V_1}{\partial b_{+1}} = \phi, \tag{5}$$

where the term $\beta\partial V_1/\partial m_{+1}$ $(\beta\partial V_1/\partial b_{+1})$ is the marginal benefit of taking one additional unit of money (bonds) into the next period, and ϕ $(\rho\phi)$ is the marginal cost of doing so. Due to the quasi-linearity of preferences, the choices of b_{+1} and m_{+1} are independent of b and m. It is straightforward to show that all agents exit the settlement market with the same portfolio of bonds and money. The envelope conditions are

$$\frac{\partial V_3}{\partial m} = \frac{\partial V_3}{\partial b} = \phi. \tag{6}$$

3.2 Goods market

Sellers Let $V_2^s(m, b)$ denote the expected value for a seller who enters the goods market with m units of money and b units of bonds. The seller's decision problem is

$$V_2^s(m, b) = \max_{q_s} \left[-q_s + V_3(m + pq_s, b)\right] \tag{7}$$

He receives disutility q_s from producing q_s units of market-2 goods and his continuation value is $V_3(m + pq_s, b)$.

Using (6), the seller's first order condition is

$$p\phi = c'(q_s) = 1. \tag{8}$$

Condition (8) means that a seller chooses to produce a quantity q_s such that the marginal cost of producing an additional unit, $c'(q_s) = 1$, is equal to the marginal benefit, $p\phi$. Note that a seller's decision is independent of his portfolio (m, b). This means that the seller's decision in the goods market is not affected by the entry shock. In what follows, we therefore assume that all sellers produce the same quantity q_s in the goods market.

Buyers Let $V_2^b(m, b)$ denote the expected value for a buyer who enters the goods market with m units of money and b units of bonds. Let p be the price of goods in the goods market. His decision problem is

$$V_2^b(m, b) = \max_q \begin{bmatrix} u(q) + V_3(m - pq, b) \\ \text{s.t. } m - pq \geq 0 \end{bmatrix}. \tag{9}$$

He receives utility $u(q)$ from consuming q units of market-2 goods, and his continuation value is $V_3(m - pq, b)$. The constraint means that he cannot spend more money than the amount he has.

Using (6) and (7), a buyer's first order conditions in market 2 is

$$u'(q) = p(\phi + \lambda_q) \tag{10}$$

where λ_q is the Lagrange multiplier of the buyer's cash constraint. If the cash constraint is not binding, then a buyer consumes the efficient quantity of goods. If the cash constraint is binding, then he spends all his money in goods purchases, and consumption is inefficiently low.

Note that a buyer's decision depends on his portfolio (m, b), since m enters his cash constraint. Furthermore, in equilibrium, an active buyer will hold more money than a passive buyer. This means that $\lambda_q > \hat{\lambda}_q$, where the "ˆ" indicates that the buyer had access to the secondary bond market. It then follows that $\hat{q} > q$.

Apply the envelope theorem to (7) and (9) to get the marginal values of bonds and the marginal values of money for buyers and sellers at the beginning of the second market:

$$\frac{\partial V_2^b}{\partial m} = \frac{u'(q)}{p}, \quad \frac{\partial V_2^s}{\partial m} = \frac{\partial V_2^b}{\partial b} = \frac{\partial V_2^s}{\partial b} = \phi. \tag{11}$$

3.3 Secondary bond market

Let $\left(\hat{m}, \hat{b}\right)$ denote the portfolio of an active agent after trading in the secondary bond market. Furthermore, let $V_1(m, b)$ denote the expected value for an agent who enters the secondary bond market with m units of money and b units of bonds before the idiosyncratic shocks are realized. The value $V_1(m, b)$ satisfies[12]

$$\begin{aligned} V_1(m,b) &= \pi(1-n)\,V_1^b(m,b) + \pi n V_1^s(m,b) \\ &\quad +(1-\pi)(1-n)\,V_2^b(m,b) + (1-\pi)nV_2^s(m,b) \end{aligned}$$

where for $j = b, s$

$$V_1^j(m,b) = \left[\begin{array}{c} \max\limits_{\hat{m},\hat{b}} V_2^j(\hat{m},\hat{b}) \\ s.t.\ \phi m + \varphi\phi b \geq \phi\hat{m} + \varphi\phi\hat{b}, \\ \hat{m} \geq 0,\ \hat{b} \geq 0 \end{array}\right]$$

is the value function of an active buyer ($j = b$) or an active seller ($j = s$). Active agents trade bonds for money in order to maximize their utility subject to the budget constraint

$$\phi m + \varphi\phi b \geq \phi\hat{m} + \varphi\phi\hat{b}. \tag{13}$$

Furthermore, there are two short-selling constraints: agents cannot sell more bonds, and they cannot spend more money, than the amount they carry from the previous period that is

$$\hat{m} \geq 0,\ \hat{b} \geq 0. \tag{14}$$

The Lagrange multiplier on (13) is denoted by λ^j, while λ_m^j, and λ_b^j are the Lagrange multipliers on (14), where $j = b, s$ indicates the agent's type (buyer or seller). The bond market first-order conditions for an active agent are

$$\frac{\partial V_2^j}{\partial \hat{m}} = \phi\lambda^j - \lambda_m^j, \text{ and} \tag{15}$$

$$\frac{\partial V_2^j}{\partial \hat{b}} = \varphi\phi\lambda^j - \lambda_b^j. \tag{16}$$

[12] Passive buyers and passive sellers cannot change their portfolios and so their value functions at the beginning of the bond market are $V_2^b(m, b)$ and $V_2^s(m, b)$, respectively.

The envelope conditions in the secondary bond market are

$$\frac{\partial V_1}{\partial m} = \pi\phi\left[(1-n)\,\lambda^b + n\lambda^s\right] + (1-\pi)\left[(1-n)\,\frac{\partial V_2^b}{\partial m} + n\frac{\partial V_2^s}{\partial m}\right], \quad (17)$$

$$\frac{\partial V_1}{\partial b} = \pi\phi\varphi\left[(1-n)\,\lambda^b + n\lambda^s\right] + (1-\pi)\left[(1-n)\,\frac{\partial V_2^b}{\partial b} + n\frac{\partial V_2^s}{\partial b}\right]. \quad (18)$$

4 Stationary monetary equilibria

We focus on symmetric, stationary monetary equilibria, where all agents follow identical strategies and where real variables are constant over time. Let $\eta \equiv B/B_{-1}$ be the gross growth rate of bonds, and let $\gamma \equiv M/M_{-1}$ be the gross growth rate of the money supply. These definitions allow us to write (2) as

$$\gamma - 1 - \tau = \frac{B_{-1}}{M_{-1}}\,(1-\rho\eta)\,. \quad (19)$$

In a stationary monetary equilibrium, the real stock of money must be constant; i.e., $\phi M = \phi_{+1} M_{+1}$ implying that $\gamma = \phi/\phi_{+1}$. Furthermore, the real amount of bonds must be constant; i.e., $\phi B = \phi_{-1} B_{-1}$. This implies $\eta = \gamma$, which we can use to rewrite (19) as

$$\gamma - 1 - \tau = \frac{B_0}{M_0}\,(1-\rho\gamma)\,. \quad (20)$$

Furthermore, in any equilibrium the market clearing condition for market 2 is

$$(1-n)\,[\pi\hat{q} + (1-\pi)q] = nq_s. \quad (21)$$

A symmetric stationary equilibrium is a sequence of quantities and prices such that: the first order conditions (8) and (10), envelope conditions (11), and clearing condition (21) in the goods market are satisfied; the first order conditions (5) and envelope conditions (6) in the settlement market hold; the first order conditions (15)-(16) and the envelope conditions (17)-(18) in the bond market are satisfied; the government budget constraint (20) holds; real variables are constant over time, and agent's decisions are symmetric.

The model has three types of equilibria. In the type I equilibrium, consumption is inefficient for passive buyers, and it is efficient for active buyers. In the type II equilibrium, consumption is inefficient for both active and passive buyers, and the constraint on bond holdings of the active buyer is not binding. In the type III equilibrium, consumption is inefficient for both active and passive buyers, and the constraint on bond holdings of the active buyers is binding.

In all these equilibria, a buyer will never spend all his money for bonds in the secondary bond market, implying that $\lambda_m^b = 0$. Furthermore, a seller will never spend all his bonds for money in the secondary bond market, implying that $\lambda_b^s = 0$.

4.1 Type I equilibrium

In the type I equilibrium, consumption is inefficient for passive buyers, and it is efficient for active buyers ($\lambda_q > 0, \hat{\lambda}_q = 0$). The constraint on bond holdings of an active buyer does not bind, and a seller's constraint on his cash holdings does not bind ($\lambda_b^b = \lambda_m^s = 0$). Using the goods market envelope conditions and the bond market first order conditions, it holds that $u'(\hat{q}) = 1/\varphi = 1$. The bond market envelope conditions can be written as

$$\begin{aligned}
\frac{\gamma}{\beta} &= \pi + (1-\pi)\left[(1-n)\,u'(q) + n\right], \\
\frac{\gamma\rho}{\beta} &= 1.
\end{aligned}$$

At this point we can define the equilibrium:

Definition 1 *A type I equilibrium is a time-independent path* $\{\tau, q, q_s, \hat{q}, \rho, \varphi\}$ *satisfying (20), (21), and*

$$\begin{aligned}
\varphi &= 1 && (22)\\
\frac{\gamma}{\beta} &= \pi + (1-\pi)\left[(1-n)\,u'(q) + n\right] && (23)\\
\rho &= \frac{\beta}{\gamma} && (24)\\
\hat{q} &= 1 && (25)
\end{aligned}$$

given parameters value $\{\gamma, \beta, B_0, M_0, n, \pi\}$.

Equation (22) comes from the goods market first order conditions, expressions (23) and (24) come from the settlement market first order conditions. In the type I equilibrium, an active buyer is not cash constrained in the goods market (i.e., he consumes the efficient quantity, $\hat{q} = 1$); i.e., equation (25) must hold. The consumed quantity for passive buyers, q comes from (23). The price of bonds is equal to 1 in the bond market, while it is equal to their fundamental value β/γ in the settlement market. The lump sum money transfer, τ, comes from (20), and the produced quantity q_s is obtained from (21).

In equilibrium I, the secondary bond market return on bonds, $1/\varphi - 1$, is zero. It cannot be negative, since otherwise both buyers and sellers want to sell bonds. It cannot be strictly positive, since, with a positive interest rate, both active sellers and active buyers would want to buy bonds, and the supply would be zero.

4.2 Type II equilibrium

In the type II equilibrium, consumption is inefficient for both active and passive buyers ($\lambda_q > 0, \hat{\lambda}_q > 0$). The active buyer's constraint on bond holdings does not bind ($\lambda_b^b = 0$), and the active seller's constraint on bond holdings binds ($\lambda_m^s > 0$). Active buyers sell bonds for money up to the point where the marginal benefit of doing so is equal to the marginal cost. Active sellers sell all their money for bonds. Using the goods market envelope conditions, and replacing $\lambda_b^s = \lambda_m^b = 0$ into the bond market first-order conditions, we obtain $u'(\hat{q})/p = \phi\lambda^b$, $\phi = \varphi\phi\lambda^b$, $\phi = \phi\lambda^s - \lambda_m^s$, $\phi = \varphi\phi\lambda^s$. Substitutions yield $\varphi = 1/u'(\hat{q}) < 1$. The bond market envelope conditions, (17) and (18), can be written as follows:

$$\begin{aligned}
\frac{\partial V_1}{\partial m} &= \pi\phi\left[(1-n)\frac{1}{\varphi} + n\frac{1}{\varphi}\right] + (1-\pi)\,\phi\left[(1-n)\,u'(q) + n\right], \\
\frac{\partial V_1}{\partial b} &= \pi\phi\varphi\left[(1-n)\frac{1}{\varphi} + n\frac{1}{\varphi}\right] + (1-\pi)\,\phi.
\end{aligned}$$

In the type II equilibrium, the following holds.

Lemma 1 *In the type II equilibrium, $q = \hat{q}(1-n)$.*

Definition 2 *A type II equilibrium is a time-independent path $\{\tau, q, q_s, \hat{q}, \rho, \varphi\}$ satisfying (20), (21), and*

$$\frac{1}{\varphi} = u'(\hat{q}) \tag{26}$$

$$\frac{\gamma}{\beta} = \pi\frac{1}{\varphi} + (1-\pi)\left[(1-n)\,u'(q) + n\right], \tag{27}$$

$$\rho = \frac{\beta}{\gamma}, \tag{28}$$

$$q = \hat{q}\,(1-n) \tag{29}$$

given parameters value $\{\gamma, \beta, B_0, M_0, n, \pi\}$.

The interpretations of the equilibrium equations in Definition 2 are similar to their respective equations in Definition 1. The model can be solved as follows. Equations (26), (27), (21), and (29) can be used to obtain φ, q, q_s, and $\hat{q}$. The price of bonds in the settlement market comes from (28), and the lump sum money transfer from (20).

4.3 Type III equilibrium

In the type III equilibrium, consumption is inefficient for both active and passive buyers ($\lambda_q > 0, \hat{\lambda}_q > 0$), the buyer's bonds constraint binds ($\lambda_b^b > 0$), and the seller's cash constraint binds ($\lambda_m^s > 0$). Using the envelope conditions in the goods-market, and replacing $\lambda_b^s = 0$ and $\lambda_m^b = 0$ into the bond market first-order conditions, one gets $u'(\hat{q})/p = \phi\lambda^b$, $\phi = \varphi\phi\lambda^b - \lambda_b^b, \phi = \phi\lambda^s - \lambda_m^s, 1 = \varphi\lambda^s$. Using (5), lagged one period, envelope conditions (17) and (18) can be written as follows:

$$\frac{\gamma}{\beta} = \pi\left[(1-n)\,u'(\hat{q}) + n\frac{1}{\varphi}\right] + (1-\pi)\left[(1-n)\,u'(q) + n\right] \tag{30}$$

$$\frac{\gamma}{\beta} = \frac{\varphi\pi}{\rho_{-1}}\left[(1-n)\,u'(\hat{q}) + \frac{n}{\varphi}\right] + \frac{1-\pi}{\rho_{-1}}. \tag{31}$$

All variables in (30) are constant. From (31), ρ must be constant too, and so the second equation can be written as follows:

$$\rho = \frac{\beta}{\gamma}\left\{1 + \pi\,(1-n)\left[u'(\hat{q})\varphi - 1\right]\right\}.$$

The price of bonds in the settlement market ρ includes two components: the *fundamental value* of bonds and the *liquidity premium* which is proportional to the access probability π. In the type III equilibrium, the buyer's constraint on bond holdings is binding, thus bonds help to relax the active buyer's cash constraint in the goods market. When $\pi = 0$, bonds are illiquid in the secondary bond market, and the liquidity premium is zero. When $\pi = 1$, they are perfectly liquid in the secondary bond market, and the liquidity premium is $\beta\,(1-n)\left[\varphi u'(\hat{q}) - 1\right]/\gamma$. For intermediate values $0 < \pi < 1$, the liquidity premium is in the interval $[0, \beta\,(1-n)\,\pi\left[\varphi u'(\hat{q}) - 1\right]/\gamma]$.

There is no liquidity premium in equilibria I and II, since an active buyer's constraint on bond holdings is not binding. Consequently, bringing one additional unit of bonds into the bond market does not help to relax the cash constraint in the goods market, and so there is no liquidity premium on bonds.

Lemma 1 still holds in the type III equilibrium. In addition, the following Lemma holds.

Lemma 2 *In the type III equilibrium,* $\frac{B_0}{M_0} = \frac{n}{\varphi(1-n)}$.

Definition 3 *A type III equilibrium is a time-independent path* $\{\tau, q, q_s, \hat{q}, \rho, \varphi\}$ *satisfying (20), (21), and*

$$\frac{B_0}{M_0} = \frac{n}{\varphi(1-n)} \tag{32}$$

$$\frac{\gamma}{\beta} = \pi\left[(1-n)\,u'(\hat{q}) + n\frac{1}{\varphi}\right] + (1-\pi)\left[(1-n)\,u'(q) + n\right] \tag{33}$$

$$\rho = \frac{\beta}{\gamma}\left\{1 + \pi\,(1-n)\left[u'(\hat{q})\varphi - 1\right]\right\} \tag{34}$$

$$q = \hat{q}(1-n) \tag{35}$$

given parameters value $\{\gamma, \beta, B_0, M_0, n, \pi\}$.

In Definition (3), φ is obtained from Lemma 2. In contrast, in Definitions (1) and (2) it was obtained from the goods-market first order conditions (26) and (22), respectively. All the other equations in Definition 3 have the same interpretation of the respective equations in Definition 2.

The problem can be solved as follows. Use (32) to get φ. Then solve equations (33), (21) and (35) for q_s, q and $\hat{q}$. Finally, solve (34) for ρ and (20) for τ.

4.4 Regions of equilibria

In the following Lemma we characterize three non-overlapping regions in which these different types of equilibria exist.

Lemma 3 *There exist critical values* γ_L *and* γ_H*, with* $\beta \leq \gamma_L \leq \gamma_H < \infty$*, such that the following is true: if* $\beta \leq \gamma < \gamma_L$*, an active buyer consumes the efficient quantity of goods and his constraint on bond holdings does not bind; if* $\gamma_L \leq \gamma < \gamma_H$*, he consumes the inefficient quantity of goods and his constraint on bond holdings does not bind; if* $\gamma_H \leq \gamma$*, he consumes the inefficient quantity of goods and his constraint on bond holdings binds. The bond prices* φ *and* ρ

satisfy:

$$\begin{array}{lll}\varphi = 1, & \rho = \beta/\gamma, & \text{if } \beta \leq \gamma < \gamma_L, \\ \varphi = 1/u'(\hat{q}), & \rho = \beta/\gamma, & \text{if } \gamma_L \leq \gamma < \gamma_H, \\ \varphi = \frac{M_0}{B_0}\frac{n}{1-n}, & \rho = \frac{\beta}{\gamma}\left\{1 + \pi(1-n)\left[u'(\hat{q})\varphi - 1\right]\right\}, & \text{if } \gamma_H \leq \gamma. \end{array}$$

The critical values are

$$\begin{aligned}\gamma_L &= \beta\left\{\pi + (1-\pi)\left[(1-n)\,u'(\frac{1}{1-n})^{-1} + n\right]\right\}, \text{ and} \\ \gamma_H &= \beta\left\{\frac{\pi}{\varphi} + (1-\pi)\left[\frac{(1-n)}{\varphi}u'(\frac{1}{1-n})^{-1} + n\right]\right\}.\end{aligned}$$

where $\frac{1}{\varphi} = \frac{1-n}{n}\frac{B_0}{M_0}$.

In the type I and II equilibria ($\beta \leq \gamma < \gamma_H$), the constraint on bond holdings of active buyers does not bind in the secondary bond market. This implies that the return on bonds in the secondary bond market, $1/\varphi$, has to be equal to the return on money, $u'(\hat{q})$, and the price of bonds in the settlement market, ρ, must equal the *fundamental value* of bonds. The economics underlying this result are straightforward. Since active agents do not sell all their bonds for money in the secondary bond market, bonds have no *liquidity premium*, for any $\beta \leq \gamma < \gamma_H$, and the Fisher equation holds.[13]

In contrast, in the type III equilibrium ($\gamma_H \leq \gamma$), the constraint on bond holdings of active buyers binds in the secondary bond market. Consequently, bonds attain a liquidity premium, and the Fisher equation does not hold; i.e., $1/\rho < \gamma/\beta$.

Figure 1 graphically characterizes the bond prices, φ and ρ, as a function of γ in the three types of equilibria. The price of bonds in the secondary bond market (φ) is constant and equal to 1 in the type I equilibrium (the region between β and γ_L); it is decreasing in the type II equilibrium (the region between γ_L and γ_H); and it is constant in the type III equilibrium (the region between γ_H and ∞). The price of bonds in the settlement market (ρ) follows a different pattern. In the type I and type II equilibrium, it is equal to the fundamental value of bonds (β/γ), whereas in the type III equilibrium it contains a liquidity premium. The higher π in the type III equilibrium is, the larger is the difference between φ and ρ.

[13] The Fisher equation requires that $1/\rho = \gamma/\beta$.

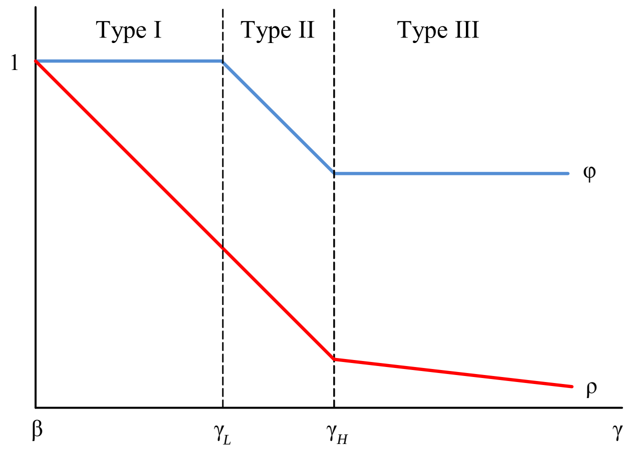

Figure 1: Bond prices when $\pi < 1$.

Why is there a positive spread $\varphi - \rho$ if $\pi < 1$? If $\pi < 1$, the price ρ reflects the fact that the bond can be traded with probability π in the secondary bond market. In contrast, the price φ reflects the fact that the bond can be traded with probability 1, since the agent has access to the secondary bond market. Thus, the positive spread is because the bond in the secondary bond market has a higher liquidity premium than the bond in the settlement market. Note that the spread vanishes as $\pi \to 1$. In this case, $\varphi = \rho$ in all equilibria.

5 Free-riding on liquidity

In this section, we make precise what we mean by the notion of free-riding on liquidity. First, we show that if agents have a choice to participate in the secondary bond market, they strictly prefer to do so. Second, we show that such a participation choice involves a negative pecuniary externality that is not internalized by market participants. Third, we show that a policy of restricting access to the secondary bond market can be welfare improving.

5.1 Endogenous participation

So far, we have assumed that participation in the bond market is determined by the exogenous idiosyncratic participation shock π. Suppose instead that each agent has a choice. Recall that $V_1^b(m, b)$ is the expected lifetime utility of a buyer at the beginning of the secondary bond market, and $V_2^b(m, b)$ is expected expected lifetime utility of a buyer at the beginning of the good market who had no access to the secondary bond market. Then, for a buyer, it is optimal to participate if

$$V_1^b(m,b) \geq V_2^b(m,b).$$

Lemma 4 *In any equilibrium,* $V_1^b(m,b) - V_2^b(m,b) > 0$.

According to Lemma 4, a buyer is always better off participating in the secondary bond market.

To develop an intuition for this result, note that, as shown in the proof of Lemma 4,

$$V_1^b(m,b) - V_2^b(m,b) = u(\hat{q}) - \hat{q} - [u(q) - q] - i(\hat{q} - q), \tag{36}$$

where $i = (1-\varphi)/\varphi$ is the nominal interest rate. A passive agent's period surplus is $u(q) - q$. He acquires q units of consumption goods which yields utility $u(q)$. Since he pays with money, the decrease in his money holdings reduces future expected consumption, which in terms of utility is valued $-q$. An active buyer's surplus is $u(\hat{q}) - \hat{q} - i(\hat{q} - q)$. He acquires $\hat{q}$ units of consumption goods, which yields utility $u(\hat{q})$. The decrease in his money holdings reduces future expected consumption, which in terms of utility is valued $-\hat{q}$. The term $i(\hat{q} - q)$ measures the utility cost of selling bonds to finance the difference $\hat{q} - q > 0$.

The term $u(\hat{q}) - \hat{q} - [u(q) - q]$ is strictly positive, while the term $-i(\hat{q} - q)$ is negative. Thus, the equilibrium interest rate cannot be too large in order for (36) to be positive. In the proof of Lemma 4 we replace i in (36) for all all three equilibria type and find that $V_1^b(m,b) - V_2^b(m,b) > 0$.

For the sellers, we also find that it is optimal to participate in the secondary bond market.

Lemma 5 *In any equilibrium,* $V_1^s(m,b) - V_2^s(m,b) \geq 0$.

In the type I equilibrium, the nominal interest rate is $i = 0$. In this case, $V_1^s(m,b) = V_2^s(m,b)$. In the type II and type III equilibria, the nominal interest rate is $i > 0$. In this case, the seller strictly prefers to enter, since $V_1^s(m,b) > V_2^s(m,b)$.

5.2 Optimal secondary bond market participation

Lemmas 4 and 5 show that if agents have a choice, they will participate in the secondary bond market. In the following section, we show that restricting participation to the bond market can be welfare improving.

In this section, we show that restricting participation to the secondary bond market can be welfare improving. The reason is straightforward. The secondary bond market provides insurance against the idiosyncratic liquidity shocks. At the end of a period in the settlement market, agents choose a portfolio of bonds and money. At this point, they do not know yet whether they will be buyers or sellers in the following period. At the beginning of the following period, this information is revealed, and they can use the secondary bond market to readjust their portfolio of liquid (money) and illiquid (bonds) assets.

From a welfare point of view, the benefit of the secondary bond market is that it allocates liquidity to the buyers and allows sellers to earn interest on their idle money holdings. The drawback of this opportunity is that the secondary bond market reduces the incentive to self-insure against the liquidity shocks. This lowers the demand for money in the settlement market, which depresses its value. This effect can be so strong that it can be optimal to restrict access to the secondary bond market. The basic mechanism can be seen from the following welfare calculations.

The welfare function can be written as follows

$$(1-\beta)\,\mathcal{W} = (1-n)\left\{\pi\left[u(\hat{q})-\hat{q}\right]+(1-\pi)\left[u(q)-q\right]\right\}+U(x^{*})-x^{*}, \qquad (37)$$

where the term in the curly brackets is an agent's expected period utility in the goods market, and $U(x^{*})-x^{*}$ is the agent's period utility in the settlement market.

Differentiating (37) with respect to π yields

$$\begin{aligned}\frac{1-\beta}{1-n}\frac{d\mathcal{W}}{d\pi} &= \left[u(\hat{q})-\hat{q}\right]-\left[u(q)-q\right] \\ &\quad +\pi\left[u'(\hat{q})-1\right]\frac{d\hat{q}}{d\pi}+(1-\pi)\left[u'(q)-1\right]\frac{dq}{d\pi},\end{aligned}$$

The contribution of the first two terms to the change in welfare is always positive, since in any equilibrium $\hat{q} \geq q$ (with strict inequality for $\gamma > \beta$). However, the derivatives $\frac{d\hat{q}}{d\pi}$ and $\frac{dq}{d\pi}$ can be negative, reflecting the fact that increasing participation reduces the incentive to self-insure against idiosyncratic liquidity risk.[14] Reducing the incentive to self-insure reduces the demand for money and hence its value, which then reduces the consumption quantities q and $\hat{q}$.

[14] We have derived a sufficient condition for $\frac{d\hat{q}}{d\pi},\frac{dq}{d\pi} < 0$ for the isoelastic utility function $u\,(q) = (1-\alpha)^{-1}\,q^{1-\alpha}$. The sufficient condition is that $\alpha \geq 1$.

Whether restricting participation is welfare improving or not depends on which of the two effects dominates. One can show that in the type I and in the type II equilibria it is always optimal to set $\pi = 1$. In contrast, restricting participation in the type III equilibrium can be welfare improving. Whether it has this effect depends on preferences and technology. In the following, we restrict our attention to a case where we can derive analytical results. That is, from now on we assume the functional forms $u(q) = \ln(q)$.

Proposition 1 *If $\gamma > \bar{\gamma}$, where $\bar{\gamma}$ is defined in the proof, it is optimal to choose $0 < \pi < 1$.*

Proposition 1 contains our key result. In the type III equilibrium, it is optimal to restrict participation in the secondary bond market if the rate of inflation is sufficiently large. We have constructed "reasonable" numerical examples, where the economy is in the type III equilibrium and the critical value $\bar{\gamma}$ is such that at two percent inflation, it is optimal to restrict access to the secondary bond market.

6 Conclusion

We constructed a general equilibrium model with a liquid and an illiquid asset where financial market participants free-ride on each other's liquidity holdings in the unique steady state equilibrium. The free-riding problem depresses the value of the liquid asset; however, by restricting access to the secondary asset market this inefficiency can be reduced.

An agent's choice of his portfolio of liquid and illiquid assets involves a pecuniary externality. An agent who increases his demand for the liquid asset not only acquires insurance against his own idiosyncratic liquidity shocks, but also affects positively the equilibrium price of this asset. This has a positive spill-over effect for all other agents in the economy.

Our results may have important policy implications. It is often argued that while measures to prevent liquidity shortages improve the resilience of the financial system to aggregate shocks, they reduce the efficiency of the financial system in normal times. Our analysis, however, shows that such a trade-off may not exist, since a policy that increases the demand for the liquid asset can also improve efficiency in normal times.

7 Appendix

Proof of Lemma 1. In the type II equilibrium, all buyers are cash constrained in the goods market. Consequently, $q = m/p$ and $\hat{q} = \hat{m}/p$ hold. The last two equations imply $q = \hat{q}m/\hat{m}$. Each active buyer exits the secondary bond market with $\hat{m}$ units of money, while an active seller exits with zero units of money. A passive agent (a seller or a buyer) exits the secondary bond market with m units of money, therefore $M_{-1} = \hat{m}(1-n)\pi + 0 * n\pi + m(1-\pi)$. Replacing $m = M_{-1}$, we get $\hat{m} = M_{-1}/(1-n)$. Use $\hat{m} = M_{-1}/(1-n)$ and $m = M_{-1}$ to replace $\hat{m}$ and m into $q = \hat{q}m/\hat{m}$, respectively, and get $q = \hat{q}(1-n)$. ■

Proof of Lemma 2. Any active agent enters the secondary bond market with a real portfolio $\phi m + \varphi\phi b$ of money and bonds. As a buyer, he sells all his bonds in a the type III equilibrium, and thus he exits the secondary bond market with a portfolio $\phi\hat{m}$. As a seller, he sells all his money thus exits this market with a portfolio $\varphi\phi\hat{b}$. Therefore $\phi m + \varphi\phi b = \phi\hat{m}$ holds for an active buyer, and $\phi m + \varphi\phi b = \varphi\phi\hat{b}$ holds for an active seller. Combining the two equations it holds that

$$\hat{m} = \varphi\hat{b}. \tag{38}$$

Immediately after the secondary bond market closes, but before the goods market opens, the stock of money in circulation is in the hands of active buyers and passive agents (sellers and buyers). Active sellers hold no money at the end of the secondary bond market. Consequently, $M_{-1} = \pi(1-n)\hat{m} + \pi n * 0 + (1-\pi)m$. Eliminate m, using $m = M_{-1}$, and rearrange to get

$$\hat{m} = \frac{M_{-1}}{1-n}. \tag{39}$$

The stock of bonds in circulation is in the hands of active sellers and passive agents (sellers and buyers), while active buyers hold no bonds at the end of the secondary bond market. Thus, the stock of bonds is equal to $B_{-1} = \pi(1-n) * 0 + \pi n\hat{b} + (1-\pi)b$. Since passive agents do not trade in the secondary bond market, they enter the goods market with the same amount of bonds they had at the beginning of the period, $b = B_{-1}$. Use this equation to eliminate b in the bond stock expression above and get

$$\hat{b} = \frac{B_{-1}}{n}. \tag{40}$$

Replace $\hat{m}$ and $\hat{b}$ in (38) using (39) and (40), respectively. Since the bonds-to-money ratio is constant over time, we can replace the time $t-1$ stock of money and bonds with their respective initial values. ■

Proof of Lemma 3. **Derivation of γ_L.** The critical value γ_L is the value of γ such that expressions (27) and (23) hold simultaneously. For easy of reference, we rewrite the two equations below:

$$\begin{aligned}\frac{\gamma}{\beta} &= \pi u'(\frac{q}{1-n}) + (1-\pi)\left[(1-n)\,u'(q) + n\right], \\ \frac{\gamma}{\beta} &= \pi + (1-\pi)\left[(1-n)\,u'(q) + n\right].\end{aligned}$$

The two expressions above are equal if $u'(\frac{q}{1-n}) = 1$. Noting that $u'(\frac{q}{1-n}) = u'(q)u'(\frac{1}{1-n})$, we can replace $u'(q)$ in the second equation to get

$$\gamma_L = \beta\left\{\pi + (1-\pi)\left[(1-n)\,u'(\frac{1}{1-n})^{-1} + n\right]\right\}.$$

Derivation of γ_H. The critical value γ_H is the value of γ such that equations (33) and (27) hold simultaneously. For easy of reference, we rewrite the two equations below:

$$\begin{aligned}\frac{\gamma}{\beta} &= \pi\left[(1-n)\,u'(\frac{q}{1-n}) + n\frac{1}{\varphi}\right] + (1-\pi)\left[(1-n)\,u'(q) + n\right], \\ \frac{\gamma}{\beta} &= \pi u'(\frac{q}{1-n}) + (1-\pi)\left[(1-n)\,u'(q) + n\right].\end{aligned}$$

The two expressions above are equal if $u'(\frac{q}{1-n}) = \frac{1}{\varphi}$. Noting that $u'(\frac{q}{1-n}) = u'(q)u'(\frac{1}{1-n})$, we have $u'(q)u'(\frac{1}{1-n}) = \frac{1}{\varphi}$. Replace $u'(q)$ in the second equation to get

$$\frac{\gamma}{\beta} = \pi\frac{1}{\varphi} + (1-\pi)\left[(1-n)\,\frac{1}{\varphi u'(\frac{1}{1-n})} + n\right].$$

■

Proof of Lemma 4. From the buyer's problem in the secondary bond market, $V_1^b(m,b) = V_2^b(\hat{m},\hat{b})$, where $\hat{m}$ and $\hat{b}$ are the quantities of money and bonds that maximize V_2^b. In any equilibrium, the buyer's budget constraint (13) holds with equality. Thus, we can use (13) to eliminate $\hat{b}$ from $V_2^b(\hat{m},\hat{b})$ and get

$$V_1^b\,(m,b) = V_2^b\left(\hat{m}, \frac{\phi m + \varphi\phi b - \phi\hat{m}}{\varphi\phi}\right) \tag{41}$$

Next, use (4), (9), and (3), to get

$$\begin{aligned} V_1^b(m,b) &= u(\hat{q}) + U(x) - x - \phi m_{+1} - \phi\rho b_{+1} + \phi(\hat{m} - p\hat{q}) \\ &\quad + \phi\left[\frac{\phi m + \varphi\phi b - \phi\hat{m}}{\varphi\phi}\right] + \phi T + \beta V_1(m_{+1}, b_{+1}). \end{aligned} \tag{42}$$

From (2), $T = M - M_{-1} + \rho B - B_{-1}$, and the budget constraint in the goods market satisfies $\hat{m} = p\hat{q}$. Furthermore, all agents exit the period with the same amount of money and bonds, hence $m_{+1} = M$ and $b_{+1} = B$. Using these equalities we can rewrite (42) as follows:

$$V_1^b(m,b) = u(\hat{q}) + U(x^*) - x^* + \frac{\phi m - \phi\hat{m}}{\varphi} - \phi m + \beta V_1(m_{+1}, b_{+1}).$$

Furthermore, $\phi m = q$ and $\phi\hat{m} = \hat{q}$ and so

$$V_1^b(m,b) = u(\hat{q}) + U(x^*) - x^* + \frac{\hat{q}}{\varphi} + \left(\frac{1-\varphi}{\varphi}\right) q + \beta V_1(m_{+1}, b_{+1}).$$

Another way to write this is

$$V_1^b(m,b) = u(\hat{q}) - \hat{q} + U(x^*) - x^* - i(\hat{q} - q) + \beta V_1(m_{+1}, b_{+1}). \tag{43}$$

The active buyer's period surplus is $u(\hat{q}) - \hat{q}$, but he has to pay interest $i = \frac{1}{\varphi} - 1$ on the difference $\hat{q} - q$.

Along the same lines, one can show that

$$V_2^b(m,b) = u(q) - q + U(x^*) - x^* + \beta V_1(m_{+1}, b_{+1}). \tag{44}$$

The difference between (43) and (44) is

$$V_1^b(m,b) - V_2^b(m,b) = u(\hat{q}) - \hat{q} - [u(q) - q] - i(\hat{q} - q). \tag{45}$$

We now need to analyze (45) for the different equilibria. For the type I equilibrium, φ comes from (22), thus

$$V_1^b(m,b) - V_2^b(m,b) = \Psi_1 \equiv u(\hat{q}) - \hat{q} - [u(q) - q] > 0$$

which is clearly strictly positive since $q^* > \hat{q} > q$.

For the type II equilibrium, φ comes from (26), thus

$$\Psi_2 \equiv u(\hat{q}) - \hat{q} - [u(q) - q] - [u'(\hat{q}) - 1](\hat{q} - q). \tag{46}$$

For the type III equilibrium, φ comes from (32), thus

$$\Psi_3 \equiv u(\hat{q}) - \hat{q} - [u(q) - q] - \left(\frac{B_0}{M_0}\frac{1-n}{n} - 1\right)(\hat{q} - q).$$

Note that in the type III equilibrium we have $u'(\hat{q}) \geq \frac{1}{\varphi} = \frac{B_0}{M_0}\frac{1-n}{n}$. Accordingly, $\Psi_3 \geq \Psi_2$. Hence, it is sufficient to show that $\Psi_2 > 0$. To do so, rewrite (46) as follows:

$$u(\hat{q}) - u(q) - \hat{q} + q > [u'(\hat{q}) - 1](\hat{q} - q).$$

Divide both sides by $\hat{q} - q > 0$ and simplify to get

$$\frac{u(\hat{q}) - u(q)}{\hat{q} - q} > u'(\hat{q}).$$

Since $\hat{q} > q$, the strict concavity of u implies that the above expression is always true. Hence, $\Psi_3 \geq \Psi_2 > 0$. ■

Proof of Lemma 5. For a seller, it is optimal to participate if

$$V_1^s(m,b) \geq V_2^s(m,b).$$

In any equilibrium, one can show along the same lines as for the buyers (see the proof of Lemma 4) that the difference $V_1^s(m,b) - V_2^s(m,b)$ satisfies

$$V_1^s(m,b) - V_2^s(m,b) = i\left(\frac{1-n}{n}\right)(\hat{q} - q), \tag{47}$$

which is clearly non-negative. Hence, sellers participate as well. ■

Proof of Proposition 1. The proof is structured as follows. First, we calculate the total derivative of the welfare function with respect π. Second, we study the sign of this derivative at $\pi = 1$ and show that it is negative for a sufficiently high γ. Third, we do the same for $\pi = 0$ and show that the derivative is positive for a sufficiently high γ.

For our functional forms, welfare satisfies

$$(1-\beta)\mathcal{W} = (1-n)\{[\pi[\log(\hat{q}) - \hat{q}] + (1-\pi)[\log(q) - q]]\} + U(x^*) - x^*. \tag{48}$$

Next, use (33) to solve for q to get

$$q = \frac{(1-n)(1-n\pi)}{\frac{\gamma}{\beta} - \pi\frac{n}{\varphi} - (1-\pi)n}. \tag{49}$$

Take the total derivative of (48) with respect to π to get

$$\frac{1-\beta}{1-n}\frac{d\mathcal{W}}{d\pi} = \log\frac{1}{1-n} - \hat{q} + q + \pi\frac{d\hat{q}}{d\pi}\frac{1-\hat{q}}{\hat{q}} + (1-\pi)\frac{dq}{d\pi}\frac{1-q}{q}. \tag{50}$$

The derivatives of the consumption quantities are

$$\begin{aligned}
\frac{dq}{d\pi} &= \frac{(1-n)\left\{(1-n\pi)\left(\frac{n}{\varphi}-n\right) - n\left[\frac{\gamma}{\beta} - \pi\frac{n}{\varphi} - (1-\pi)n\right]\right\}}{\left[\frac{\gamma}{\beta} - \pi\frac{n}{\varphi} - (1-\pi)n\right]^2} \text{ and} \\
\frac{d\hat{q}}{d\pi} &= \frac{(1-n\pi)\left(\frac{n}{\varphi}-n\right) - n\left[\frac{\gamma}{\beta} - \pi\frac{n}{\varphi} - (1-\pi)n\right]}{\left[\frac{\gamma}{\beta} - \pi\frac{n}{\varphi} - (1-\pi)n\right]^2}.
\end{aligned}$$

The next step is to calculate the value of (50) at $\pi = 1$, and show that it is negative for a sufficiently high γ. Let $\Phi \equiv \frac{\gamma}{\beta} - \frac{n}{\varphi}$. After simple substitutions we get

$$\begin{aligned}
\frac{1-\beta}{1-n}\left.\frac{d\mathcal{W}}{d\pi}\right|_{\pi=1} &= \log\frac{1}{1-n} - \frac{1-n}{\Phi} + \frac{(1-n)(1-n)}{\Phi} \\
&\quad + \frac{(1-n)n\left(\frac{1}{\varphi}-1\right) - n\Phi}{(1-n)\Phi} - \frac{(1-n)n\left(\frac{1}{\varphi}-1\right) - n\Phi}{\Phi^2}
\end{aligned}$$

which can be simplified as follows:

$$\frac{1-\beta}{1-n}\left.\frac{d\mathcal{W}}{d\pi}\right|_{\pi=1} = \log\frac{1}{1-n} - \frac{n}{1-n} + \frac{n\left[\frac{1}{\varphi} - (1-n)\right]}{\Phi} - \frac{(1-n)n\left(\frac{1}{\varphi}-1\right)}{\Phi^2}. \tag{51}$$

The first two terms of the right-hand side, $\log\frac{1}{1-n}$ and $-n/(1-n)$, do not depend on γ. In absolute value, they are both increasing in n, but the first term increases slower than the second term, since $\frac{\partial \frac{n}{1-n}}{\partial n} = 1/(1-n)^2 > 1/(1-n) = \frac{\partial \log\frac{1}{1-n}}{\partial n}$ for $0 < n \leq 1$. Therefore, their sum is decreasing in n. Since $\log\frac{1}{1-n} - n/(1-n) = 0$ at $n = 0$, $\log\frac{1}{1-n} - n/(1-n) < 0$ for any $0 < n \leq 1$.

In absolute value, the third and fourth term are decreasing in γ, and their sum tends to zero as γ goes to infinity. Denote the value of γ such that $\left.\frac{d\mathcal{W}}{d\pi}\right|_{\pi=1} =$

0 by $\gamma_{\overline{\pi}}$. Then, $\frac{d\mathcal{W}}{d\pi}\big|_{\pi=1} < 0$ for any $\gamma \geq \gamma_{\overline{\pi}}$. Finally, the type III equilibrium only exists for $\gamma > \gamma_H$. Hence, both conditions $\gamma > \gamma_{\overline{\pi}}$ and $\gamma > \gamma_H$ must be satisfied for $\pi < 1$ to be optimal. However, one can show that $\gamma_{\overline{\pi}} > \gamma_H$ at $\pi = 1$.

We now calculate the value of (50) at $\pi = 0$, and show that $\frac{d\mathcal{W}}{d\pi}$ is positive for a sufficiently high γ. The derivative of the welfare function with respect to π, at $\pi = 0$, is

$$\begin{aligned}
\frac{1-\beta}{1-n}\left.\frac{d\mathcal{W}}{d\pi}\right|_{\pi=0} &= \log\frac{1}{1-n} - \frac{1}{\frac{\gamma}{\beta}-n} + \frac{(1-n)}{\frac{\gamma}{\beta}-n} \\
&+ \frac{n\left(\frac{1}{\varphi}-1\right) - n\left(\frac{\gamma}{\beta}-n\right)}{\frac{\gamma}{\beta}-n} - (1-n)\frac{n\left(\frac{1}{\varphi}-1\right) - n\left(\frac{\gamma}{\beta}-n\right)}{\left(\frac{\gamma}{\beta}-n\right)^2}
\end{aligned}$$

which can be simplified further as

$$\frac{1-\beta}{1-n}\left.\frac{d\mathcal{W}}{d\pi}\right|_{\pi=0} = \log\frac{1}{1-n} - n + \frac{n\left[\frac{1}{\varphi}-(1+n)\right]}{\frac{\gamma}{\beta}-n} - \frac{(1-n)\,n\left(\frac{1}{\varphi}-1\right)}{\left(\frac{\gamma}{\beta}-n\right)^2}. \quad (52)$$

In absolute value, the first and the second terms, $\log\frac{1}{1-n}$ and $-n$, are increasing in n, but the first term increases faster than the second term. Therefore, the difference $\log\frac{1}{1-n} - n$ is increasing in n. Since $\log\frac{1}{1-n} - n = 0$ for $n = 0$, then $\log\frac{1}{1-n} - n > 0$ for any $0 < n \leq 1$.

In absolute value, the third and fourth term are both decreasing in γ and they tend to zero as γ approaches infinity. Denote the value of γ such that $\frac{d\mathcal{W}}{d\pi}\big|_{\pi=0} = 0$ by $\gamma_{\underline{\pi}}$. Then, $\frac{d\mathcal{W}}{d\pi}\big|_{\pi=0} > 0$ for any $\gamma \geq \gamma_{\underline{\pi}}$. Since the type III equilibrium only exists for $\gamma > \gamma_H$, it is optimal to set $\pi > 0$ for any $\gamma > \max\left\{\gamma_{\underline{\pi}}, \gamma_H\right\}$.

Combining the second and third part, it is optimal to set $0 < \pi < 1$ for any $\gamma > \bar{\gamma}$, where $\bar{\gamma} \equiv \max\left\{\gamma_{\overline{\pi}}, \gamma_{\underline{\pi}}, \gamma_H\right\}$. ■

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