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Abstract: We study political influence in institutions where members choose from among several options their levels of support to a collective goal, these individual choices determining the degree to which the goal is reached. Influence is assessed by newly defined binary relations, each of which compares any two individuals on the basis of their relative performance at a corresponding level of participation. For institutions with three levels of support (e.g., voting games in which each voter may vote "yes", "abstain", or vote "no"), we obtain three influence relations, and show that the strict component of each of them may be cyclical. The cyclicity of these relations contrasts with the transitivity of the unique influence relation of binary voting games. Weak conditions of anonymity are sufficient for each of them to be transitive. We also obtain a necessary and sufficient condition for each of them to be complete. Further, we characterize institutions for which the rankings induced by these relations, and the Banzhaf-Coleman and Shapley-Shubik power indices coincide. We argue that the extension of these relations to firms would be useful in efficiently allocating workers to different units of production. Applications to various forms of political and economic organizations are provided.

Keywords and Phrases: Level-based influence relations, Multi-choice institutions, cyclicity, anonymity, transitivity

1 INTRODUCTION

We study political influence in institutions where members choose from among several options their levels of support to a collective goal, these individual choices determining the degree to which the goal is reached. There have been generally two approaches to this question. The first approach is quantitative, and it consists of assigning to each individual a numerical value measuring his power. It dates back to Penrose's (1946) pioneering work on voting power, and it has been widely adopted by most of the subsequent researches in the field (see, e.g., Shapley and Shubik 1954; Banzhaf 1965; Rae 1969; Coleman 1971; Deegan and Packel 1978; Holler and Packel 1983; Diffo Lambo and Moulen 2000; Moulen and Diffo Lambo 2001; also see Andjiga, Chantreuil and Lepelley (2003) for a complete review of the literature on numerical power theories). The second approach introduced by Isbell (1958) is qualitative. It consists of a binary relation called "replacement relation" or "influence relation", which ranks players according to their *a priori* influence in a vote.[1] The two approaches have been essentially developed in the basic framework of binary voting games, where a voter may only vote "yes" or "no" (see, e.g., Laruelle and Valenciano 2001; Carreras and Freixas 2005; Freixas and Pons 2008).

In this paper, we extend the notion of influence relation to institutions that account for more than two levels of participation. The concept of (u, v) simple games serve as a useful mathematical model for such institutions (Freixas and Zwicker 2003). These are games in which players choose from among u options their levels of support to a collective goal, these individual choices partitioning all the society into u coalitions, and each possible partition facing v levels of collective approval in the output; u and v are linearly ordered.[2] This class of games emerged from the observation that in real life, several levels of participation in public decisions or collective productions are often observed.[3] The model of (u, v) simple games builds on and generalizes the notion of binary voting games where each voter may vote "yes" or "no", the final outcome being "win" or "lose" depending on whether the final bipartition resulting from individual choices is "winning" or "losing" ($u = v = 2$) (von Neumann and Morgenstern 1944); it also generalizes the model of voting games with abstention (VGAs) ($u = 3$ and $v = 2$) proposed by Rubinstein (1980) and Felsenthal and Machover (1997).[4] Felsenthal and Machover (1997) generalized the Banzhaf-Coleman and Shapley-Shubik power indices to VGAs, and Freixas

[1] Also see Maschler and Peleg (1966), Allingham (1975) and Taylor (1995).

[2] These games are monotonic in the sense that a player who increases his support to a collective goal also increases the degree to which the goal is achieved, which degree is measured by one of the v values.

[3] For instance, in a presidential election opposing the status quo to a challenger, a voter may vote "for" or "against" the challenger, or may "abstain" ($u = 3$); the final outcome is the victory or the defeat of the challenger ($v = 2$); allowing for the possibility of the two candidates being tied implies $v = 3$. One may also be interested in the magnitude of a victory or defeat. The challenger may *lose*, be *tied* with the status quo, or win by a *weak*, *strong*, or *overwhelming* margin ($v = 5$).

Another example of a game with several possible levels of participation is a public goods game consisting of raising \$1M to build a bridge. Each member of the community may contribute up to \$1M ($u = 10^6 + 1$ if one can only contribute a multiple of \$1), the final outcome being the construction of the bridge if the necessary funds are raised, or its postponement if not ($v = 2$).

[4] Prior to the introduction of the more general models of VGAs by Rubinstein (1980) and Felsenthal and Machover (1997), Fishburn (1973) had already modeled "abstention" as an intermediate option to "yes" and "no", but his study was restricted to self-dual weighted voting games.

(2005a,b) further generalized these notions to (u, v) simple games. As mentioned earlier, we extend the concept of influence relation to (u, v) simple games, and study its properties for VGAs.

In defining the notion of influence relation in (u, v) simple games, we exploit the u possible levels of support from which a player can choose, and the $\frac{u(u-1)}{2}$ possible ways by which an upward shift in the level of support of a player within a u-partition can increase the collective value of the resulting u-partition. Formally, a player i is said to be at least as influential or desirable as a player j at levels (s, r) if given that i and j have the same level of support s in any u-partition $\mathcal{S}_1$, the increase in the value of the u-partition $\mathcal{S}_2$ resulting from an upward shift of i from s to r must be larger than the increase in the value of the u-partition $\mathcal{S}_2'$ resulting from an identical shift of j. We obtain $\frac{u(u-1)}{2}$ influence relations in total.[5] For binary voting games ($u = v = 2$), we therefore have a unique influence relation, which is defined as follows: a player i is at least as influential as a player j if whenever j can transform a losing coalition S_1 into a winning coalition by joining it, player i can do the same *ceteris paribus*.[6][7] In a VGA ($u = 3$) we obtain three distinct relations.[8]

We study the properties of the influence relations of VGAs. The symmetric components of these relations are reflexive, symmetric, but not transitive in general (Proposition 3). So, none of them is an equivalence relation in general, unlike the influence relation of binary voting games. Their strict components are not transitive either. More interestingly, they may be cyclical: it might be the case that for each of these relations, a player i is more influential than another player j, j is more influential than another player k, and k is more influential than i (Proposition 4). This property, referred to as the "paradox of power", is somewhat reminiscent of the "paradox of voting", which depicts a form of political instability (Condorcet 1785; also see Tchantcho et al. (2009) on the stability of multi-choice institutions). None of these influence relations is therefore a preorder. But we show that each of them becomes a preorder under a corresponding condition of level-based anonymity (Proposition 5).[9] None of these relations is complete either, however we obtain a complete characterization of VGAs for which each of them is complete (Proposition 6). These findings inspire a characterization of VGAs for which each of these relations is a complete preorder (Theorem 1). We explain the cyclicity of the strict components of these relations. Combining all the three relations leads to a more conservative generalization of the influence

[5] We call these relations $(s, r)-$ influence relations or level-based influence relations.

[6] Note that a coalition S uniquely identifies with the bipartition $(S, N \setminus S)$ so that joining S is equivalent to departing from $N \setminus S$.

[7] This unique relation is the one that was discovered by Isbell (1958), generalized to any cooperative game by Maschler and Peleg (1966), and later studied by Allingham (1975) and Taylor (1995). Our definition therefore can be viewed as a generalization of Isbell's.

[8] These three relations are combined into a single relation in a companion paper (Tchantcho et al. 2008). As we will see, the level-based influence relations proposed here and the combined relation in Tchantcho et al. have very different properties and appeal, so that the two studies complete each other. Also, referring to an earlier version of the current paper (Pongou et al. 2007), we acknowledge in Tchantcho et al. that a unique feature the level-based influence relations is that they make it possible to examine the relative influence of a player at intermediate levels of participation in a multi-choice institution. We also argue that the extension of these relations to firms are useful in efficiently allocating workers to different units of production.

[9] Level-based anonymity means that voters play interchangeable roles only at certain levels of support. For instance, a VGA is anonymous at the levels (s, r) if in any tripartition, switching two voters whose levels of support are s and r respectively does not change the value of the resulting tripartition. This notion of anonymity is a weakening of the one proposed in a recent study by Freixas and Zwicker (2009).

relation, which is intransitive, but is not cyclical (see Tchantcho et al. 2008).

Along the line of Tomiyama (1987), Diffo Lambo and Moulen (2002) and Tchantcho et al. (2008), we conduct an ordinal comparison of four influence relations (the three obtained in this study and the combined relation in Tchantcho et al.) with the preorderings (SS) and (BC) induced on the set of players by the Shapley-Shubik and Banzhaf-Coleman power indices (Felsenthal and Machover 1997, Freixas 2005a, Freixas 2005b).[10] We provide a partial characterization of VGAs for which all these power theories are ordinally equivalent (Theorem 2).[11]

Finally, we provide some applications of the influence relations to real-life instances of VGAs. Applications to the United Nations Security Council show that while a permanent member of the Council is overall more influential than a non-permanent or a rotating member, this domination does not hold at all levels of support. This is also true for the United States Senate where a senator is overall more influential than the vice-president, but not at all levels. This highlights the relative usefulness of the level-based influence relations in accurately describing the structure of power in an organization.

The remainder of this paper is organized as follows. We introduce some notation and preliminary definitions in Section 2. The extensions of the influence relation to (u, v) simple games are presented in Section 3. Their properties for VGAs are studied in Section 4. In Section 5, we introduce a further generalization of the influence relation, and study the ordinal equivalence of power theories in Section 6. Applications of the influence relations to real-life examples of VGAs follow in Section 7. Finally, we discuss and conclude our study in Section 8.

2 Preliminaries

In this section, we introduce some notation and preliminary definitions. The set N denotes a non-empty finite set of players. A non-empty subset of N is called a coalition, and the set 2^N denotes the set of coalitions. For any set E, $|E|$ denotes the cardinality or the number of elements of E. Following Freixas and Zwicker (2003), an ordered $u-$partition of N is a sequence $\mathcal{S} = (S_1, S_2, ...S_u)$ of mutually disjoint subsets of N whose union is N. The subset S_l $(l \in I_u)$ refers to the set of players of N whose vote approval level is l, and may be empty. The members of S_1 and S_u respectively have the highest and the lowest degree of support. We denote by $\mathcal{N}^u$ the set of all ordered $u-$partitions of N. If $\mathcal{S}_1, \mathcal{S}_2 \in \mathcal{N}^u$, we write $\mathcal{S}_1 \subseteq^u \mathcal{S}_2$ to mean that either $\mathcal{S}_1 = \mathcal{S}_2$ or $\mathcal{S}_1$ may be transformed into $\mathcal{S}_2$ by shifting one or more players to higher levels of support.

A (u, v) hypergraph $G = (N, V)$ consists of a finite set N together with a value function $V : \mathcal{N}^u \to \{w_1, w_2, ..., w_v\}$ where $\{w_1, w_2, ..., w_k\}$ is the value set of G whose elements are any v objects equipped with a

[10] The works of Tomiyama (1987) and Diffo and Moulen (2002) have inspired several other studies on the topic of ordinal equivalence and the subject of achievable hierarchies (Carreras and Freixas 2008; Freixas and Pons 2008; Bean et al. 2008; Friedman et al. 2006), but all have been conducted in the basic framework of binary voting games.

[11] Tchantcho et al. (2008) compare the preorderings (SS) and (BC) with the combined influence relation; so our contribution in this regard is in providing a basis for the ordinal equivalence of six relations, which is a richer result.

strict linear ordering $w_1 \succ w_2 \succ ... \succ w_v$. For any ordered $u-$partition $\mathcal{S}_1$ and $\mathcal{S}_2$, $V(\mathcal{S}_1) \preceq V(\mathcal{S}_2)$ means that the degree to which a collective goal is reached when $\mathcal{S}_2$ forms is at least as high as when $\mathcal{S}_1$ forms. $\prec$ and ˜˜ denote respectively the strict and symmetric components of $\preceq$. To illustrate this definition of $\preceq$, consider an absolute majority voting system where everyone votes "yes" or "no"; a candidate who is elected with the support of 100% of the electorate ($\mathcal{S}_2 = (N, \emptyset)$) enjoys a higher degree of collective support than a candidate who is elected with the support of 55% of the electorate ($\mathcal{S}_1 = (S_1, S_2)$ with $\frac{|S_1|}{|N|}$ =0.55 and $\frac{|S_2|}{|N|}$ =0.45).

Definition 1 *: Let $u, v \geq 2$. A (u, v) simple game is a (u, v) hypergraph $G = (N, V)$ which is monotonic : that is, for all ordered $u-$partition $\mathcal{S}_1$ and $\mathcal{S}_2$, if $\mathcal{S}_1 \subseteq^u \mathcal{S}_2$ then $V(\mathcal{S}_1) \preceq V(\mathcal{S}_2)$.*

In a (u, v) simple game, each player expresses one of the u possible levels of input support, and the output consists of one of the v possible levels of collective support.

It will be useful in the sequel to define the notions of simple games (or binary voting games) and voting games with abstention.

Definition 2 *A simple game is a couple $G = (N, \mathcal{V})$ where $\mathcal{V}$ is a non-empty subset of 2^N representing the set of winning coalitions or majorities such that: $\forall S, T \in 2^N$, $(S \in \mathcal{V}$ and $S \subset T) \Rightarrow T \in \mathcal{V}$. G is said to be proper if $\forall S \in 2^N$, $S \in \mathcal{V} \Rightarrow N \setminus S \notin \mathcal{V}$.*

In a simple game, one may only vote "yes" or "no" ($u = 2$), the outcome being "win" or "lose" ($v = 2$). Note a coalition S uniquely identifies with the bipartition $(S, N \setminus S)$, so that $\mathcal{V}$ can be defined as the set of winning bipartitions. A simple game is therefore a $(2, 2)$ simple game.

Pose $\mathcal{N} = \{(S_1, S_2) / S_1 \subset N, S_2 \subset N$ and $S_1 \cap S_2 = \emptyset\}$. We recall below the definition of a social decision system, a model of VGAs introduced by Rubinstein (1980).

Definition 3 *A social decision system (SDS) is a couple $\mathcal{H} = (N, \mathcal{W})$ where $\mathcal{W}$ is a subset of $\mathcal{N}$ satisfying the following properties:*

(i) $\mathcal{W} \neq \emptyset$

(ii) $\forall (S_1, S_2), (T_1, T_2) \in \mathcal{N}$, *if* $(S_1, S_2) \in \mathcal{W}$, $S_1 \subset T_1$ *and* $S_1 \cup S_2 \subset T_1 \cup T_2$, *then* $(T_1, T_2) \in \mathcal{W}$.

(iii) $\forall (S_1, S_2) \in \mathcal{N}$, $(S_1, S_2) \in \mathcal{W} \Rightarrow (N \setminus (S_1 \cup S_2), \emptyset) \notin \mathcal{W}$.

In an SDS, one may vote "yes", "no" or "abstain" ($u = 3$), resulting in a tripartition $(S_1, S_2, N \setminus (S_1 \cup S_2))$ which is "winning" if $(S_1, S_2) \in \mathcal{W}$ or "losing" if not ($v = 2$). S_1, S_2 and $N \setminus (S_1 \cup S_2)$ are respectively the set of "yes" voters, abstainers, and "no" voters. (ii) is the condition of monotonicity, and (iii) is a condition of properness, which implies that when a tripartition is decisive or winning, its complementary set cannot be decisive.

We also recall the notion of ternary voting games introduced by Felsenthal and Machover (1997) as another model of VGAs.

Definition 4 *: A tripartition or ternary division of N is a map S from N to the set {-1, 0, 1}. The inverse image of {-1}, {0} and {1} under S are respectively :*

$S_3 = \{i \in N : S(i) = -1\}$, $S_2 = \{i \in N : S(i) = 0\}$ *and* $S_1 = \{i \in N : S(i) = 1\}$.

A ternary voting game (TVG) on N is a mapping $\mathcal{U}$ *from the set {-1, 0, 1}*^{N} *of all tripartitions of N to {-1, 1} such that :*

(1) If S is the tripartition such that $S_3 = N$, *then* $\mathcal{U}(S) = -1$.

(2) If S is the tripartition such that $S_1 = N$, *then* $\mathcal{U}(S) = 1$.

(3) If S and T are two tripartitions such that $\forall i \in N$, $S(i) \leq T(i)$, *then* $\mathcal{U}(S) \leq \mathcal{U}(T)$.

A tripartition S uniquely identifies with (S_1, S_2, S_3), which under the mapping $\mathcal{U}$ can be a losing tripartition (i.e. $\mathcal{U}((S_1, S_2, S_3)) = -1$) or a winning tripartition (i.e. $\mathcal{U}((S_1, S_2, S_3)) = 1$)). S_1, S_2 and S_3 are respectively the set of "yes" voters, abstainers, and "no" voters.

We note that the notion of TVGs relaxes the properness condition of an SDS, and thus is more general.[12] We will refer to an SDS as a proper VGA and to a TVG as a VGA (note however that we shall not really need the properness condition of an SDS in the sequel). Also, given that a tripartition (S_1, S_2, S_3) uniquely identifies with the couple (S_1, S_2), in the sequel, we shall abuse language and call (S_1, S_2) a tripartition as well.

A tripartition $(S_1, S_2) \in \mathcal{W}$ is said to be a minimal winning tripartition if $\forall (K_1, K_2) \in \mathcal{N}$ such that $K_1 \subset S_1$ and $K_1 \cup K_2 \subset S_1 \cup S_2$, if $(K_1, K_2) \neq (S_1, S_2)$, then $(K_1, K_2) \notin \mathcal{W}$. We denote by $\mathcal{W}_m$ the set of minimal winning tripartitions.

3 The Influence relations of (u, v) simple games

In this section, we introduce the notion of influence relation in a (u, v) simple game, present an illustration of this concept, and show how it generalizes the influence relation of binary voting games.

3.1 Definition

Let $\mathcal{S} = (S_1, S_2, ...S_u)$ be a u-partition of N, r, $s \in I_u$ two levels of support with $r < s$, and i, $j \in N$ two players. We write $(\mathcal{S}(i(s))_{i \in N})$ to mean that i is in the coalition S_s of the u-partition $\mathcal{S}$; $\mathcal{S}(i(s \to r))$ to denote the u-partition resulting from i shifting from s to r, and $\mathcal{S}(i(s \to r), j(s))$ to denote the u-partition in which i

[12] Condition (iii) of Definition 2 is a properness condition absent in Definition 3. We also observe that (ii) and (3) are equivalent.

shifts from s to r and j supports at level s. We will say that i is at least as $(s,r)-$ influential than j in a (u,v) simple game G if i and j having the same initial level of approval s in any u-partition $\mathcal{S}$, the increase in the value of the u-partition $\mathcal{S}_1$ resulting from i shifting from s to r is at least as large as the increase in the value of the u-partition $\mathcal{S}_1'$ resulting from an identical shift by j. This definition is formalized as follows.

Definition 5 *Let $G = (N, V)$ be a (u,v) simple game and i and j two players.*

1) i is said to be at least as $(s,r)-$ influential as j, denoted $i \geq_{Tsr,G} j$, if for any u-partition $\mathcal{S}$ such that $i, j \in S_s$, $V(\mathcal{S}(i(s), j(s \to r)) \preceq V(\mathcal{S}(i(s \to r), j(s))$.

2) i is said to be as $(s,r)-$ influential as j, denoted $i \sim_{Tsr,G} j$, if $i \geq_{Tsr,G} j$ and $j \geq_{Tsr,G} i$.

3) i is said to be more $(s,r)-$influential than j, denoted $i >_{Tsr,G} j$, if $i \geq_{Tsr,G} j$ and non($j \geq_{Tsr,G} i$).

In 1), $V(\mathcal{S}(i(s), j(s \to r)) \preceq V(\mathcal{S}(i(s \to r), j(s))$ expresses the requirement that the marginal contribution of i measured by the shift from s to r be at least as large as that of j after an identical shift *ceteris paribus*. $\sim_{Tsr,G}$ and $>_{Tsr,G}$ denote respectively the symmetric and the strict components of $\geq_{Tsr,G}$. Given that only unidirectional upward shifts are relevant, we have in total $\frac{u(u-1)}{2}$ $(s,r)-$ influence relations, that we also call level-based influence relations.

In a (2, 2) simple game, we thus have the unique influence relation of simple games introduced by Isbell (1958), and in a VGA, we have three relations, that we define more explicitly below.

Definition 6 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA, and i and j two players.*

1) i is said to be at least as $(3,1)-$ influential as j, denoted $i \geq_{T_{31},\mathcal{H}} j$, (or $i \geq_{T_{31}} j$ for short), if $\forall (S_1, S_2) \in \mathcal{N}$ such that $i, j \notin S_1 \cup S_2$, $(S_1 \cup \{j\}, S_2) \in \mathcal{W} \Rightarrow (S_1 \cup \{i\}, S_2) \in \mathcal{W}$.

2) i is said to be at least as $(3,2)-$ influential as j, denoted $i \geq_{T_{32}} j$, if $\forall (S_1, S_2) \in \mathcal{N}$ such that $i, j \notin S_1 \cup S_2$, $(S_1, S_2 \cup \{j\}) \in \mathcal{W} \Rightarrow (S_1, S_2 \cup \{i\}) \in \mathcal{W}$.

3) i is said to be at least as $(2,1)-$ influential as j, denoted $i \geq_{T_{21}} j$, if $\forall (S_1, S_2) \in \mathcal{N}$ such that $i, j \notin S_1 \cup S_2$, $(S_1 \cup \{j\}, S_2 \cup \{i\}) \in \mathcal{W} \Rightarrow (S_1 \cup \{i\}, S_2 \cup \{j\}) \in \mathcal{W}$.

4) Let r, $s \in I_3$ such that $r < s$. i is said to be as $(s,r)-$ influential as j, denoted $i \sim_{Tsr} j$, if $i \geq_{Tsr} j$ and $j \geq_{Tsr} i$.

5)Let r, $s \in I_3$ such that $r < s$. i is said to be more $(s,r)-$ influential than j, denoted $i >_{Tsr} j$, if $i \geq_{Tsr} j$ and non($j \geq_{Tsr} i$).

Interpretation : A player i is said to be at least as $(3,1)-$ influential as a player j if whenever j can transform a tripartition (S_1, S_2) into a winning tripartition by shifting from total disapproval to total approval

$((S_1 \cup \{j\}, S_2, S_3 \setminus \{j\}) \in \mathcal{W})$, player i can do the same by an identical shift $((S_1 \cup \{i\}, S_2, S_3 \setminus \{i\}) \in \mathcal{W})$. i is said to be at least as $(3,2)-$ influential as j if whenever j can transform a tripartition (S_1, S_2) by shifting from total disapproval to abstention, i can do the same by an identical shift. i is said to be at least as $(2,1)-$ influential as j if whenever j can transform a tripartition (S_1, S_2^*) where $S_2^* = S_2 \cup \{i,j\}$ and $i, j \notin S_2$ (i and j are initially abstainers in the tripartition (S_1, S_2^*)) into a winning tripartition by shifting from total disapproval to abstention, i can do the same by an identical shift.

3.2 An illustrating example

Below is an illustration of the $(s,r)-$ influence relations in the context of the United States Senate.

Example 1 *The United States Senate (USS) contains 100 senators and the vice-president of the United States who leads the Senate. Voting rules vary depending on the nature of the decision to be made. In the case of passing an ordinary bill or amendment, which is the case that we consider here, a decision is made when the number of senators who cast a "yes" vote is strictly greater than the number of senators who cast a "no" vote, and vice-versa; in case of equality, the vice-president (vp for short) casts a tie-breaking vote. This social choice context is a voting game with abstention modelled by Freixas and Zwicker (2003) as follows:*

Let $\mathcal{S} = (S_1, S_2, S_3)$ be an ordered 3-partition of N $(N = USS)$:

$$V_{USS}(\mathcal{S}) = \begin{cases} win \text{ } if \text{ } |S_1| > |S_3 \setminus \{vp\}| \\ \quad lose \text{ } otherwise. \end{cases}$$

We have the following structure of power among the members of the Senate yielded by the $(s,r)-$ influence relations :

$\forall \{i,j\} \subset N \setminus \{vp\}$,

(1) $i >_{T_{31}} vp$ and $i \sim_{T_{31}} j$.

(2) $i >_{T_{32}} vp$ and $i \sim_{T_{32}} j$.

(3) $i \sim_{T_{21}} vp$ and $i \sim_{T_{21}} j$.

We note that while a senator is more influential than the vice-president at levels (3,1) and (3,2), both are equally influential at levels (2,1). This illustrates the usefulness of the $(s,r)-$ influence relations in providing a full picture of power relations among the members of the Senate.

3.3 $(s,r)-$influence relations of VGAs as generalizations

We show that the $(3,2)-$influence relation and the $(3,1)-$influence relation of VGAs generalize the influence relation of binary voting games. We need a few preliminary definitions.

Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA satisfying : $\forall (S,T) \in \mathcal{W}$, $\forall L \subset N \setminus S$, $(S,L) \in \mathcal{W}$. Thus, the coalition S of any winning tripartition (S,L) has absolute power. This power is equivalent to the power enjoyed by a winning coalition of a simple game. This comparison inspires the following definition of a simple VGA.

Definition 7 *A VGA is simple if :* $\forall (S,T) \in \mathcal{W}$, $\forall L \subset N \setminus S$, *we have* $(S,L) \in \mathcal{W}$.

Let $G = (N, \mathcal{V})$ be a simple game. We associate to G the couple $\mathcal{H}_G = (N, \mathcal{W}_{\mathcal{V}})$ where $\mathcal{W}_{\mathcal{V}}$ is defined as $\mathcal{W}_{\mathcal{V}} = \{(S,T) \in \mathcal{N} \setminus S \in \mathcal{V}$ and $T \subset N \setminus S\}$. Conversely, we associate to a VGA $\mathcal{H} = (N, \mathcal{W})$ the list $G^{\mathcal{H}} = (N, \mathcal{V}^{\mathcal{W}})$ where $\mathcal{V}^{\mathcal{W}}$ is defined as $\mathcal{V}^{\mathcal{W}} = \{S \in 2^N \mid \forall T \subset N \setminus S,\ (S,T) \in \mathcal{W}\}$. Tchantcho et al. (2008) show that if $\mathcal{H}$ is a proper VGA, then $G^{\mathcal{H}}$ is a proper simple game, and that a necessary and sufficient condition on G for $\mathcal{H}_G$ to be a proper VGA is that G be proper.

We show below that the $(3,2)-$influence relation and the $(3,1)-$influence relation generalize the influence relation of simple games.

Proposition 1 *For any players* $i, j \in N$, *we have :*

(1) $i \geq_{T,G} j \Longleftrightarrow i \geq_{T_{31},\mathcal{H}_G} j$.

(2) $i \geq_{T,G} j \Rightarrow i \geq_{T_{32},\mathcal{H}_G} j$.

(3) $i \geq_{T,G} j \Longleftrightarrow i \geq_{T_{21},\mathcal{H}_G} j$.

Proof : (1) Assume that s=3 and r=1. Suppose that $i \geq_{T_{31},\mathcal{H}_G} j$ in $(N, \mathcal{W}_{\mathcal{V}})$ and show that $i \geq_{T,G} j$. Suppose $S \in 2^N$ such that $i,j \notin S$ and $S \cup \{j\} \in \mathcal{V}$. Then $(S \cup \{j\}, \emptyset) \in \mathcal{W}_{\mathcal{V}}$. But $i,j \notin S = S \cup \emptyset$, $i \geq_{T_{31},\mathcal{H}_G} j$, and $(S \cup \{j\}, \emptyset) \in \mathcal{W}_{\mathcal{V}}$ imply $(S \cup \{i\}, \emptyset) \in \mathcal{W}_{\mathcal{V}}$, implying $S \cup \{i\} \in \mathcal{V}$.

Suppose now that $i \geq_{T,G} j$ and let show that $i \geq_{T_{31},\mathcal{H}_G} j$. Suppose $(S_1, S_2) \in \mathcal{N}$ such that $i,j \notin S_1 \cup S_2$ and $(S_1 \cup \{j\}, S_2) \in \mathcal{W}_{\mathcal{V}}$. It follows from the definition of $\mathcal{W}_{\mathcal{V}}$ that $S_1 \cup \{j\} \in \mathcal{V}$, therefore implying $S_1 \cup \{i\} \in \mathcal{V}$ because $i,j \notin S_1$ and $i \geq_{T,G} j$. But $S_1 \cup \{i\} \in \mathcal{V}$ implies $(S_1 \cup \{i\}, S_2) \in \mathcal{W}_{\mathcal{V}}$ by the definition of $\mathcal{W}_{\mathcal{V}}$ since $S_2 \subset (N \setminus [S_1 \cup \{i\}])$.

(2) Assume that s=3 and r=2. Suppose now that $i \geq_{T,G} j$ and let show that $i \geq_{T_{32},\mathcal{H}_G} j$. Suppose $(S_1, S_2) \in \mathcal{N}$ such that $i,j \notin S_1 \cup S_2$ and $(S_1, S_2 \cup \{j\}) \in \mathcal{W}_{\mathcal{V}}$. It follows from the definition of $\mathcal{W}_{\mathcal{V}}$ that $S_1 \in \mathcal{V}$, therefore implying that $(S_1, S_2 \cup \{i\}) \in \mathcal{W}_{\mathcal{V}}$ by the definition of $\mathcal{W}_{\mathcal{V}}$.

(3) Assume that s=2 and r=1. Suppose that $i \geq_{T_{21},\mathcal{H}_G} j$ in $(N, \mathcal{W}_{\mathcal{V}})$ and let show that $i \geq_{T,G} j$. Suppose $S \in 2^N$ such that $i,j \notin S$ and $S \cup \{j\} \in \mathcal{V}$. Then $(S \cup \{j\}, \emptyset) \in \mathcal{W}_{\mathcal{V}}$, which implies $(S \cup \{j\}, \{i\}) \in \mathcal{W}_{\mathcal{V}}$. But $i,j \notin S = S \cup \emptyset$, $i \geq_{T_{21},\mathcal{H}_G} j$ and $(S \cup \{j\}, \{i\}) \in \mathcal{W}_{\mathcal{V}}$ imply $(S \cup \{i\}, \{j\}) \in \mathcal{W}_{\mathcal{V}}$, implying $S \cup \{i\} \in \mathcal{V}$.

Suppose now that $i \geq_{T,G} j$ and let show that $i \geq_{T_{21},\mathcal{H}_G} j$. Suppose $(S_1, S_2) \in \mathcal{N}$ such that $i,j \notin S_1 \cup S_2$ and $(S_1 \cup \{j\}, S_2 \cup \{i\}) \in \mathcal{W}_{\mathcal{V}}$. It follows from the definition of $\mathcal{W}_{\mathcal{V}}$ that $S_1 \cup \{j\} \in \mathcal{V}$, therefore implying

$S_1 \cup \{i\} \in \mathcal{V}$ since $i, j \notin S_1$ and $i \geq_{T,G} j$. This implies $(S_1 \cup \{i\}, \emptyset) \in \mathcal{W}_\mathcal{V}$, implying $(S_1 \cup \{i\}, S_2 \cup \{j\}) \in \mathcal{W}_\mathcal{V}$ since $S_2 \cup \{j\} \subset (N \setminus [S_1 \cup \{i\}])$. $\square$

Note that $i \geq_{T_{32},\mathcal{H}_G} j$ does not necessarily imply $i \geq_{T,G} j$ because in evaluating influence, $\geq_{T_{32},\mathcal{H}_G}$ only considers shifts from total disapproval to abstention, but such a shift is not possible in a simple game.

4 Properties of the influence relations of VGAs

In this section, we study some properties of the $(s,r)-$influence relations of VGAs.

4.1 Cyclicity, anonymity and transitivity

We show below that the symmetric components of the relations $\geq_{Tsr}$ are reflexive, symmetric, but not transitive in general.

Proposition 2 *For any $(s,r) \in \{(3,1),(3,2),(2,1)\}$, $\sim_{Tsr}$ is reflexive, symmetric, but not transitive in general.*

Proof : Since the proof is similar for all the three relations, the argument will be shown only for $\sim_{T_{31}}$.[13]

(1) Reflexivity is obvious.

(2) Symmetry: $\sim_{T_{31}}$ is symmetric because for any players i and j, $i \sim_{T_{31}} j$ obviously implies $j \sim_{T_{31}} i$.

(3) Non-transitivity: Consider the VGA $\mathcal{H} = (N, \mathcal{W})$ where $N = \{1,2,3,4\}$ and $\mathcal{W}_m = \{(3,24),(2,34),(1,34),(3,14),(2,1$ (note that we write (3,24) for instance for ({3},{2,4})). We have: $1 \sim_{T_{31}} 2$ and $2 \sim_{T_{31}} 3$, but $non(1 \sim_{T_{31}} 3)$ because $(3,24) \in \mathcal{W}_m$ and $(1,24) \notin \mathcal{W}_m$. Thus, $\sim_{T_{31}}$ is not transitive. $\square$

We provide below a sufficient condition for each relation $\sim_{Tsr}$ to be transitive. We first introduce the definition of the $(s,r)-$ anonymity of a VGA.

Definition 8 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA, and $s, r \in I_3$. $\mathcal{H}$ is said to be $(s,r)-$ anonymous if: $\forall \{i,j\} \subset N$, $\forall \mathcal{S}(i(r), j(s)) \in \mathcal{N}^3$, $\mathcal{S}(i(r), j(s)) \in \mathcal{W}$ if and only if $\mathcal{S}(i(r \to s), j(s \to r)) \in \mathcal{W}$.*

A VGA is $(s,r)-$ anonymous if the permutation of two players whose levels of support in a tripartition $\mathcal{S}$ are r and s, respectively, does not change the value of the resulting tripartition.

We have the following result.

Proposition 3 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA.*

(1) If $\mathcal{H}$ is (2,1)-anonymous, then $\sim_{T_{31}}$ is transitive.

[13] The proof for $\sim_{T_{32}}$ and $\sim_{T_{21}}$ is available from the authors upon request.

(2) If $\mathcal{H}$ is (3,1)-anonymous, then $\sim_{T_{32}}$ is transitive.

(3) If $\mathcal{H}$ is (3,2)-anonymous, then $\sim_{T_{21}}$ is transitive.

Proof : The proof will be shown only for (1) since the proof for (2) and (3) is similar. Let $\mathcal{H} = (N, \mathcal{W})$ be a (2,1)-anonymous VGA. Let us show that $\sim_{T_{31}}$ is transitive. Suppose $\{i, j, k\} \subset N$ such that $i \sim_{T_{31}} j$ and $j \sim_{T_{31}} k$. Show that $i \sim_{T_{31}} k$. Let us first show that $i \geq_{T_{31}} k$. Let $(S_1, S_2) \in \mathcal{N}$ such that $i \notin S_1 \cup S_2$, $k \notin S_1 \cup S_2$ and $(S_1 \cup \{k\}, S_2) \in \mathcal{W}$. We need to show that $(S_1 \cup \{i\}, S_2) \in \mathcal{W}$. There are two cases: (a) $j \notin S_1 \cup S_2$ and (b) $j \in S_1 \cup S_2$.

(a) : Suppose that $j \notin S_1 \cup S_2$. Because of $i \notin S_1 \cup S_2$, $k \notin S_1 \cup S_2$ and $j \sim_{T_{31}} k$, $(S_1 \cup \{k\}, S_2) \in \mathcal{W}$ implies $(S_1 \cup \{j\}, S_2) \in \mathcal{W}$. Likewise, because of $i \notin S_1 \cup S_2$, $j \notin S_1 \cup S_2$ and $i \sim_{T_{31}} j$, $(S_1 \cup \{j\}, S_2) \in \mathcal{W}$ implies $(S_1 \cup \{i\}, S_2) \in \mathcal{W}$.

(b) : Suppose that $j \in S_1 \cup S_2$. There are two cases: $(b1)$ $j \in S_1$ and $(b2)$ $j \in S_2$.

$(b1)$: Suppose that $j \in S_1$. Pose $S_1' = S_1 \setminus \{j\}$. Then $S_1 = S_1' \cup \{j\}$. Our assumption $(S_1 \cup \{k\}, S_2) \in \mathcal{W}$ therefore becomes $(S_1' \cup \{j\} \cup \{k\}, S_2) \in \mathcal{W}$ or $(S_1' \cup \{k\} \cup \{j\}, S_2) \in \mathcal{W}$. But because of $i \notin (S_1' \cup \{k\}) \cup S_2$, $j \notin (S_1' \cup \{k\}) \cup S_2$ and $i \sim_{T_{31}} j$, $(S_1' \cup \{k\} \cup \{j\}, S_2) \in \mathcal{W}$ implies $(S_1' \cup \{k\} \cup \{i\}, S_2) \in \mathcal{W}$ or equivalently $(S_1' \cup \{i\} \cup \{k\}, S_2) \in \mathcal{W}$. Likewise, because of $j \notin (S_1' \cup \{i\}) \cup S_2$, $k \notin (S_1' \cup \{i\}) \cup S_2$ and $j \sim_{T_{31}} k$, $(S_1' \cup \{i\} \cup \{k\}, S_2) \in \mathcal{W}$ implies $(S_1' \cup \{i\} \cup \{j\}, S_2) \in \mathcal{W}$, therefore implying $(S_1 \cup \{j\}, S_2) \in \mathcal{W}$ since $S_1 = S_1' \cup \{j\}$.

$(b2)$: Suppose that $j \in S_2$. Pose $S_2' = S_2 \setminus \{j\}$. Then $S_2 = S_2' \cup \{j\}$. Our assumption $(S_1 \cup \{k\}, S_2) \in \mathcal{W}$ therefore becomes $(S_1 \cup \{k\}, S_2' \cup \{j\}) \in \mathcal{W}$, thus implying $(S_1 \cup \{j\}, S_2' \cup \{k\}) \in \mathcal{W}$ by the assumption that $\mathcal{H}$ is (2,1)-anonymous. But because of $i \notin (S_1 \cup \{k\}) \cup S_2'$, $j \notin (S_1 \cup \{k\}) \cup S_2'$ and $i \sim_{T_{31}} j$, $(S_1 \cup \{j\}, S_2' \cup \{k\}) \in \mathcal{W}$ implies $(S_1 \cup \{i\}, S_2' \cup \{k\}) \in \mathcal{W}$, which by the assumption that $\mathcal{H}$ is (2,1)-anonymous implies $(S_1 \cup \{k\}, S_2' \cup \{i\}) \in \mathcal{W}$. Similarly, because of $j \notin (S_1 \cup \{i\}) \cup S_2'$, $k \notin (S_1 \cup \{i\}) \cup S_2'$ and $j \sim_{T_{31}} k$, $(S_1 \cup \{k\}, S_2' \cup \{i\}) \in \mathcal{W}$ implies $(S_1 \cup \{j\}, S_2' \cup \{i\}) \in \mathcal{W}$, which by the assumption that $\mathcal{H}$ is (2,1)-anonymous implies $(S_1 \cup \{i\}, S_2' \cup \{j\}) \in \mathcal{W}$ or equivalently $(S_1 \cup \{i\}, S_2) \in \mathcal{W}$.

We just showed that $i \geq_{T_{31}} k$. We obtain $k \geq_{T_{31}} i$ by a circular permutation of i, j and k. So $i \sim_{T_{31}} k$. □

We show below that the strict component of each of the $(s, r)-$ influence relations of a VGA may be cyclical, which implies that it is not transitive in general.

Proposition 4 *For any $(s, r) \in \{(3, 1), (3, 2), (2, 1)\}$, there exists a VGA $\mathcal{H} = (N, \mathcal{W})$ for which $>_{Tsr}$ is cyclical.*

Proof : For $(s, r) = (3, 1)$, consider the VGA $\mathcal{H} = (N, \mathcal{W})$ where $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{W}_m = \{(15, 34),$ $(24, 15), (34, 25)\}$. We have : $1 >_{T_{31}} 2$, $2 >_{T_{31}} 3$ and $3 >_{T_{31}} 1$. So $>_{T_{31}}$ is cyclical.

For $(s, r) = (3, 2)$, consider the VGA $\mathcal{H} = (N, \mathcal{W})$ where $N = \{1, 2, 3, 4, 5, 6, 7\}$ and $\mathcal{W}_m = \{(34, 15),$

$(15, 26), (26, 37), (16, 37), (16, 27)\}$. We have : $1 >_{T_{32}} 2$, $2 >_{T_{32}} 3$ and $3 >_{T_{32}} 1$. So $>_{T_{32}}$ is cyclical

For $(s, r) = (2, 1)$, consider the VGA $\mathcal{H} = (N, \mathcal{W})$ where $N = \{1, 2, 3, 4, 5\}$ and $\mathcal{W}_m = \{(25, 1), (15, 2), (35, 1),$ $(14, 2), (24, 3), (14, 3), (34, 1)\}$. We have : $1 >_{T_{21}} 2$, $2 >_{T_{21}} 3$ and $3 >_{T_{21}} 1$. So $>_{T_{21}}$ is cyclical. □

Proposition 4 implies a "paradox of power" in voting games with abstention. This type of paradox does not occur in binary voting games where voters may cast only a "yes" or a "no" vote.[14] It also follows from Propositions 2 and 4 that none of the $(s, r)-$ influence relations of a VGA is a preorder (i.e. reflexive and transitive) in general, as stated in the following corollary.

Corollary 1 *For any $(s, r) \in \{(3, 1), (3, 2), (2, 1)\}$, $\geq_{Tsr}$ is not a preorder in general.*

Next, we provide a sufficient condition for each $(s, r)-$influence relation to be a preorder.

Proposition 5 *Let $\mathcal{H} = (N, \mathcal{W})$ be VGA.*

(1) If $\mathcal{H}$ is (2,1)-anonymous, then $\geq_{T_{31}}$ is a preorder.

(2) If $\mathcal{H}$ is (3,1)-anonymous, then $\geq_{T_{32}}$ is a preorder.

(3) If $\mathcal{H}$ is (3,2)-anonymous, then $\geq_{T_{21}}$ is a preorder.

Proof : For $(s, r) = (3, 1)$, given that $\sim_{T_{31}}$is reflexive, $\geq_{T_{31}}$is too. The proof of transitivity is similar to that of Proposition 3. The proof is similar for $\geq_{T_{32}}$ and $\geq_{T_{21}}$. □

4.2 Completeness

It can be shown that in a VGA, it is not always possible to compare any two voters by the $(s, r)-$ influence relations, which means that these relations are not complete in general. Our goal in this section is to provide a necessary and sufficient condition for each of these relations to be complete. Prior to this, we need to introduce some notions of swap-robustness.[15]

Definition 9 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA, and $(S_1, S_2) \in \mathcal{W}$ and $(T_1, T_2) \in \mathcal{W}$.*

1-a) A permutation of two voters $i \in S_1 \setminus (T_1 \cup T_2)$ and $j \in T_1 \setminus (S_1 \cup S_2)$ between (S_1, S_2) and (T_1, T_2) results in the tripartitions $(S'_1, S'_2) = ([S_1 \setminus \{i\}] \cup \{j\}, S_2)$ and $(T'_1, T'_2) = ([T_1 \setminus \{j\}] \cup \{i\}, T_2)$.

1-b) The VGA $\mathcal{H} = (N, \mathcal{W})$ is said to be (3, 1)-swap-robust if at least one of the tripartitions (S'_1, S'_2) and (T'_1, T'_2) is winning.

2-a) A permutation of two voters $i \in S_2 \setminus (T_1 \cup T_2)$ and $j \in T_2 \setminus (S_1 \cup S_2)$ between (S_1, S_2) and (T_1, T_2) results in the tripartitions $(S'_1, S'_2) = (S_1, [S_2 \setminus \{i\}] \cup \{j\})$ and $(T'_1, T'_2) = (T_1, [T_2 \setminus \{i\}] \cup \{j\})$.

[14] In fact, Taylor (1995) shows that the influence relation of binary voting games is transitive.

[15] Also see Taylor and Zwicker (1999) for binary voting games, and Tchantcho et al. (2008) for a stronger version of swap-robustness in VGAs.

2-b) The VGA $\mathcal{H} = (N, \mathcal{W})$ is said to be (3, 2)-swap-robust if at least one of the tripartitions (S'_1, S'_2) and (T'_1, T'_2) is winning.

3-a) A permutation of two voters $i \in S_1 \cap T_2$ and $j \in S_2 \cap T_1$ between (S_1, S_2) and (T_1, T_2) results in the tripartitions $(S'_1, S'_2) = ([S_1 \setminus \{i\}] \cup \{j\}, [S_2 \setminus \{j\}] \cup \{i\})$ and $(T'_1, T'_2) = ([T_1 \setminus \{j\}] \cup \{i\}, [T_2 \setminus \{i\}] \cup \{j\})$.

3-b) The VGA $\mathcal{H} = (N, \mathcal{W})$ is said to be (2, 1)-swap-robust if at least one of the tripartitions (S'_1, S'_2) and (T'_1, T'_2) is winning.

Given a VGA $\mathcal{H} = (N, \mathcal{W})$ and two winning tripartitions (S_1, S_2) and (T_1, T_2), 1-a) and 1-b) mean that if a "yes" voter i in (S_1, S_2) and a "yes" voter j in (T_1, T_2) are permuted, and if at least one of the resulting tripartition is still winning, then $\mathcal{H}$ is said to be (3, 1)-swap-robust. The interpretation of (3, 2) and (2,1)-swap-robustness is similar.

We have the following Lemma.

Lemma 1 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA, (S_1, S_2), $(T_1, T_2) \in \mathcal{W}$, and $i, j \in N$ two voters.*

1- Let $(S'_1, S'_2) = ([S_1 \setminus \{i\}] \cup \{j\}, S_2)$ and $(T'_1, T'_2) = ([T_1 \setminus \{j\}] \cup \{i\}, T_2)$. Then,

1-a) if $i \sim_{T_{31}} j$, then $(S'_1, S'_2) \in \mathcal{W}$ and $(T'_1, T'_2) \in \mathcal{W}$.

1-b) if $i >_{T_{31}} j$, then $(T'_1, T'_2) \in \mathcal{W}$.

1-c) if $(S'_1, S'_2) \notin \mathcal{W}$ and $(T'_1, T'_2) \notin \mathcal{W}$, then i and j are not comparable by the influence relation $\geq_{T_{31}}$.

2- Let $(S'_1, S'_2) = (S_1, [S_2 \setminus \{i\}] \cup \{j\})$ and $(T'_1, T'_2) = (T_1, [T_2 \setminus \{i\}] \cup \{j\})$. Then,

2-a) if $i \sim_{T_{32}} j$, then $(S'_1, S'_2) \in \mathcal{W}$ and $(T'_1, T'_2) \in \mathcal{W}$.

2-b) if $i >_{T_{32}} j$, then $(T'_1, T'_2) \in \mathcal{W}$.

2-c) if $(S'_1, S'_2) \notin \mathcal{W}$ and $(T'_1, T'_2) \notin \mathcal{W}$, then i and j are not comparable by the influence relation $\geq_{T_{32}}$

3- Let $(S'_1, S'_2) = (S_1, [S_2 \setminus \{i\}] \cup \{j\})$ and $(T'_1, T'_2) = (T_1, [T_2 \setminus \{i\}] \cup \{j\})$. Then,

3-a) if $i \sim_{T_{21}} j$, then $(S'_1, S'_2) \in \mathcal{W}$ and $(T'_1, T'_2) \in \mathcal{W}$.

3-b) if $i >_{T_{21}} j$, then $(T'_1, T'_2) \in \mathcal{W}$.

3-c) if $(S'_1, S'_2) \notin \mathcal{W}$ and $(T'_1, T'_2) \notin \mathcal{W}$, then i and j are not comparable by the influence relation $\geq_{T_{21}}$.

Proof : Again, the proof will be shown only for part 1). We assume all the hypotheses of Lemma 1.

1-a) Suppose $i \sim_{T_{31}} j$ and let us show that $(S'_1, S'_2) \in \mathcal{W}$ and $(T'_1, T'_2) \in \mathcal{W}$. Pose $K'_1 = S_1 \setminus \{i\}$ and $L'_1 = T_1 \setminus \{j\}$. Then $(S'_1, S'_2) = (K'_1 \cup \{j\}, S_2)$ and $(T'_1, T'_2) = (L'_1 \cup \{i\}, T_2)$, $(S_1, S_2) = (K'_1 \cup \{i\}, S_2) \in \mathcal{W}$ and $(T_1, T_2) = (L'_1 \cup \{j\}, T_2) \in \mathcal{W}$. Given that $i \notin K'_1 \cup S_2$, $j \notin K'_1 \cup S_2$ and $i \sim_{T_{31}} j$, we have $(K'_1 \cup \{i\}, S_2) \in \mathcal{W} \Rightarrow (K'_1 \cup \{j\}, S_2) \in \mathcal{W}$, which implies $(S'_1, S'_2) \in \mathcal{W}$. Likewise, because of $i \notin L'_1 \cup T_2$, $j \notin L'_1 \cup T_2$ and $i \sim_{T_{31}} j$, we have $(L'_1 \cup \{j\}, T_2) \in \mathcal{W} \Rightarrow (L'_1 \cup \{i\}, T_2) \in \mathcal{W}$, which implies $(T'_1, T'_2) \in \mathcal{W}$.

1-b) is obtained similarly as 1-a).

1-c) We use the notation in part 1-a). If $(S'_1, S'_2) \notin \mathcal{W}$ and $(T'_1, T'_2) \notin \mathcal{W}$, then $(K'_1 \cup \{j\}, S_2) \notin \mathcal{W}$ and $(L'_1 \cup \{i\}, T_2) \notin \mathcal{W}$. But $(K'_1 \cup \{i\}, S_2) \in \mathcal{W}$ and $(L'_1 \cup \{j\}, T_2) \in \mathcal{W}$. Because of $i \notin K'_1 \cup S_2$, $j \notin K'_1 \cup S_2$, $(K'_1 \cup \{i\}, S_2) \in \mathcal{W}$ and $(K'_1 \cup \{j\}, S_2) \notin \mathcal{W}$, we have $non(j \geq_{T_{31}} i)$. Also $i \notin L'_1 \cup T_2$, $j \notin L'_1 \cup T_2$, $(L'_1 \cup \{j\}, T_2) \in \mathcal{W}$ and $(L'_1 \cup \{j\}, T_2) \notin \mathcal{W}$ imply $non(i \geq_{T_{31}} j)$. It follows that i and j are not comparable by the influence relation $\geq_{T_{31}}$. $\square$

We are now ready to provide a full characterization of VGAs for which the $(s, r)-$influence relations are complete.

Proposition 6 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA. For any $(s, r) \in \{(3, 1), (3, 2), (2, 1)\}$, the $(s, r)-$influence relation is complete if and only if $\mathcal{H}$ is $(s, r)-$swap-robust.*

Proof : We show the proof only for $(s, r) = (3, 1)$. Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA.

Suppose that $\mathcal{H}$ is (3, 1)-swap-robust and let us show that $\geq_{T_{31}}$is complete. If $\geq_{T_{31}}$is not complete, there exist two players i and j such that $non(i \geq_{T_{31}} j)$ and $non(j \geq_{T_{31}} i)$. It follows that there exist $(S_1, S_2) \in \mathcal{W}$ and $(T_1, T_2) \in \mathcal{W}$ such that $i \notin [S_1 \cup S_2] \cup [T_1 \cup T_2]$, $j \notin [S_1 \cup S_2] \cup [T_1 \cup T_2]$, $(T_1 \cup \{j\}, T_2) \in \mathcal{W}$ and $(T_1 \cup \{i\}, T_2) \notin \mathcal{W}$, and $(S_1 \cup \{i\}, S_2) \in \mathcal{W}$ and $(S_1 \cup \{j\}, S_2) \notin \mathcal{W}$. Pose $T^*_1 = T_1 \cup \{j\}$, $T^*_2 = T_2$, $S^*_1 = S_1 \cup \{i\}$ and $S^*_2 = S_2$. Then we have $(S^*_1, S^*_2) \in \mathcal{W}$, $(T^*_1, T^*_2) \in \mathcal{W}$, $i \in S^*_1 \setminus (T^*_1 \cup T^*_2)$ and $j \in T^*_1 \setminus (S^*_1 \cup S^*_2)$. Given that $\mathcal{H}$ is (3, 1)-swap-robust, it follows that $([S^*_1 \setminus \{i\}] \cup \{j\}, S^*_2) \in \mathcal{W}$ or $([T^*_1 \setminus \{j\}] \cup \{i\}, T^*_2) \in \mathcal{W}$, which is equivalent to $(S_1 \cup \{j\}, S_2) \in \mathcal{W}$ or $(T_1 \cup \{i\}, T_2) \in \mathcal{W}$, contradicting $(S_1 \cup \{j\}, S_2) \notin \mathcal{W}$ and $(T_1 \cup \{i\}, T_2) \notin \mathcal{W}$. We then conclude that $\geq_{T_{31}}$ is complete.

Suppose that $\geq_{T_{31}}$ is complete and let us show that $\mathcal{H}$ is (3, 1)-swap-robust. Assume by contradiction that $\mathcal{H}$ is not (3, 1)-swap-robust. Then there exist two winning tripartitions $(S_1, S_2) \in \mathcal{W}$ and $(T_1, T_2) \in \mathcal{W}$, and two players i and j such that $i \in S_1 \setminus (T_1 \cup T_2)$, $j \in T_1 \setminus (S_1 \cup S_2)$, $([S_1 \setminus \{i\}] \cup \{j\}, S_2) \notin \mathcal{W}$ and $([T_1 \setminus \{j\}] \cup \{i\}, T_2) \notin \mathcal{W}$. Pose $S'_1 = S_1 \setminus \{i\}$, $S'_2 = S_2$, $T_1 = T_1 \setminus \{j\}$ and $T'_2 = T_2$. It follows that $i \notin [S'_1 \cup S'_2] \cup [T'_1 \cup T'_2]$, $j \notin [S'_1 \cup S'_2] \cup [T'_1 \cup T'_2]$, $(S'_1 \cup \{j\}, S'_2) \notin \mathcal{W}$ and $(S'_1 \cup \{i\}, S'_2) \in \mathcal{W}$, and $i \notin [S'_1 \cup S'_2] \cup [T'_1 \cup T'_2]$, $j \notin [S'_1 \cup S'_2] \cup [T'_1 \cup T'_2]$, $(T'_1 \cup \{j\}, T'_2) \in \mathcal{W}$ and $(T'_1 \cup \{i\}, T'_2) \notin \mathcal{W}$, implying respectively $non(j \geq_{T_{31}} i)$ and $non(i \geq_{T_{31}} j)$. It follows that i and j are not comparable by the influence relation $\geq_{T_{31}}$, which contradicts the fact that $\geq_{T_{31}}$ is complete. We conclude that $\mathcal{H}$ is (3, 1)-swap-robust. $\square$

Let f be a permutation on the set $\{(3, 1), (3, 2), (2, 1)\}$ defined as follows: $f((3, 1)) = (2, 1)$, $f((3, 2)) = (3, 1)$, and $f((2, 1)) = (3, 2)$. As a direct consequence of Propositions 5 and 6, we have the following characterization of VGAs for which each of the $(s, r)-$influence relations is a complete preorder.

Theorem 1 *Let $(s, r) \in \{(3, 1), (3, 2), (2, 1)\}$, and $\mathcal{H} = (N, \mathcal{W})$ be an $f((s, r))-$anonymous VGA. The $(s, r)-$influence* *relation is a complete preorder on the set of voters if and only if $\mathcal{H}$ is $(s,r)-$swap-robust.*

4.3 Understanding cycles in the $(s,r)-$ influence relations of VGAs

We have shown that the strict components of the $(s,r)-$ influence relations of VGAs may be cyclical (Proposition 4). While this property may be justified and even seen as positive for certain types of organizations, it requires some explanation in the context of VGAs. In VGAs, cycles in the $(s,r)-$ influence relations are due to the fact that these relations evaluate the relative influence of voters only at specific levels of participation, abstracting away from other levels. To be clear, let us consider the example given in the proof of Proposition 4 showing that the strict component of the $(3,1)-$ influence relation may be cyclical. We have $1 >_{T_{31}} 2$, $2 >_{T_{31}} 3$ and $3 >_{T_{31}} 1$. Note that in that example, we also have $non(3 \geq_{T_{32}} 1)$ and $non(2 \geq_{T_{32}} 3)$. It can be easily shown that if we had $3 \geq_{T_{32}} 1$ and $2 \geq_{T_{32}} 3$, given $1 >_{T_{31}} 2$, it wouldn't have been possible to have $3 >_{T_{31}} 1$ (because other minimal winning tripartitions would have been generated, which would have eventually cancelled $3 >_{T_{31}} 1$), and there would have been no cycle. This shows that $>_{T_{31}}$is cyclical because it evaluates the relative influence of voters only at the first and third levels of support. The same reasoning helps understand why $>_{T_{32}}$ and $>_{T_{21}}$may be cyclical, and why the combined influence relation defined in Tchantcho et al. (2008) is not. A unique advantage of the $(s,r)-$ influence relations however is that they allow a full assessment of the structure of power among the members of an organization, by evaluating their relative influence at various levels of participation (see Section 7 for illustration).

5 A further generalization of the influence relation

In this section, we introduce a more conservative generalization of the influence relation in $(u,v)-$ simple games. We have the following definition.

Definition 10 *Let $G = (N,V)$ be an (u,v) simple game and i and j two players.*

1) i is said to be at least as influential as j, denoted $i \geq_{T,G} j$, if for any u-partition $\mathcal{S}$ and any r, $s \in I_u$ such that $r < s$ and $i, j \in S_s$, $V(\mathcal{S}(i(s), j(s \to r)) \preceq V(\mathcal{S}(i(s \to r), j(s))$.

2) i is said to be as influential as j, denoted $i \sim_{T,G} j$, if $i \geq_{T,G} j$ and $j \geq_{T,G} i$.

3) i is said to be more influential than j, denoted $i >_{T,G} j$, if $i \geq_{T,G} j$ and non($j \geq_{T,G} i$).

Interpretation: i is said to be at least as influential as j if i is at least as $(s,r)-$ influential as j for all r, $s \in I_u$ such that $r < s$. This extension is simply a combination of the $(s,r)-$ influence relations.

Restricting this generalization to voting games with abstention leads to the following influence relation proposed in Tchantcho et al. (2008).

Definition 11 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA and i and j two players.*

1) i is said to be at least as influential as j, denoted $i \geq_{T,\mathcal{H}} j$, if $\forall (S_1, S_2) \in \mathcal{N}$ such that $i, j \notin S_1 \cup S_2$,

$$\begin{cases} (S_1 \cup \{j\}, S_2) \in \mathcal{W} \Rightarrow (S_1 \cup \{i\}, S_2) \in \mathcal{W} \\ (S_1, S_2 \cup \{j\}) \in \mathcal{W} \Rightarrow (S_1, S_2 \cup \{i\}) \in \mathcal{W} \\ (S_1 \cup \{j\}, S_2 \cup \{i\}) \in \mathcal{W} \Rightarrow (S_1 \cup \{i\}, S_2 \cup \{j\}) \in \mathcal{W} \end{cases}$$

2) i is said to be as influential as j, denoted $i \sim_{T,\mathcal{H}} j$, if $i \geq_{T,\mathcal{H}} j$ and $j \geq_{T,\mathcal{H}} i$.

3) i is said to be more influential than j, denoted $i >_{T,\mathcal{H}} j$, if $i \geq_{T,\mathcal{H}} j$ and non($j \geq_{T,\mathcal{H}} i$).

It is shown in Tchantcho et al. (2008) that the indifference component of this relation is an equivalence relation; its strict component is not transitive in general, but is not cyclical either. For a detailed examination of the structure of power among players, the $(s,r)-$ influence relations would be more useful than this combined relation, as we will show in Section 7.

6 Ordinal equivalence of power theories in VGAs

In this section, we compare the $(s,r)-$ influence relations with the preorderings (SS) and (BC) respectively induced on the set of players by the Shapley-Shubik and Banzhaf-Coleman indices. These indices were obtained by Felsenthal and Machover (1997) and Freixas (2005a, 2005b) for VGAs.[16] We characterize VGAs for which these power theories yield similar rankings.

Let χ_i be the indicator function of i, that is, $\chi_i(k) = 1$ if $k = i$, and 0 if $k \neq i$. Let $S \in 2^N$, $(S_1, S_2) \in \mathcal{N}$, $i, j \in N$.

Pose $\pi_{ij} S = \begin{cases} S \text{ if } \{i,j\} \subset S \text{ or } \{i,j\} \cap S = \emptyset \\ [S \setminus \{i\}] \cup \{j\} \text{ if } i \in S \text{ and } j \notin S \\ [S \setminus \{j\}] \cup \{i\} \text{ if } i \notin S \text{ and } j \in S \end{cases}$; and $\pi_{ij}(S_1, S_2) = (\pi_{ij} S_1, \pi_{ij} S_2)$.

$\pi_{ij} S$ and $\pi_{ij}(S_1, S_2)$ are respectively the permutation of i and j over coalition S and tripartition (S_1, S_2).

We define the degree of support of i in a tripartition (S_1, S_2) as $a(i, (S_1, S_2)) = 1$ if $i \in S_1$, 0 if $i \in S_2$, and -1 if $i \in S_3 = N \setminus (S_1 \cup S_2)$. If $a(i, (S_1, S_2)) = 1$ and $a(j, (S_1, S_2)) = -1$, we say that i and j are totally or strongly opposed in the tripartition (S_1, S_2, S_3).

We recall below the notion of equitable VGAs first introduced in Tchantcho et al. (2008).

[16] Felsenthal and Machover (1997) and Freixas (2005a,b) obtained the same generalizations of these notions, but Freixas (2005b) obtained an additional generalization of the Banzhaf-Coleman index, which can be shown to have the same ordinal ranking as the other one. For our purpose, we therefore only use the generalizations obtained by Felsenthal and Machover (1997).

Definition 12 *: 1) A VGA $\mathcal{H}$ is said to be* equitable *if* $\forall i, j \in N, \forall T \in \{-1,0,1\}^N$ *such that* $T(i) = 1$ *and* $T(j) = -1$*, the following condition is satisfied:*

$$\mathcal{H}(T) = \mathcal{H}(\pi_{ij}T) \Rightarrow \begin{cases} \mathcal{H}(T - \chi_i) = \mathcal{H}(\pi_{ij}(T - \chi_i)) \text{ and} \\ \mathcal{H}(T + \chi_j) = \mathcal{H}(\pi_{ij}(T + \chi_j)) \end{cases}$$

2) A VGA $\mathcal{H}$ is said to be weakly equitable *if* $\forall i, j \in N, \forall T \in \{-1,0,1\}^N$ *such that* $T(i) = 1$ *and* $T(j) = -1$*, (a) and (b) are satisfied:*

$$a) \quad \left.\begin{array}{c} \mathcal{H}(T) = 1 \\ \mathcal{H}(\pi_{ij}T) = 1 \end{array}\right\} \Rightarrow \mathcal{H}(T - \chi_i) = \mathcal{H}(\pi_{ij}(T - \chi_i)),$$

$$b) \quad \left.\begin{array}{c} \mathcal{H}(T) = -1 \\ \mathcal{H}(\pi_{ij}T) = -1 \end{array}\right\} \Rightarrow \mathcal{H}(T + \chi_j) = \mathcal{H}(\pi_{ij}(T + \chi_j))$$

Interpretation : A VGA is equitable if any two voters who are equally desirable when strongly opposed in a tripartition are still equally desirable when their views come closer. In a weakly equitable VGA, two voters who are equally desirable when strongly opposed in a winning tripartition remain equally desirable when the voter with the strongest support lowers her support, and two voters who are equally desirable when strongly opposed in a losing tripartition remain equally desirable when the voter with the weakest support raises her support. An equitable VGA is also weakly equitable.

The class of equitable VGAs, and thus weakly equitable VGAs, is a very large class of VGAs that include the class of simple VGAs, the class of relative majority voting games, but expands beyond these well-known classes of games to include real-life institutions such as the United Nations Security Council (also see Tchantcho et al. for other examples).

We recall below the notion of swap-robustness introduced in Tchantcho et al. (2008).

Definition 13 *: A VGA $\mathcal{H} = (N, \mathcal{W})$ is said to be swap-robust if for any (S_1, S_2), $(T_1, T_2) \in \mathcal{W}$, and $(i, j) \in N^2$ such that $a(i, (S_1, S_2)) > a(j, (S_1, S_2))$ and $a(j, (T_1, T_2)) > a(i, (T_1, T_2))$, at least one of the tripartitions $\pi_{ij}(S_1, S_2)$ and $\pi_{ij}(T_1, T_2)$ is winning.*

Felsenthal and Machover (1997) extend to VGAs the Banzhaf-Coleman power index as follows.

Definition 14 *: (Banzhaf-Coleman power index) Let $\mathcal{H}$ be a VGA, $T \in \{-1,0,1\}^N$ and $i \in N$.*

1) Player i is said to be positively critical (resp. negatively critical) if $T(i) \geq 0, \mathcal{H}(T) = 1$ and $\mathcal{H}(T - \chi_i) = -1$ (resp. $T(i) \leq 0, \mathcal{H}(T) = -1$ and $\mathcal{H}(T + \chi_i) = 1$)

2) The Banzhaf score of i in $\mathcal{H}$, denoted $n_i(\mathcal{H})$, is the number of tripartitions for which i is critical either positively or negatively. The relative and absolute Banzhaf-Coleman indices are respectively defined as follows: $\beta_i = \frac{n_i(\mathcal{H})}{\sum_{i \in N} n_i(\mathcal{H})}$ *and* $\beta'_i = \frac{n_i(\mathcal{H})}{3^{n-1}}$.

The relative and the absolute Banzhaf-Coleman indices define on N a complete preordering denoted (BC).

Let Γ be the set of bijective mappings from N onto $\{1, ..., n\}$. The following extension of the Shapley-Shubik power index to VGAs is also due to Felsenthal and Machover (1997).

Definition 15 *: Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA, $\mathcal{R} = (s\mathcal{R}, d\mathcal{R}) \in \Gamma \times \{-1, 0, 1\}^N$ and $i \in N$.*

1) We say that $\mathcal{S} \in \Gamma \times \{-1, 0, 1\}^N$ is in agreement with $\mathcal{R}$ up to i if $s\mathcal{R} = s\mathcal{S}$ and $d\mathcal{R}(x) = d\mathcal{S}(x)$ for any x such that $s\mathcal{R}(x) \leq s\mathcal{R}(i)$.

Voter i is the pivot of $\mathcal{R}$ in $\mathcal{H}$, denoted $i = Piv(\mathcal{R}, \mathcal{H})$, if i is the first voter in the ordering $s\mathcal{R}$ satisfying: $\mathcal{H}(d\mathcal{S}) = \mathcal{H}(d\mathcal{R})$ for any $\mathcal{S}$ in agreement with $\mathcal{R}$ up to i.

2) The Shapley-Shubik index of i is : $\phi_i(\mathcal{H}) = \frac{\left|\{\mathcal{R} \in \Gamma \times \{-1,0,1\}^N | i = Piv(\mathcal{R}, \mathcal{H})\}\right|}{3^n n!}$.

The Shapley-Shubik index induces on the set of voters N a preordering (SS).

Diffo Lambo and Moulen (2002) show that the influence relation and the preorderings (SS) and (BC) coincide in a simple game if and only if the game is swap-robust. Tchantcho et al. (2008) show that in a weakly equitable VGA, the influence relation $\geq_T$ and the preorderings (SS) and (BC) if and only if the VGA is swap-robust. We show below a similar result for the $(s, r)-$ influence relations.

Theorem 2 *Let $\mathcal{H} = (N, \mathcal{W})$ be a weakly equitable and swap-robust VGA. The $(s, r)-$ influence relations $\geq_{T_{sr}}$, the influence relation $\geq_T$, and the preorderings (SS) and (BC) coincide.*

Proof : Let $\mathcal{H} = (N, \mathcal{W})$ be a weakly equitable and swap-robust VGA. It is easy to see that the influence relation $\geq_T$ is included in each of the $(s, r)-$ influence relations. In addition, since $\mathcal{H}$ is weakly equitable and swap-robust, according to Theorem 1 in Tchantcho et al. (2008), the influence relation $\geq_T$, the (SS) and the (BC) preorderings coincide. Given that the influence relation $\geq_T$ is included in each of the $(s, r)-$ influence relations, each $(s, r)-$ influence relation coincides with $\geq_T$ and consequently with the preorderings (SS) and (BC). $\square$

This result provides a characterization of VGAs for which six power theories have the same rankings. It is intriguing because the $(s, r)-$ influence relations $\geq_{T_{sr}}$ and the influence relation $\geq_T$ have very different properties. It is also interesting to observe that an argument in the proof of Theorem 2 implies that swap-robustness is a sufficient condition for each of the $(s, r)-$ influence relations to be a complete preorder in a VGA.

7 Some applications of the influence relations

In this section, we show some applications of the influence relations of VGAs. The first application is to the United Nations Security Council.

Example 2 *(The United Nations Security Council) The United Nations Security Council (UNSC) contains* 5 *permanent members and* 10 *non-permanent members. A decision is made if at least* 9 *members support it and no permanent member is explicitly opposed to it. Let us denote by* $P = \{1, 2, 3, 4, 5\}$ *the set of all permanent members and* $NP = \{6, 7, ..., 15\}$ *the set of non-permanent members. Following Freixas and Zwicker (2003), the vote of the UNSC can be modelled as follows.*

Let $\mathcal{S} = (S_1, S_2, S_3)$ *be an ordered* 3*-partition of* N $(N = UNSC)$*:*

$$V_{UNSC}(\mathcal{S}) = \begin{cases} \textit{win if } |S_1| \geq 9 \textit{ and } S_3 \cap P = \varnothing \\ \textit{lose otherwise.} \end{cases}$$

V_{UNSC} *is clearly a voting game with abstention. We have the following result:*

$\forall \{p, p'\} \times \{np, np') \subset P \times NP$,

(1) $\forall (s,\ r) \in \{3\} \times \{1, 2\}$, $p >_{Tsr} np$, $p \sim_{Tsr} p'$ *and* $np \sim_{Tsr} np'$.

(2) $p \sim_{T_{21}} np$, $p \sim_{T_{21}} p'$ *and* $np \sim_{T_{21}} np'$

(3) $p >_T np$, $p \sim_T p'$, *and* $np \sim_T np'$.

Our second application is to the simple majority vote.

Example 3 *(A simple majority vote) A simple majority game is a social choice context in which a candidate is elected if the number of players who vote for her is greater than the number of players who vote against her no matter how many players abstain. This game can be modelled as follows.*

Let $\mathcal{S} = (S_1, S_2, S_3)$ *be an ordered* 3*-partition :*

$$V_{maj}(\mathcal{S}) = \begin{cases} \textit{win if } |S_1| > |S_3| \\ \textit{lose otherwise.} \end{cases}$$

In a simple majority game, all players are equally influential by any measure of influence:

$\forall \{i, j\} \subset N$,

(1) $\forall (s,\ r) \in \{2, 3\} \times \{1, 2\}$ *with* $r < s$, $i \sim_{Tsr} j$.

(2) $i \sim_T j$.

Our third application is a thought experiment.

Example 4 *Let $\mathcal{H} = (N, \mathcal{W})$ be a VGA where $N = \{1, 2, 3\}$ and $\mathcal{W}_m$={(1,2,3)} (note that we write (1,2,3) for (1,2)). $\mathcal{H}$ might be a firm whose final products are judged as "good" or "bad", and 1, 2, 3 are workers with different ability. The three levels of participation might refer to the different levels of effort (e.g., high, medium, low) that a worker might exert, or to the amount of working time (e.g., working full time, part time or occasionally), or might refer to three different tasks with different levels of difficulty.*

We have the following performance structure:

(1) $1 \sim_{T_{31}} 2$, $1 >_{T_{31}} 3$ and $2 \sim_{T_{31}} 3$.

(2) $1 \sim_{T_{32}} 2$, $1 \sim_{T_{32}} 3$ and $2 >_{T_{32}} 3$.

(3) $1 >_{T_{21}} 2$, $1 >_{T_{21}} 3$ and $2 \sim_{T_{21}} 3$.

(4) $1 >_T 2$, $1 >_T 3$ and $2 >_T 3$.

It follows from Examples 2 and 4 that it might be misleading to rely only on the combined influence relation $\geq_{T,\mathcal{H}}$ to evaluate the relative influence of a player. In Example 1, while a permanent member of the U.N. Security Council is overall more influential than a non-permanent member, we note that this domination is not true at all levels of participation since $p \sim_{T_{21}} np$. In Example 4, while we have $1 >_T 2$, we have $1 \sim_{T_{31}} 2$, $1 \sim_{T_{32}} 2$ and $1 >_{T_{21}} 2$. This demonstrates that worker 1 is not more performant than worker 2 at all levels of participation in the firm despite the fact that $\geq_{T,\mathcal{H}}$ gives 1 as strictly more performant than 2.

8 CONCLUSION

We have studied political influence in multi-choice institutions, which are institutions in which members choose from among several options their levels of support to a collective goal, these individual choices determining the degree to which the goal is reached. (u, v) simple games have served as a useful mathematical model for such institutions. Influence was assessed by the $(s, r)-$ influence relations, which are newly defined binary relations, each of which compares any two individuals of an organization on the basis of their relative performance at a corresponding level of participation. In voting games with abstention, we have found that the strict components of the $(s, r)-$ influence relations may be cyclical, unlike in binary voting games. We have provided sufficient conditions of anonymity under which these relations are transitive. We have also obtained a necessary and sufficient condition for each of them to be complete, and have provided a partial characterization of institutions for which the rankings induced by these relations, and the Banzhaf-Coleman and Shapley-Shubik power indices coincide. This latter result particularly extends the ordinal equivalence theorem obtained by Tchantcho et al. (2008) as it involves additional power relations.

We note that in general, the properties the influence relation of binary voting games do not extend to (u, v)

simple games. The cyclicity of the $(s, r)-$ influence relations is particularly meaningful. One however wonders about the presence of cycles in weighted (u, v) simple games.[17] It is well-known that in a weighted binary voting game, if i has more weight than j, then i is at least as influential as j. It would be interesting in the future to check whether this property holds for the $(s, r)-$ influence relations of (u, v) simple games.

It is important to observe that the $(s, r)-$ influence relations readily extend to economic organizations that can also be modeled as (u, v) simple games, but where the u different roles that members can play are not ordered.[18] An example of such an organization is a soccer team where *a priori*, strikers cannot be viewed as contributing more than the midfielders or the defenders to the collective performance of the team. In this type of (u, v) simple games, one would have $u(u-1)$ $(s, r)-$ influence relations as bidirectional shifts should be considered in the evaluation of players' relative performance, instead of $\frac{u(u-1)}{2}$ as in organizations where u denotes the number of ordered roles or production units.

In an organization with non-ordered roles, the $(s, r)-$ influence relations would also be useful in efficiently allocating workers to different units of production, contrary to the combined influence relation. In a soccer team for example, the $(s, r)-$ influence relations will make it possible to establish that player i is more desirable than player j as a defender while j is more desirable than i as a striker, if this is really the case. But in such a situation of conflicting competence, the combined influence relation will not be able to compare i and j at all, and will not be useful in assigning these players to their right positions in the team.

The combined influence relation may be particularly appealing when units of production are ordered such as in a voting game with abstention, although the $(s, r)-$ influence relations in this context would still evaluate with useful details the relative influence of each player. For example, we note that in the application of the influence relations to the United States Senate (Example 1), a senator is overall more influential than the vice-president, but this is not the case if the relative contribution of a senator and the vice-president is evaluated on the basis of a shift from "abstention" to a "yes" vote, given that $\forall \{i\} \subset N \setminus \{vp\}$, $i \sim_{T_{21}} vp$. On all accounts, the extensions of the influence relation as shown in this study provide useful tools for assessing individual performance in team work. A study of their properties yields interesting and unexpected results, especially when compared to those obtained in the basic framework of binary voting games.

[17] See Freixas and Zwicker (2003) for a characterization of weighted (u, v) simple games.

[18] They also extend to a more general model of organizations $G = (N, V)$, where V maps any $u-$ partition of N into a real number. A $u-$ partition may be ordered or not.

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