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Disappointment Models: an axiomatic approach

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Abstract: In this paper, a fully choice-based theory of disappointment is developed. It encompasses, as particular cases, EU theory, Gul's theory of disappointment (1991) and the models of Loomes and Sugden (1986). According to the new theory, the risk premium of a random prospect is the sum of two premiums: a concavity premium that is nothing but the usual Arrow-Pratt premium and a second premium that may be identified to expected disappointment. The corresponding representing functional belongs to the class of lottery-dependent utility models (Becker and Sarin 1987) since disappointment is the deficit between the utility of the realized outcome and its expected value. However, unlike the lottery-dependent approach, the theory is choice-based and its axioms are experimentally testable.

JEL classification: D81

Key-words: *axiomatization, disappointment aversion, random prospect, risk premium, expected utility.*

Résumé : Nous proposons, dans cette étude, une axiomatisation du comportement d'un individu en univers incertain. Cette axiomatisation permet d'effectuer une synthèse de nombreuses théories antérieures: théorie de l'utilité espérée, théorie de la déception (Gul 1991), modèles à la Loomes et Sugden (1986) ou modèles à fonctionnelle "loterie-dépendante" (Becker and Sarin 1987). La prime de risque exigée par un investisseur y est la somme d'une prime d'Arrow-Pratt et d'une seconde prime, égale à l'espérance de la déception. Celle-ci n'est autre que la différence entre la satisfaction éprouvée *ex ante* et l'utilité espérée du revenu aléatoire.

Classification JEL : D81

Mots-clés : *axiomatisation, aversion pour la déception, revenu aléatoire, prime de risque, utilité espérée.*

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1 Introduction

It is often emphasized, in psychological literature, that (*i*) disappointment (elation) is experimented, once a decision has been taken, when the chosen option turns out to be worse (better) than expected (see, for instance, Mellers 2000), (*ii*) that it is the most frequently experimented emotion (Weiner *et alii* 1979) and (*iii*) that disappointment is the most powerful among the negative emotions that are experimented (Schimmack and Diener 1997). There is a lot of empirical evidence that supports this view (Van Dijk and Van der Pligt 1996, Zeelenberg *et alii* 2002, Van Dijk *et alii* 2003). Moreover, emotions are anticipated by decision makers. As Frijda (1994) points out, "actual emotion, affective response, anticipation of future emotion can be regarded as the primary source of decisions". In other words, expected elation or disappointment plays an important role in decision making.

This role was first formalized independently by Bell (1985) and Loomes and Sugden (1986). Despite its earliness, their approach has revealed surprisingly close to the analyses mentioned above. Indeed, according to Loomes and Sugden, the satisfaction experimented by an investor after a lottery has been run includes two elements: (*i*) the satisfaction resulting from the ownership of the outcome that actually occurred and (*ii*) elation or disappointment that depends on the difference between the above satisfaction and the utility of a reference outcome that will be called, from now on, zero-disappointment (henceforth ZD) outcome.[1]

Despite their psychological relevance, the models of Bell (1985) and of Loomes and Sugden (1986) have been somewhat neglected in the economic literature, probably because they lack an axiomatic framework. In the meantime, other disappointment models have been developed, in which elation or disappointment is measured somewhat differently; Delquie and Cillo (2006) use all the outcomes of the prospect; Jia *et alii* (2001) generalize Bell's (1985) approach and advocate the use of its expected value; Grant and Kajii (1998) adapt the setting of the rank-dependent expected utility model (Quiggin 1982 among others) to highlight the dependence on the best possible outcome. However these approaches also lack an axiomatic basis. By contrast, Gul (1991) develops an axiomatized theory of disappointment where the ZD outcome is the certainty equivalent of the prospect.

[1] Since its occurring implies neither elation nor disappointment.

In this article, an axiomatization of decision making under risk is developed. Preferences are defined over a set of bounded random prospects. As usual, they are assumed to be a continuous total order. Next, a partition of the set of the random prospects is made. Each element of the partition is a subset of prospects exhibiting the same ZD outcome. When constant marginal utility is assumed the ZD outcome coincides with the expected outcome. In the general case, the utility of a ZD outcome is the expected utility of the prospect. The independence axiom is set over each subset of prospects exhibiting the same ZD outcome and a standard representation theorem is used to define a lottery-dependent functional. The consistency of the functionals is obtained through setting an additional axiom. The intuition is that risk premia, –in the sense of Markowitz (1952)– combine linearly if the prospects that are mixed exhibit the same ZD outcome. Finally, preferences will be represented by a functional that encompasses, as particular cases, that of EU theory, that of Gul and that of Loomes and Sugden. Clearly, disappointment models belong to the very general class of lottery-dependent utility models (Becker and Sarin 1987). However, unlike its predecessors our lottery-dependent models preserve a fully choice-based approach and are testable. Indeed, the elementary utility function that is used to define the ZD outcome, can be elicited, and, consequently the subsets over which the independence axiom is set, may be identified. An important part of this article is thus devoted to the presentation of an elicitation process that can be implemented. The rest of this article is organized as follows: first, preferences are assumed to exhibit constant marginal utility and a simplified axiomatics is proposed (Section 2). Next, the general case is considered (Section 3). Section 4 concludes.

2 Constant marginal utility.

2.1 The framework

From now on, we consider a set of random prospects, labelled $\mathbb{W}$, whose outcomes are monetary and belong to a bounded interval of $\mathbb{R}$, say $[a, b]$. An element of $\mathbb{W}$ will be labelled $\widetilde{w}$ and its cumulative distribution function $F_{\widetilde{w}}(.)$. If a random prospect $\widetilde{w}$ has a discrete support $\{w_1, w_2, ..., w_K\}$, it will also be denominated $[w_1, w_2, ..., w_K\ ;\ p_1, p_2, ..., p_K]$ where $p_k = \Pr(\widetilde{w} = w_k)$. A probability mixture of $\widetilde{w}_1 \in \mathbb{W}$ and $\widetilde{w}_2 \in \mathbb{W}$, will be denoted $\alpha\widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2$, where α belongs to $[0,1]$. Binary lotteries will be labelled

$$\widetilde{w}_{\pi}^{w_1,w_2} \overset{def}{=} [w_1, w_2; 1-\pi, \pi]$$

where $\pi \in [0,1]$. Probability $1-\pi$ (π) will be called the first (second) weight of $\widetilde{w}_{\pi}^{w_1,w_2}$. The degenerate lottery whose outcome is w with certainty is $\delta(w)$. Preferences over prospects will be denoted $\precsim$, with $\prec$ (strict preference) and $\sim$ (indifference). The certainty equivalent of the prospect $\widetilde{w} \in \mathbb{W}$ is labelled $\mathbf{c}(\widetilde{w})$, that is $\widetilde{w} \sim \delta(\mathbf{c}(\widetilde{w}))$.

In Loomes and Sugden (1986), preferences are represented by the following functional:

$$\mathcal{U}(\widetilde{w}) \overset{def}{=} \mathbf{E}[u(\widetilde{w})] + \mathbf{E}[DIS(\widetilde{w})]$$

where $\widetilde{w} \in \mathbb{W}$, $\mathcal{U}(.)$ maps $\mathbb{W}$ on to $\mathbb{R}$, $\mathbf{E}[.]$ stands for the expectation operator, $u(.)$ is an elementary utility function –that is a strictly increasing function mapping $[a,b]$ on to $[0,1]$– and $DIS(w)$ is elation or disappointment when the outcome happens to be w. Elation (disappointment) is experimented if $DIS(w)$ is positive (negative). $\mathcal{U}(\widetilde{w})$ is the sum of (i) expected elementary utility –that is $\mathbf{E}[u(\widetilde{w})]$– and (ii) expected elation/disappointment –that is $\mathbf{E}[DIS(\widetilde{w})]$–. In their seminal work, Loomes and Sugden (1986) set

$$DIS(w) = \mathcal{E}(u(w) - \mathbf{E}[u(\widetilde{w})])$$

where $\mathcal{E}(.)$ is strictly increasing and fulfills the self-explanatory condition $\mathcal{E}(0) = 0$, what will be assumed in the rest of this article. We now focus on the case when constant marginal utility is assumed. Preferences are represented by the below functional:

$$\mathcal{U}(\widetilde{w}) = \mathbf{E}[\widetilde{w}] + \mathbf{E}[\mathcal{E}(\widetilde{w} - \mathbf{E}[\widetilde{w}])] \tag{1}$$

Moreover, if investors are risk averse, that is if they always demand a negative risk premium, we get the below inequality:

$$\forall \widetilde{w} \in \mathbb{W}, \qquad \mathbf{E}[\mathcal{E}(\widetilde{w} - \mathbf{E}[\widetilde{w}])] \leq 0 = \mathbf{E}[\widetilde{w} - \mathbf{E}[\widetilde{w}]] \tag{2}$$

that implies that function $\mathcal{E}(.)$ is concave, what will be assumed from now on.

If preferences are represented by (1), then the binary relation "$\widetilde{w}_1$ and $\widetilde{w}_2$ have the same expected value" is clearly an equivalence relation over $\mathbb{W}$, whose classes are the subsets $\mathbb{W}_z = \{\widetilde{w} \in \mathbb{W} \,|\mathbf{E}[\widetilde{w}] = z\}$. Any subset

$\mathbb{W}_z$ possesses exactly one degenerate lottery $\delta(z)$ and one binary lottery of the $\widetilde{w}_p^{a,b}$ type. The binary lottery will be denominated $\widetilde{w}_{\pi(z)}^{a,b}$ where $\pi(z) = (z-a)/(b-a)$. The family of subsets $\{\mathbb{W}_z\}_{z\in[a,b]}$ is endowed with three other important properties. The first two ones may be stated as indicated below:

Property A (extremal elements). Any element of $\mathbb{W}_z$ is such that: $\widetilde{w}_{\pi(z)}^{a,b} \prec \widetilde{w} \prec \delta(z)$.[2]

Proof. See the Appendix.■

Property B (independence). The independence property is met over any subset $\mathbb{W}_z$, that is, for any triple of prospects $(\widetilde{w}_1, \widetilde{w}_2, \widetilde{w}_3) \in \mathbb{W}_z \times \mathbb{W}_z \times \mathbb{W}_z$ and for any real number $\alpha \in [0,1]$, the following implication holds: $\widetilde{w}_1 \preceq \widetilde{w}_2 \Rightarrow \alpha\widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_3 \preceq \alpha\widetilde{w}_2 \oplus (1-\alpha)\widetilde{w}_3$.

Proof. See the Appendix.■

Let $\widetilde{w}(a)$ be the probability mixture of $\delta(\mathbf{E}[\widetilde{w}])$ and of $\widetilde{w}_{\pi(\mathbf{E}[\widetilde{w}])}^{a,b}$ whose weights are $(a, 1-a)$, that is let $\widetilde{w}(a) \overset{def}{=} a\delta(\mathbf{E}[\widetilde{w}]) \oplus (1-a)\widetilde{w}_{\pi(\mathbf{E}[\widetilde{w}])}^{a,b}$. Now, consider an arbitrary prospect $\widetilde{w} \in \mathbb{W}$ and its certainty equivalent $\mathbf{c}(\widetilde{w})$. Clearly, there exists a unique real number, $a_{\widetilde{w}} \in [0,1]$, that is defined by the following equivalence : $\widetilde{w} \sim \widetilde{w}(a_{\widetilde{w}})$. The third property may then be stated as follows:

Property C (consistency). The certainty equivalent of a prospect $\widetilde{w} \in \mathbb{W}$ is a convex combination the certainty equivalents of the extremal elements of $\mathbb{W}_{\mathbf{E}[\widetilde{w}]}$ whose weights are $(a_{\widetilde{w}}, 1-a_{\widetilde{w}})$. that is:

$$\mathbf{c}(\widetilde{w}) = a_{\widetilde{w}}\mathbf{E}[\widetilde{w}] + (1-a_{\widetilde{w}})\,\mathbf{c}(\widetilde{w}_{\pi(\mathbf{E}[\widetilde{w}])}^{a,b}). \tag{3}$$

Proof. From (1) we get that $\mathcal{U}(\delta(\mathbf{c}(\widetilde{w}))) = \mathbf{c}(\widetilde{w})$, for any $\widetilde{w} \in \mathbb{W}$. Moreover, since $\widetilde{w} \sim \mathbf{c}(\widetilde{w})$, then $\mathcal{U}(\widetilde{w}) = \mathcal{U}(\delta(\mathbf{c}(\widetilde{w})))$. Finally, we get:

$$\mathcal{U}(\widetilde{w}) = \mathbf{c}(\widetilde{w}) \tag{4}$$

Next, from (1) we get that $\mathcal{U}(\widetilde{w}(a_{\widetilde{w}}))$ expresses as:

$$\mathcal{U}(\widetilde{w}(a_{\widetilde{w}})) = a_{\widetilde{w}}\mathbf{E}[\widetilde{w}] + (1-a_{\widetilde{w}})\,\mathbf{c}(\widetilde{w}_{\pi(\mathbf{E}[\widetilde{w}])}^{a,b}) \tag{5}$$

and since $\mathcal{U}(\widetilde{w}) = \mathcal{U}(\widetilde{w}(a_{\widetilde{w}}))$, we also get the below equality:

$$\mathcal{U}(\widetilde{w}) = a_{\widetilde{w}}\mathbf{E}[\widetilde{w}] + (1-a_{\widetilde{w}})\,\mathbf{c}(\widetilde{w}_{\pi(\mathbf{E}[\widetilde{w}])}^{a,b}) \tag{6}$$

Combining (4) and (6) yields (3).■ As a consequence, we set the following definition:

[2] Note that Property A is due to the concavity of $\mathcal{E}(.)$.

Definition 1 (consistency property). *Preferences are endowed with the consistency property if Property C is met,*

A graphical illustration of the consistency property is given on Figure 1. Certainty equivalents are plotted on the horizontal axis whereas ZD outcomes, that is expected values, are plotted on the vertical axis. The subset $\mathbb{W}_z^c$ of prospects whose expected value is z and whose certainty equivalent is c, is represented by point $\mathbf{L} = (c, z)$. The subset $\mathbb{W}_z$ ($\mathbb{W}^c$) of prospects whose expected value is z (whose certainty equivalent is c) is represented by the horizontal (vertical) segment whose bounds are $\mathbf{M} = (\mathbf{c}(\widetilde{w}^{a,b}_{\pi(z)}), z)$ and $\mathbf{N} = (z, z)$ ($\mathbf{Q} = (c, c)$ and $\mathbf{P} = (c, z_p)$ where z_p is such that $\mathbf{c}(\widetilde{w}^{a,b}_{\pi(z_p)}) = c$).[3] Since the investor is assumed to be risk-averse, any prospect must be plotted in the area lying between segment $\mathbf{OB}$, that is the locus of the degenerate lotteries $\delta(c)$ and the curve whose points correspond the binary lotteries $\widetilde{w}^{a,b}_{\pi(z)}$ that is the minimal elements of the subsets $\mathbb{W}_z$. Let $\widetilde{w} \in \mathbb{W}_z^c$. From Property C, we get that:

$$a_{\widetilde{w}} = \frac{\mathbf{LN}}{\mathbf{MN}} = \frac{\mathbf{QN}}{\mathbf{TN}} \ ; \ 1 - a_{\widetilde{w}} = \frac{\mathbf{ML}}{\mathbf{MN}} = \frac{\mathbf{TQ}}{\mathbf{TN}}$$

Insert Figure 1 about here

In the next subsection, we develop an axiomatics corresponding to investors whose preferences exhibit constant marginal utility.

2.2 An axiomatics for disappointment models with constant marginal utility.

First, any investor will be assumed to have preferences that obey the two first axioms of EU theory. These two axioms are recalled below:

Axiom A1 (total ordering of $\preceq$). *The binary relation $\preceq$ is a complete weak order.*

Axiom A2 (continuity of $\preceq$). *For any prospect $\widetilde{w} \in \mathbb{W}$ the sets $\{\widetilde{v} \in \mathbb{W} \mid \widetilde{v} \preceq \widetilde{w}\}$ and $\{\widetilde{v} \in \mathbb{W} \mid \widetilde{w} \preceq \widetilde{v}\}$ are closed in the topology of weak convergence.*

Under the axioms of total ordering and continuity, there exists a numerical representation for the preference relation. It consists in a continuous utility function $\mathbf{U}(.)$ mapping $\mathbb{W}$ on to $[\mathbf{U}(\delta(a)), \mathbf{U}(\delta(b))] \subset \mathbb{R}$. $\mathbf{U}(.)$ is defined up to a continuous and strictly increasing transformation. To get a stronger result, additional axioms must be set. In EU theory, it is the independence axiom.

[3] Recall that $\pi(z) \overset{def}{=} (z - a)/(b - a)$.

By contrast, some weaker axioms will now be set. *They correspond to the three last properties that were presented above*. Indeed, we need to set an axiom to get Property A. We now assume that the subset of the prospects exhibiting the same expected value z has two extremal elements.

Axiom A3 (extremal elements) *The subset* $\mathbb{W}_z = \{\widetilde{w} \in \mathbb{W}\,|\mathbf{E}[\widetilde{w}] = z\}$ *has a maximal element* $\delta(z)$ *and a minimal element* $\widetilde{w}^{a,b}_{\pi(z)}$, *where* $\pi(z) \overset{def}{=} (z-a)/(b-a)$.

Clearly, $\widetilde{w}^{a,b}_{\pi(z)}$ $(\delta(z))$ is the most (less) risky –in the sense of Rothschild and Stiglitz (1970)– among the prospects exhibiting the same expected value. Hence, the above axiom means that the less risky prospect is preferred to the other ones and that any prospect is preferred to the most risky one. Actually, Axiom 3 is not very restrictive since a standard second-order dominance axiom can be substituted for it to get the same result. The next axiom is the independence axiom, that is set for any subset of prospects exhibiting the same expected value.

Axiom A4 (independence over any $\mathbb{W}_z$). *The independence property is met over any subset* $\mathbb{W}_z$.

From Axiom 4, we get the following standard result:

Proposition 1. *Under Axioms A1 to A4, the weak order of preferences* $\preceq$ *may be represented over* $\mathbb{W}_z$ *by the lottery-dependent functional* $\mathcal{U}_z(\widetilde{w}) \overset{def}{=} \int_a^b \boldsymbol{v}_z(x) dF_{\widetilde{w}}(x)$, *where* $z = \mathbf{E}[\widetilde{w}]$ *and where* $\boldsymbol{v}_z(.)$ *is an increasing function that is defined up to an affine and positive transformation. In other words, for any* $(\widetilde{w}_1, \widetilde{w}_2) \in \mathbb{W}_z \times \mathbb{W}_z$, *the following equivalence is valid:*

$$\widetilde{w}_1 \preceq \widetilde{w}_2 \Leftrightarrow \int_a^b \boldsymbol{v}_z(x) dF_{\widetilde{w}_1}(x) \leq \int_a^b \boldsymbol{v}_z(x) dF_{\widetilde{w}_2}(x) \tag{7}$$

Proof. Since the axioms of EU theory are valid over each subset $\mathbb{W}_z$, there exists a continuous and increasing function $\boldsymbol{v}_z(.)$ mapping $[a, b]$ on to $[\boldsymbol{v}_z(a), \boldsymbol{v}_z(b)]$ that numerically represents preferences over $\mathbb{W}_z$. Function $\boldsymbol{v}_z(.)$ is defined up to a positive and affine transformation. See, for instance, Fishburn (1970).■

From now on, we set the following normalization conditions:

$$\boldsymbol{v}_z(\mathbf{c}(\widetilde{w}^{a,b}_{\pi(z)})) = \mathbf{c}(\widetilde{w}^{a,b}_{\pi(z)}) \ ; \ \boldsymbol{v}_z(z) = z \tag{8}$$

and $\boldsymbol{v}_z(.)$ is now unambiguously defined. Let us call $\boldsymbol{v}_z(.)$ $(\mathcal{U}_z(.), \mathcal{U}_z(\widetilde{w}))$ the lottery-dependent elementary utility (functional, value) of $\widetilde{w} \in \mathbb{W}_z$. The normalization conditions mean that the the lottery-dependent elementary utilities of the extremal elements of $\mathbb{W}_z$ coincide with their certainty equivalents.[4] Now, let

[4] The certainty equivalent of $\delta(z)$ is z itself.

$\widetilde{w}(a)$ be defined according to the below definition:

$$\widetilde{w}(a) \overset{def}{=} a\delta(\mathbf{E}[\widetilde{w}]) \oplus (1-a)\,\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])} \tag{9}$$

Clearly $\widetilde{w}(a) \in \mathbb{W}_{\mathbf{E}[\widetilde{w}]}$. Moreover, Axiom 2 implies that there exists a unique real number $a_{\widetilde{w}} \in [0,1]$ such that $\widetilde{w}(a_{\widetilde{w}}) \sim \widetilde{w}$. Next, Proposition 1 and the normalization conditions imply that

$$\mathcal{U}_{\mathbf{E}[\widetilde{w}]}(\widetilde{w}) = \mathcal{U}_{\mathbf{E}[\widetilde{w}]}(\widetilde{w}(a_{\widetilde{w}})) = a_{\widetilde{w}}\mathbf{E}[\widetilde{w}] + (1-a_{\widetilde{w}})\mathbf{c}(\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])})$$

that is the lottery-dependent valuation of $\widetilde{w} \in \mathbb{W}_{\mathbf{E}[\widetilde{w}]}$, namely $\mathcal{U}_{\mathbf{E}[\widetilde{w}]}(\widetilde{w})$, is a convex function of those of $\delta(\mathbf{E}[\widetilde{w}])$ and of $\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])}$, namely $\mathcal{U}_{\mathbf{E}[\widetilde{w}]}(\delta(\mathbf{E}[\widetilde{w}])) = \mathbf{E}[\widetilde{w}]$ and $\mathcal{U}_{\mathbf{E}[\widetilde{w}]}(\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])}) = \mathbf{c}(\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])})$. Now consider the certainty equivalent of $\widetilde{w}$: since $\delta(\mathbf{E}[\widetilde{w}])$ and $\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])}$ are the extremal elements of $\mathbb{W}_{\mathbf{E}[\widetilde{w}]}$, it is a convex combination of $\delta(\mathbf{E}[\widetilde{w}])$ and of $\widetilde{w}^{a,b}_{\pi(\mathbf{E}[\widetilde{w}])}$ that is:

$$\mathbf{c}(\widetilde{w}) = \alpha_{\widetilde{w}}\mathbf{E}[\widetilde{w}] + (1-\alpha_{\widetilde{w}})\,\mathbf{c}(\widetilde{w}^{a,b}_{\pi(z)}). \tag{10}$$

The lottery-dependent functionals will be consistent with one another if and only if their value coincides with the that of the relevant certainty equivalent, that is if and only if $a_{\widetilde{w}} = \alpha_{\widetilde{w}}$ for any $\widetilde{w} \in \mathbb{W}$. Hence we set the following axiom:

Axiom A5 (consistency). *Preferences are endowed with the consistency property.*

The above axiom means that the weights of the compound lottery $\widetilde{w}(a_{\widetilde{w}})$ are those of the convex combination the certainty equivalents of the extremal elements of $\mathbb{W}_{\mathbf{E}[\widetilde{w}]}$ that is equal to $\mathbf{c}(\widetilde{w})$. Finally, from Axiom 5 we get that condition (7) will also hold in the case when $\widetilde{w}_1 \in \mathbb{W}_{z_1}$ and $\widetilde{w}_2 \in \mathbb{W}_{z_2}$with $z_1 \neq z_2$.

Proposition 2. *Under Axioms A1 to A5, the preferences of a disappointment averse investor over* $\mathbb{W}$ *are represented by the lottery-dependent functional:*

$$\mathcal{U}(\widetilde{w}) = \int_a^b \boldsymbol{\upsilon}_{\mathbf{E}[\widetilde{w}]}(x)dF_{\widetilde{w}}(x) \tag{11}$$

where $\widetilde{w} \in \mathbb{W}$ *and where* $\boldsymbol{\upsilon}_{\mathbf{z}(\widetilde{w})}(.)$ *is defined from Proposition 1 and the normalization conditions* (8).

Proof. It is a direct consequence of Proposition 1, of the normalization conditions (8) and of the consistency property. ■

Three additional remarks must now be made: first there exists a non-empty set of models satisfying Axioms A1 to A5 since the functional (1) is clearly a particular case of (11).[5] Last, $\boldsymbol{\upsilon}_{\mathbf{E}[\widetilde{w}]}(x)$ may be

[5] To check that point just set $\boldsymbol{\upsilon}_{\mathbf{E}[\widetilde{w}]}(x) = x + \mathcal{E}(x - \mathbf{E}[\widetilde{w}])$

substituted for $a + b\boldsymbol{v}_{\mathbf{E}[\widetilde{w}]}(x)$, where $a \in \mathbb{R}$, and $b \in \mathbb{R}_{+}^{*}$. In other words, $\boldsymbol{v}_{\mathbf{E}[\widetilde{w}]}(.)$ is defined up to an affine and positive transformation.

3 Variable marginal utility

Now what happens if variable marginal utility is assumed? Actually the above axiomatics has to be –only slightly– modified. First, since $u(.)$ is strictly increasing Axiom 3 remains unchanged. Next, the equivalence relation "$\widetilde{w}_1$ and $\widetilde{w}_2$ have the same expected value" must be substituted for: "$\widetilde{w}_1$ and $\widetilde{w}_2$ have the same expected utility".

Let $\mathbb{W}_z^u$ be defined as the subset of prospects whose expected utility is $u(z)$, that is $\mathbb{W}_z^u = \{\widetilde{w} \in \mathbb{W} \mid \mathbf{E}[u(\widetilde{w})] = u(z)\}$, and, consequently, $\pi(z) = u(z)$. The independence axiom is set on each subset $\mathbb{W}_z^u$. The lottery-dependent valuation of $\widetilde{w}$ is $\mathcal{U}_z^u(\widetilde{w}) \overset{def}{=} \int_a^b \boldsymbol{v}_z^u(x) dF_{\widetilde{w}}(x)$, where $z = \mathbf{E}[\widetilde{w}]$ and where $\boldsymbol{v}_z^u(.)$ is, here again, an increasing function that is defined up to an affine and positive transformation. The normalization conditions (8) now read:

$$\boldsymbol{v}_z^u(\mathbf{c}(\widetilde{w}_{u(z)}^{a,b})) = u(\mathbf{c}(\widetilde{w}_{u(z)}^{a,b}))\ ;\ \boldsymbol{v}_z^u(z) = u(z) \tag{12}$$

and, finally, the definition of the consistency property must be restated as indicated below:

Definition 2 (generalized consistency property). *Preferences are endowed with the consistency property if the weights of the utility of the compound lottery $\widetilde{w}(a_{\widetilde{w}})$ are those of the convex combination the utilities of the certainty equivalents of the extremal elements of $\mathbb{W}_{\mathbf{E}[\widetilde{w}]}$ that is equal to $u(\mathbf{c}(\widetilde{w}))$, that is if*

$$u(\mathbf{c}(\widetilde{w})) = a_{\widetilde{w}} u(z) + (1 - a_{\widetilde{w}})\, u(\mathbf{c}(\widetilde{w}_{\pi(z)}^{a,b})) \tag{13}$$

where $a_{\widetilde{w}}$ is defined by $\widetilde{w}(a_{\widetilde{w}}) \sim \widetilde{w}$.

If function $u(.)$ is known, the so modified axiomatics may be used to get a representation of preferences similar to (11). Indeed, the next proposition may be substituted for Propositions 1 and 2.

Proposition 3. *Under Axioms A1 to A4, and given that $u(.)$ is known, the weak order of preferences $\preceq$ may be represented over $\mathbb{W}$ by the lottery-dependent representing functional $\mathcal{U}_z^u(\widetilde{w}) \overset{def}{=} \int_a^b \boldsymbol{v}_z^u(x) dF_{\widetilde{w}}(x)$, where $z = u^{-1}(\mathbf{E}[u(\widetilde{w})])$ and where $\boldsymbol{v}_z^u(.)$ is an increasing function that is defined up to an affine and* *positive transformation.*

If Axiom A5 is set, the preferences of a disappointment averse investor over $\mathbb{W}$ *are represented by the lottery-dependent functional* $\mathcal{U}^u(\widetilde{w}) = \int_a^b \boldsymbol{v}^u_{\mathbf{z}(\widetilde{w})}(x)dF_{\widetilde{w}}(x)$, *where* $\boldsymbol{v}^u_{\mathbf{z}(\widetilde{w})}(.)$ *is defined according to the normalization conditions* (12).

Proof. The proof is similar to those of Propositions 1 and 2.■

Clearly, the above results are of interest if and only if $u(.)$ can be elicited. To get an elicitation property we must set an additional axiom. As a preliminary to this setting, we show that some models *à la* Loomes and Sugden are endowed with such a property. Actually, we focus on the case when preferences are represented by the below functional

$$\mathcal{U}(\widetilde{w}) = \mathbf{E}[u(\widetilde{w})] + \mathbf{E}[\mathcal{E}(u(\widetilde{w}) - \mathbf{E}[u(\widetilde{w})])]$$

where $\mathcal{E}(.)$ is strictly increasing and concave, and fulfills the self-explanatory condition $\mathcal{E}(0) = 0$. Although Loomes and Sugden (1986) considered more general models, we call them LS-models.

3.1 The elicitation property of LS-models

In LS-models, the betweenness property, and, consequently, the independence axiom may be violated. Hence, it is of interest to set the below definition.

Definition 3 (strong indifference). *Two prospects* $\widetilde{w}_1$ *and* $\widetilde{w}_2$ *are strongly indifferent if and only if* (a) *they are indifferent and* (b) *they meet the betweenness property. The binary relation "*$\widetilde{w}_1$ *and* $\widetilde{w}_2$ *are strongly indifferent" will be labelled "*$\widetilde{w}_1 \approx \widetilde{w}_2$*".*

Clearly, strong indifference implies indifference in the usual sense that will be called, from now on, weak indifference. An important property of LS-models is given in the following proposition:

Proposition 4. *In LS-models, two prospects* $\widetilde{w}_1$ *and* $\widetilde{w}_2$ *are strongly indifferent if and only if they exhibit the same certainty equivalent and the same ZD-outcome, what formally reads:*

$$\widetilde{w}_1 \ \approx \ \widetilde{w}_2 \Longleftrightarrow \mathbf{c}(\widetilde{w}_1) = \mathbf{c}(\widetilde{w}_2) \quad \textit{and} \quad \mathbf{z}(\widetilde{w}_1) = \mathbf{z}(\widetilde{w}_2)$$

Proof. It is given in the Appendix.■

The above proposition makes sense because the certainty equivalent of $\widetilde{w} \in \mathbb{W}$ now generically differs from the zero-disappointment outcome $\mathbf{z}(\widetilde{w}) = u^{-1}(\mathbf{E}[u(\widetilde{w})])$. The binary relation $\approx$ is obviously reflexive and symmetric. From Proposition 4, we get that it is also transitive. Hence, it is an equivalence relation over $\mathbb{W}$. An equivalence class will be denoted $\mathbb{W}_z^c = \mathbb{W}^c \cap \mathbb{W}_z$ where $\mathbb{W}^c = \{\widetilde{w} \in \mathbb{W} \mid \mathbf{c}(\widetilde{w}) = c\}$ and $\mathbb{W}_z = \{\widetilde{w} \in \mathbb{W} \mid \mathbf{z}(\widetilde{w}) = z\}$. An important property of LS-models is that each equivalence class $\mathbb{W}_z^c$ possesses exactly one binary lottery of the $\widetilde{w}_p^{a,x}$ type and one of the $\widetilde{w}_{1-q}^{y,b}$ type.[6]

Proposition 5 (strong equivalents). *There exists exactly one binary lottery of the $\widetilde{w}_p^{a,x}$ type (of the $\widetilde{w}_{1-q}^{y,b}$ type) that is strongly indifferent to $\widetilde{w}$. Lottery $\widetilde{w}_p^{a,x}$ ($\widetilde{w}_{1-q}^{y,b}$) will be called the left (right) strong equivalent of $\widetilde{w}$. The degenerate lottery $\delta(z)$ (the binary lottery $\widetilde{w}_{u(z)}^{a,b}$) is a maximal (minimal) element in $\mathbb{W}_z$, that is $\widetilde{w}_{u(z)}^{a,b} \preceq z \preceq \delta(z)$.*

Proof. The proof is given in the Appendix. The intuition behind the proof is as follows: among the $\widetilde{w}_p^{a,x}$s, the prospects that exhibit the same expected utility $u(z)$ are all the more valuable that the scattering of their outcomes is more narrow. Clearly, the degenerate lottery $\delta(z)$, where $u(z) = pu(b) + (1-p)u(a) = p$, is endowed with the minimal dispersion of outcomes whereas the maximal one is obtained with the discrete distribution $\widetilde{w}_{u(z)}^{a,b} = [a, b : 1 - u(z), u(z)]$.■

Another important property of LS-models is the *elicitation property*. Let $w \in [a,b]$ ($\pi \in [0,1]$) be an arbitrary level of wealth (probability). Consider the sequence of binary lotteries labelled $\{\widetilde{w}_{p_n}^{a,x_n}\}_{n\in\mathbb{N}}$ that meets the below requirements:

$$x_0 = w,\ p_0 = \pi \text{ and } \widetilde{w}_{1-p_{n+1}}^{x_{n+1},b} \approx \widetilde{w}_{p_n}^{a,x_n} \tag{14}$$

where $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ is the right strong equivalent of $\widetilde{w}_{p_n}^{a,x_n}$. Since $\widetilde{w}_{p_n}^{a,x_n}$ has exactly one right strong equivalent, the function which is defined by the below equality:

$$(x_{n+1}, p_{n+1}) = \mathbf{F}(x_n, p_n)$$

is one-to-one. Clearly, $\{x_n\}_{n\in\mathbb{N}}$ is a strictly decreasing sequence. The difference between the expected utilities of two consecutive binary lotteries, $\widetilde{w}_{p_n}^{a,x_n}$ and $\widetilde{w}_{p_{n+1}}^{a,x_{n+1}}$, is equal to the second weight $(1 - p_{n+1})$ of the right strong equivalent $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ of $\widetilde{w}_{p_n}^{a,x_n}$, what formally reads:

$$\mathbf{E}\left[u(\widetilde{w}_{p_n}^{a,x_n})\right] - \mathbf{E}\left[u(\widetilde{w}_{p_{n+1}}^{a,x_{n+1}})\right] = 1 - p_{n+1}$$

[6] Recall that we have: $\widetilde{w}_p^{a,x} \overset{def}{=} [a, x\ ;\ 1-p, p]$ and $\widetilde{w}_{1-q}^{y,b} \overset{def}{=} [y, b\ ;\ q, 1-q]$.

consequently, the expected utility of the initial lottery –that is $\pi u(w)$– satisfies the following equality:

$$\pi u(w) = \mathbf{E}\left[u(\widetilde{w}_{p_n}^{a,x_n})\right] + \sum_{i=1}^{n}(1-p_i)$$

Alternatively, one may consider a sequence of binary lotteries, $\{\widetilde{w}_{q_n}^{y_n,b}\}_{n\in\mathbb{N}}$, that are defined as indicated below:

$$y_0 = w; q_0 = \pi \text{ and } \widetilde{w}_{q_{n+1}}^{a,y_{n+1}} \approx \widetilde{w}_{1-q_n}^{y_n,b} \tag{15}$$

and the elements of the sequence are endowed with the following property:

$$\mathbf{E}\left[u(\widetilde{w}_{q_{n+1}}^{a,y_{n+1}})\right] - \mathbf{E}\left[u(\widetilde{w}_{q_n}^{a,y_n})\right] = 1 - q_{n+1} \Longrightarrow \pi u(w) = \mathbf{E}\left[u(\widetilde{w}_{q_n}^{a,y_n})\right] - \sum_{i=1}^{n}(1-q_i)$$

From now on, the sequences $\{\widetilde{w}_{p_n}^{a,x_n}\}_{n\in\mathbb{N}}$ and $\{\widetilde{w}_{1-q_n}^{y_n,b}\}_{n\in\mathbb{N}}$, will be called the canonical sequences generated by (w,π). As shown below, they respectively converge, in LS-models, towards $\delta(a)$ or $\delta(b)$. The result holds whatever the value of π.

Proposition 6. *Let $\{\widetilde{w}_{p_n}^{a,x_n}\}_{n\in\mathbb{N}}$ and $\{\widetilde{w}_{q_n}^{y_n,b}\}_{n\in\mathbb{N}}$ be the canonical sequences of binary lotteries generated by $(w,\pi) \in]a,b[\times]0,1[$. Then, in LS-models where investors are disappointment averse, $\{x_n\}_{n\in\mathbb{N}}$ ($\{y_n\}_{n\in\mathbb{N}}$) is a decreasing (increasing) sequence of real numbers converging towards a (b). The sequence $\{1-p_n\}_{n\in\mathbb{N}}$ ($\{1-q_n\}_{n\in\mathbb{N}}$) is increasing and converges towards $\pi\ell$ ($1-\pi\ell$) where ℓ does not depend on π and is a strictly increasing function of w, mapping $[a,b]$ on to $[0,1]$.*

Proof. It is given in the Appendix.■

Finally, in LS-models, we have the following equalities:

$$\ell = \lim_{n\to\infty}(\sum_{i=1}^{n}(1-p_i))/\pi = \lim_{n\to\infty}(1-\sum_{i=0}^{n}(1-q_i))/\pi \tag{16}$$

and from now on we shall set $\ell = u(w)$. Preferences are endowed with the elicitation property, that is $u(w)$ can be known with as much accuracy as desired, for any outcome $w \in]a,b[$. Indeed, one may choose an arbitrary probability $\pi \in]0,1[$ and build, from the answers of an investor facing lotteries of the $\widetilde{w}_p^{a,x}$ type and/or of the $\widetilde{w}_{1-q}^{y,b}$ type, the two canonical sequences generated by (w,π). An accurate ranging of $u(w)$ may be obtained since we have:

$$\sum_{i=1}^{n}(1-p_i))/\pi \leq u(w) \leq (1-\sum_{i=0}^{n}(1-q_i))/\pi \tag{17}$$

Finally we shall define the elicitation property as indicated below:

Definition 4 (elicitation property). *The equivalence relation $\approx$ over $\mathbb{W}$ is endowed with the elicitation property iff the canonical sequences of binary lotteries generated by an arbitrary couple of real numbers $(w, \pi) \in \left]a, b\right[\times \left]0, 1\right[$ satisfy Proposition 6.*

3.2 The additional axiom

Finally, before setting Axiom A3 and after having set Axiom A2, we can set the following one:

Axiom AE (elicitation property). *The strong indifference relation is an equivalence relation over the set of random prospects $\mathbb{W}$ that is endowed with the elicitation property.*

We provisionally rule out the cases when the quotient set of $\mathbb{W}$ by $\approx$ is empty or equal to $\mathbb{W}$ itself. The latter case is clearly of no interest whereas the former one corresponds to that of EU theory since strong indifference then coincides with weak indifference.

Axiom AE states that the strong indifference relation is endowed with the elicitation property. Hence, for any level of wealth w, one can choose an arbitrary probability $\pi \in [0, 1]$ and build two canonical sequences $\{\widetilde{w}^{a,x_n}_{p_n}\}_{n\in\mathbb{N}}$ and $\{\widetilde{w}^{y_n,b}_{1-q_n}\}_{n\in\mathbb{N}}$, according to (14) and (15). Let $(w, \pi) \in [a, b] \times \left]0, 1\right[$, $P_n \overset{def}{=} \sum_{i=1}^{n} (1 - p_i))/\pi$ and $Q_n \overset{def}{=} (1 - \sum_{i=0}^{n} (1 - q_i))/\pi$. The canonical sequences of binary lotteries generated by (w, π) converge respectively towards $\delta(a)$ and $\delta(b)$ and the sequence $\{P_n\}_{n\in\mathbb{N}}$ ($\{Q_n\}_{n\in\mathbb{N}}$) converges towards a limit $u(w)$ that is an increasing function of w.

We now set: $\mathbf{z}(\widetilde{w}) = u^{-1}(\mathbf{E}[u(\widetilde{w})])$ and $\mathbf{z}(\widetilde{w})$ will be called the ZD-outcome of $\widetilde{w}$. A particular case occurs if, for any $\widetilde{w} \in \mathbb{W}$, $\mathbf{z}(\widetilde{w}) = \mathbf{c}(\widetilde{w})$. For ease of exposition, we provisionally rule out this particular case. Hence, as before, we set $\mathbb{W}^c_z = \mathbb{W}^c \cap \mathbb{W}_z$ where $\mathbb{W}_z$ ($\mathbb{W}^c$) consists in the prospects whose ZD-outcome (certainty equivalent) is z (c). Using the above results we may state the below proposition:

Proposition 7. *Under Axioms A1 to A5 and axiom AE, the preferences of a disappointment averse investor over $\mathbb{W}$ are represented by the lottery-dependent functional:*

$$\mathcal{U}(\widetilde{w}) = \int_a^b \boldsymbol{v}_{\mathbf{z}(\widetilde{w})}(x) dF_{\widetilde{w}}(x) \tag{18}$$

where $\widetilde{w} \in \mathbb{W}$ and where $\boldsymbol{v}_{\mathbf{z}(\widetilde{w})}(.)$ is defined according to Proposition 3 and the normalization conditions (12).

Proof. The proof is similar to that of Proposition 2.■

Some additional remarks must now be made. First, *note that LS-models satisfy Axioms A1 to A5 and Axiom AE.* Hence, setting these axioms does not lead to an empty set of models. In other words, one may be sure that the axiomatization makes sense. Next, note that the new axioms are, at least in principle, experimentally testable since their checking comes down to making choices between binary loteries. The number of experiments obviously depends on the desired accuracy of the elementary utility function $u(.)$. The third remark is that EU theory is a degenerate case of the above disappointment theory. It corresponds to the case that was provisionally ruled out at the beginning of the section. In EU theory, strong indifference and weak indifference coincide and the independence property is met over $\mathbb{W}$.

3.3 A review of some early disappointment models

We now go back to some early lottery-dependent utility models (henceforth LDU-models).

3.3.1 Back to LDU-models (Becker and Sarin 1987 and Schmidt 2001)

The preference functional of a LDU-model reads:[7]

$$\mathcal{U}_{LDU}\left(\widetilde{w}\right) = \sum_{k=1}^{K} p_k v\left(h(\widetilde{w}), w_k\right) \tag{19}$$

where $v\left[.,.\right]$ is a function defined over $[a, b] \times \mathbb{R}^{+}$ and whose values belong to $[0, 1]$ and where $h(\widetilde{w})$ is a function defined over $\mathbb{W}$ and whose values belong to $\mathbb{R}$.

The functional (19) may be derived from three axioms that have been provided by Becker and Sarin (1987): total ordering, continuity and monotonicity. Their first two axioms are those of EU theory and the third one is but the stochastic dominance principle. However, as pointed out by Starmer (2000) "the basic model is conventional theory for minimalists as, without further restriction, it has virtually no empirical content."[8] Finally, almost any non-EU model can be viewed as a LDU-model, once an appropriate functional form of $v\left(.,.\right)$ has been chosen. According to Becker and Sarin, a LDU-model may be particularized in the following way: one may assess $h\left(\widetilde{w}\right)$ to be linear with respect to the probabilities p_k, that one may set

[7] For ease of exposition, we focus on discrete prospects.

[8] Starmer: *Developments in Non Expected Utility Theory*, JEL, p. 345.

$h(\widetilde{w}) = \sum_{k=1}^{K} h_k p_k$ and define a function $H(.)$ such that $H(w_k) = h_k$. Actually, the authors set:

$$h(\widetilde{w}) = \sum_{k=1}^{K} H(w_k)\, p_k = \mathbf{E}\left[H\left[\widetilde{w}\right]\right] \tag{20}$$

and the new model then belongs to a subset of LDU-models called lottery-dependent expected utility models (henceforth LDEU-models). The functions $h(.)$ and/or $H(.)$, may be chosen arbitrarily but they have to be specified before testable implications of the model be derived. As a consequence, LDEU-models are not choice-based.

Schmidt (2001) considers somewhat more general models called "lottery-dependent convex utility models" (henceforth LDCU-models). A condition less restrictive than (20) is fulfilled by LDCU-models. It reads:

$$h(\widetilde{w}_i) = \lambda \text{ and } \alpha_i \geq 0 \text{ and } \sum_{i=1}^{N} \alpha_i = 1 \Rightarrow h\left(\sum_{i=1}^{N} \alpha_i \widetilde{w}_i\right) = \lambda \tag{21}$$

Four axioms are necessary to develop this class of models. The first two axioms (total ordering and continuity) are, again, those of EU theory. The author then substitutes for the independence axiom two new axioms: the first one, called the lottery dependent independence axiom, states that the independence property is met over any subset $\mathbb{W}_\lambda$ of prospects fulfilling (21). To derive LDCU-models, the author considers the functionals $\mathbf{U}(\widetilde{w})$ that satisfy the above axiom that are linear in every subset $\mathbb{W}_\lambda$ for all λ. A linear $\mathbf{U}(.)$ is obtained if and only if there exists a sequence of functions $\{\varphi_\lambda, \lambda \in [\mathbf{U}(\delta(a)), \mathbf{U}(\delta(b))]\}$ where $\varphi_\lambda : \mathbb{R} \to \mathbb{R}^+$ is continuous and strictly increasing so that:

$$\forall \lambda \in \left[\mathbf{U}(\delta(a)), \mathbf{U}(\delta(b))\right], \mathbf{U}(\widetilde{w}) = \varphi_\lambda\left[u_\lambda(\widetilde{w})\right] \;\; if \;\; \widetilde{w} \in \mathbb{W}_\lambda$$

This result is guaranteed by an additional axiom that enables the author to select one particular function $\mathbf{U}(.)$ from all the candidates. Here again, function $h(.)$ may be chosen arbitrarily and has to be specified before testable implications of the model be derived. As a consequence, LDCU-models are not choice-based.[9]

3.3.2 Other examples

LS-models constitute a subset of disappointment models. Indeed, if we set:

$$\boldsymbol{v}_{\mathbf{z}(\widetilde{w})}(x) = u(x) + \mathcal{E}\left(u(x) - u(\mathbf{z}(\widetilde{w}))\right) \quad ; \quad u(\mathbf{z}(\widetilde{w})) = \mathbf{E}\left[u(\widetilde{w})\right]$$

[9] Finally, as most of the models deriving from non-EU theories, our models clearly belong to the LDU class of models. Nevertheless, they are neither included in the subset of LDEU-models nor in that of LDCU-models. Symmetrically, they include neither all the models of the LDEU type nor all the models of the LDCU type.

we get:

$$\mathcal{U}\left(\widetilde{w}\right) \overset{def}{=} \int_a^b \boldsymbol{\upsilon}_{\mathbf{z}(\widetilde{w})}(x)dF_{\widetilde{w}}\left(x\right) = u\left(\mathbf{z}\left(\widetilde{w}\right)\right) + \int_a^b \mathcal{E}\left(u\left(x\right) - u(\mathbf{z}\left(\widetilde{w}\right))\right)dF_{\widetilde{w}}\left(x\right)$$

and the representing functional well coincides with (18). Hence, our axiomatization can be viewed as grounding the model of Loomes and Sugden. To make apparent the potential of these models for management applications, we now highlight a particular case of (11) that we call the *disappointment weighted utility model*. It is obtained by putting the following restrictions on function $\boldsymbol{\upsilon}_{\mathbf{z}(\widetilde{w})}(.)$:

$$\boldsymbol{\upsilon}_z(w) = u(w)\left[1 - A\left(u(w) - u\left(z\right)\right)\right]$$

where $u(.)$ is a standard utility function and A a positive parameter that controls for disappointment aversion. The preference functional then reads:

$$\mathcal{U}\left(\widetilde{w}\right) = \int_a^b u(x)\left[1 - A\left(u(x) - \mathbf{E}\left[u(\widetilde{w})\right]\right)\right]dF_{\widetilde{w}}(x) \tag{22}$$

The above equation can be rewritten to show that the investor's satisfaction $\mathcal{U}\left(\widetilde{w}\right)$ can be expressed as the sum of a standard von Neumann and Morgenstern (henceforth VNM) utility and of a penalty that is equal to the covariance between the utility of the lottery and the disappointment premium, that is:

$$\mathcal{U}\left(\widetilde{w}\right) = \mathbf{E}\left[u(\widetilde{w})\right] - COV\left[u(\widetilde{w}), A * \left(u(\widetilde{w}) - \mathbf{E}\left[u(\widetilde{w})\right]\right)\right] = \mathbf{E}\left[u(\widetilde{w})\right] - A * Var\left[u(\widetilde{w})\right] \tag{23}$$

Such a result is connected with Allais' conjecture. Indeed Allais (1979) argued that a positive theory of choice should contain two basics elements: (i) the existence of a cardinal utility function that is independent of risk attitudes and (ii) a valuation functional of risky lotteries that depends on the second moment of the probability distribution of uncertain utility. In EU theory, only the first moment is relevant to determine the attitude towards risk. By contrast, in this setting, risk attitudes are determined by the second moment of the probability distribution of utility as in Allais' theory.

Another example of disappointment model is the descriptive model provided by Jia *et alii* (2001). It generalizes Bell's (1985) approach by considering the expected value of the lottery as the reference point for measuring disappointment. Their representing functional can be expressed as:

$$\mathcal{U}\left(\widetilde{w}\right) = \int_a^b (1 + c\mathbf{1}_{[x<\mathbf{E}(\widetilde{w})]} - d\mathbf{1}_{[x>\mathbf{E}(\widetilde{w})]})xdF_{\widetilde{w}}(x) \tag{24}$$

where c and d are two positive parameters. The above functional is, here again, a particular case of (11). To see this point, just set:

$$u(x) = x \; ; \; \mathbf{z}(\widetilde{w}) = \mathbf{E}(\widetilde{w}) \; ; \; \boldsymbol{v}_{\mathbf{E}(\widetilde{w})}(x) = (1 + c\mathbf{1}_{[x<\mathbf{E}(\widetilde{w})]} - d\mathbf{1}_{[x>\mathbf{E}(\widetilde{w})]})x$$

3.3.3 Generalization

An alternative point of view may be selected. Recall that Gul (1991) argues that elation/disappointment should depend on the difference between the certainty equivalent of a random prospect –that is the price the investor is willing to pay– and the actual outcome. Following Gul, we now allow for the case when the utility of the ZD-outcome of the prospect is a convex combination of the elementary utility of the prospect and of that of its certainty equivalent. We get:

$$\mathbf{z}(\widetilde{w}) \stackrel{def}{=} u^{-1}\left(\zeta u(\widetilde{w}) + (1-\zeta)\, u(\mathbf{c}(\widetilde{w}))\right) \tag{25}$$

where $\zeta \in [0, 1]$, is an arbitrary scalar. Now what happens if $\zeta \to 0$? The consistency property is met over subsets whose certainty equivalent is the same. This is precisely the case that was ruled out at the beginning of subsection 3.2. It encompasses Gul's theory, since Gul's representing functional reads:

$$\mathcal{U}(\widetilde{w}) = u(\mathbf{c}(\widetilde{w})) = \int_a^b \boldsymbol{v}_{\mathbf{c}(\widetilde{w})}(x) dF_{\widetilde{w}}(x) = \int_a^b \frac{1 + \beta\mathbf{1}_{[x<\mathbf{c}(\widetilde{w})]}}{1 + \beta\mathbf{E}[\mathbf{1}_{[x<\mathbf{c}(\widetilde{w})]}]} u(x) dF_{\widetilde{w}}(x)$$

4 Concluding remarks

In this paper, a fully choice-based theory of disappointment has been developed that can be viewed as an axiomatic foundation of LS-models. The axiomatization has been grounded on works by psychologists, showing that anticipatory emotions, and in particular disappointment, play an important role in decision making. The axiomatic model is general enough to encompass as particular cases the models of Loomes and Sugden and that of Gul (1991). It allows for violations of betweenness. Moreover, the reference point for measuring disappointment, namely the ZD outcome, may differ from the certainty equivalent of the prospect under review.

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5 Appendix

5.1 Proofs

Proof of Property A.

From (2) we get that the risk premium of $\widetilde{w}$ is positive, or, equivalently, that the difference between the certainty equivalent of $\widetilde{w}$ and that of z is negative. Hence $\widetilde{w} \prec \delta(z)$.

The next step of the proof consists in showing that any random prospect $\widetilde{w} \in \mathbb{W}_z$ is preferred to $\widetilde{w}_z^{a,b}$. Let $\pi \stackrel{def}{=} u(z)$; by definition, we have:

$$\mathcal{U}(\widetilde{w}_{u(z)}^{a,b}) = \pi + \pi\mathcal{E}(1-\pi) + (1-\pi)\mathcal{E}(-\pi)$$

and, using a Taylor's expansion gives:

$$\mathcal{U}(\widetilde{w}_{u(z)}^{a,b}) = \pi + \pi(\mathcal{E}(-\pi) + \mathcal{E}'(\theta-\pi)) + (1-\pi)\mathcal{E}(-\pi) = \pi + \mathcal{E}(-\pi) + \pi\mathcal{E}'(\theta-\pi)$$

where $\theta \in]0,1[$. Now $\mathcal{U}(\widetilde{w})$ expresses as:

$$\begin{aligned}\mathcal{U}(\widetilde{w}) &= \pi + \textstyle\sum_{n=1}^{N} p_n \mathcal{E}(u(w_n)-\pi)\\ &= \pi + \textstyle\sum_{n=1}^{N} p_n (\mathcal{E}(-\pi) + u(w_n)\mathcal{E}'(\varsigma_n u(w_n)-\pi))\\ &= \pi + \mathcal{E}(-\pi) + \textstyle\sum_{n=1}^{N} p_n u(w_n)\mathcal{E}'(\varsigma_n u(w_n)-\pi)\end{aligned}$$

Moreover, since $1 > u(w_n)$ and that $\mathcal{E}(.)$ is concave, the inequality $\theta > \varsigma_n u(w_n)$ holds, and so does the next one: $\mathcal{E}'(\varsigma_n u(w_n)-\pi) > \mathcal{E}'(\theta-\pi)$. Finally, we get that $\mathcal{U}(\widetilde{w}_{u(z)}^{a,b}) < \mathcal{U}(\widetilde{w})$ or, equivalently, $\widetilde{w}_{u(z)}^{a,b} \prec \widetilde{w}$.$\blacksquare$

Proof of Property B.

Consider three random prospects $\widetilde{w}_1$, $\widetilde{w}_2$ and $\widetilde{w}_3$ exhibiting the same ZD-outcome z. We get, for $i = 1,2,3$:

$$\mathcal{U}(\widetilde{w}_i) = \mathbf{E}[\widetilde{w}_i] + \mathbf{E}[\mathcal{E}(\widetilde{w}_i - \mathbf{E}[\widetilde{w}_i])] = z + \mathbf{E}[\mathcal{E}(\widetilde{w}_i - z)]$$

that implies that:

$$\mathcal{U}(\widetilde{w}_1) - \mathcal{U}(\widetilde{w}_2) = \mathbf{E}[\mathcal{E}(\widetilde{w}_1 - z)] - \mathbf{E}[\mathcal{E}(\widetilde{w}_2 - z)]$$

hence if we set:

$$\widetilde{w}_i^* = \alpha\widetilde{w}_i \oplus (1-\alpha)\widetilde{w}_3$$

we get, for $i = 1, 2$:

$$\mathbf{E}\left[\widetilde{w}_i^*\right] = \alpha \mathbf{E}\left[\widetilde{w}_i\right] + (1-\alpha)\,\mathbf{E}\left[\widetilde{w}_3\right] = \alpha z + (1-\alpha)\,z = z$$

and, consequently:

$$\mathcal{U}\left(\widetilde{w}_i^*\right) = z + \mathbf{E}\left[\mathcal{E}\left(\widetilde{w}_i^* - z\right)\right] = u(z) + \alpha \mathbf{E}\left[\mathcal{E}\left(\widetilde{w}_i - z\right)\right] + (1-\alpha)\,\mathbf{E}\left[\mathcal{E}\left(\widetilde{w}_3 - z\right)\right]$$

Finally, the difference $\mathcal{U}\left(\widetilde{w}_1^*\right) - \mathcal{U}\left(\widetilde{w}_2^*\right)$ comes down to:

$$\mathcal{U}\left(\widetilde{w}_1^*\right) - \mathcal{U}\left(\widetilde{w}_2^*\right) = \alpha\left\{\mathbf{E}\left[\mathcal{E}\left(\widetilde{w}_1 - z\right)\right] - \mathbf{E}\left[\mathcal{E}\left(\widetilde{w}_2 - z\right)\right]\right\} = \alpha\left(\mathcal{U}\left(\widetilde{w}_1\right) - \mathcal{U}\left(\widetilde{w}_2\right)\right)$$

Hence the sign of $\mathcal{U}\left(\widetilde{w}_1^*\right) - \mathcal{U}\left(\widetilde{w}_2^*\right)$ is that of $\mathcal{U}\left(\widetilde{w}_1\right) - \mathcal{U}\left(\widetilde{w}_2\right)$.■

Proof of Proposition 4.

The first part of the proof consists in proving that if the two indifferent prospects $\widetilde{w}_1$ and $\widetilde{w}_2$ have the same ZD-outcome z and the same certainty equivalent c, then they are strongly indifferent. Let $\widetilde{w}_1$ and $\widetilde{w}_2$ exhibit the same ZD-outcome z and the same certainty equivalent c. We have, for $i = 1, 2$:

$$u(c) = u(z) + \textstyle\sum_{n=1}^{N} p_n^i \left(\mathcal{E}\left(u\left(w_n\right) - u(z)\right)\right) \tag{26}$$

and, consequently:

$$\textstyle\sum_{n=1}^{N} p_n^1 \mathcal{E}\left(u\left(w_n\right) - u(z)\right) - \sum_{n=1}^{N} p_n^2 \mathcal{E}\left(u\left(w_n\right) - u(z)\right) = 0 \tag{27}$$

where $\widetilde{w}_i = \left[w_1, ..., w_N\ ;\ p_1^i, ..., p_N^i\right]$ $(i = 1, 2)$. Consider the compound lottery $\alpha\widetilde{w}_1 \oplus (1-\alpha)\,\widetilde{w}_2 = \widetilde{w}_\alpha$. Its ZD-outcome is also equal to z since its expected elementary utility is $u(z) = \alpha u(z) + (1-\alpha)\,u(z)$. Hence, we get:

$$u\left(\mathbf{c}\left(\widetilde{w}_\alpha\right)\right) = u(z) + \textstyle\sum_{n=1}^{N} \left(\alpha p_n^1 + (1-\alpha)\,p_n^2\right) \mathcal{E}\left(u\left(w_n\right) - u(z)\right)$$

where $\mathbf{c}\left(\widetilde{w}_\alpha\right)$ is the certainty equivalent of $\widetilde{w}_\alpha$. Finally we get:

$$u\left(\mathbf{c}\left(\widetilde{w}_\alpha\right)\right) - u(c) = \alpha\left(\textstyle\sum_{n=1}^{N} p_n^1 \mathcal{E}\left(u\left(w_n\right) - u(z)\right) - \sum_{n=1}^{N} p_n^2 \mathcal{E}\left(u\left(w_n\right) - u(z)\right)\right) = 0$$

The proof of the converse is now given. Consider two discrete prospects:

$$\widetilde{w}_i = \left[w_1, ..., w_N\ ;\ p_1^i, ..., p_N^i\right] \quad i = 1, 2$$

and their probability mixture:

$$\alpha\widetilde{w}_1 \oplus (1-\alpha)\,\widetilde{w}_2 = \left[w_1, ..., w_N\ ;\ \alpha p_1^1 + (1-\alpha)\,p_1^2, ..., \alpha p_N^1 + (1-\alpha)\,p_N^2\right]$$

where $\alpha \in [0,1]$. We must show that if $\widetilde{w}_1$ and $\widetilde{w}_2$ are strongly indifferent –that is if they have the same certainty equivalent and if they exhibit the betweenness property–, then they share the same ZD-outcome. We have, for $i = 1, 2$:

$$u(c) = u(\mathbf{c}(\widetilde{w}_i)) = u(z_i) + \sum_{n=1}^{N} p_n^i \mathcal{E}\left(u_n^i\right)$$

where:

$$u(z_i) = \sum_{n=1}^{N} p_n^i u(w_n) \text{ and } u_n^i = u(w_n) - u(z_i) \tag{28}$$

By definition, we have:

$$u(\mathbf{c}(\alpha\widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2)) = u(\alpha z_1 + (1-\alpha) z_2) + \sum_{n=1}^{N} \left[\alpha p_n^1 + (1-\alpha) p_n^2\right] \mathcal{E}\left(\alpha u_n^1 + (1-\alpha) u_n^2\right)$$

and, since $u(\alpha z_1 + (1-\alpha) z_2) = \alpha u(z_1) + (1-\alpha) u(z_2)$ (see (28)), we get:

$$u(\mathbf{c}(\alpha\widetilde{w}_1 \oplus (1-\alpha)\widetilde{w}_2)) = \alpha u(z_1) + (1-\alpha) u(z_2) + \sum_{n=1}^{N} \left[\alpha p_n^1 + (1-\alpha) p_n^2\right] \mathcal{E}\left(\alpha u_n^1 + (1-\alpha) u_n^2\right)$$

Now, from (26), we get:

$$\alpha u(\mathbf{c}(\widetilde{w}_1)) + (1-\alpha) u(\mathbf{c}(\widetilde{w}_2)) = \alpha u(z_1) + (1-\alpha) u(z_2) + \sum_{n=1}^{N} \alpha p_n^1 \mathcal{E}\left(u_n^1\right) + \sum_{n=1}^{N} (1-\alpha) p_n^2 \mathcal{E}\left(u_n^2\right)$$

Substracting the above equation from the previous one and using the betweenness property yields:

$$\sum_{n=1}^{N} p_n^1 \alpha \mathcal{E}\left(u_n^1\right) + \sum_{n=1}^{N} p_n^2 (1-\alpha) \mathcal{E}\left(u_n^2\right) = \alpha \left(\sum_{n=1}^{N} p_n^1 \mathcal{E}\left(\alpha u_n^1 + (1-\alpha) u_n^2\right)\right) + (1-\alpha) \left(\sum_{n=1}^{N} p_n^2 \mathcal{E}\left(\alpha u_n^1 + (1-\alpha) u_n^2\right)\right)$$

or:

$$\alpha \sum_{n=1}^{N} p_n^1 \left[\mathcal{E}\left(u_n^1\right) - \mathcal{E}\left(\alpha u_n^1 + (1-\alpha) u_n^2\right)\right] + (1-\alpha) \sum_{n=1}^{N} p_n^2 \left[\mathcal{E}\left(u_n^2\right) - \mathcal{E}\left(\alpha u_n^1 + (1-\alpha) u_n^2\right)\right] = 0$$

or, using a Taylor's expansion of $\mathcal{E}(.)$ around $\alpha u_n^1 + (1-\alpha) u_n^2$:

$$\begin{aligned} 0 &= \alpha \sum_{n=1}^{N} p_n^1 (1-\alpha) \left(u_n^1 - u_n^2\right) \mathcal{E}'\left(\alpha u_n^1 + (1-\alpha) u_n^2 + \theta_n^\alpha \alpha \left(u_n^1 - u_n^2\right)\right) \\ &\quad + (1-\alpha) \sum_{n=1}^{N} p_n^2 \alpha \left(u_n^1 - u_n^2\right) \mathcal{E}'\left(\alpha u_n^1 + (1-\alpha) u_n^2 + \zeta_n^\alpha \alpha \left(u_n^2 - u_n^1\right)\right) \end{aligned}$$

where $\theta_n^\alpha \in]0,1[$ and $\zeta_n^\alpha \in]0,1[$. Since $u_n^1 - u_n^2 = u(z_2) - u(z_1)$, the above equality is equivalent to the following one:

$$(u(z_2) - u(z_1)) \left[\sum_{n=1}^{N} p_n^1 \mathcal{E}'\left(u(w_n) - \overline{u}_n^\alpha\right) - \sum_{n=1}^{N} p_n^2 \mathcal{E}'\left(u(w_n) - \overline{u}_2^\alpha\right)\right] = 0 \tag{29}$$

where $\overline{u}_n^\alpha = \alpha\left(1+\theta_n^\alpha\right)u\left(z_1\right)+\left(\left(1-\alpha\right)-\alpha\theta_n^\alpha\right)u\left(z_2\right)$ and $\overline{u}_2^\alpha = \left(1-\alpha+\alpha\zeta_n^\alpha\right)u\left(z_2\right)+\alpha\left(1-\zeta_n^\alpha\right)u\left(z_1\right)$. Since the quantity between brackets is not identically zero, (29) implies $z_1 = z_2$.■

Proof of Proposition 5.

We now prove that if $\widetilde{w} \in \mathbb{W}_z$, there exists exactly one binary lottery $\widetilde{w}_p^{a,x}$ that is strongly indifferent to $\widetilde{w}$. Recall that, by definition, we have:

$$\mathcal{U}\left(\widetilde{w}_p^{a,x}\right) = pu\left(x\right)+p\mathcal{E}\left(u\left(x\right)\left(1-p\right)\right)+\left(1-p\right)\mathcal{E}\left(-pu\left(x\right)\right)$$

and, once $u\left(x\right)$ has been substituted for $u\left(z\right)/p$, we get:

$$\mathcal{U}\left(\widetilde{w}_p^{a,x}\right) = \mathrm{z}+p\mathcal{E}\left(\frac{\mathrm{z}}{p}-\mathrm{z}\right)+\left(1-p\right)\mathcal{E}\left(-\mathrm{z}\right)$$

where $\mathrm{z}= u\left(z\right)$. We get:

$$\frac{d\mathcal{U}\left(\widetilde{w}_p^{a,x}\right)}{dp} = \mathcal{E}\left(\frac{\mathrm{z}}{p}-\mathrm{z}\right)-\mathcal{E}\left(-\mathrm{z}\right)+p\mathcal{E}'\left(\frac{\mathrm{z}}{p}-\mathrm{z}\right)\left(-\frac{\mathrm{z}}{p^2}\right) = \frac{\mathrm{z}}{p}\left[\mathcal{E}'\left(\theta\frac{\mathrm{z}}{p}-\mathrm{z}\right)-\mathcal{E}'\left(\frac{\mathrm{z}}{p}-\mathrm{z}\right)\right]$$

where $\theta \in \left]0,1\right[$. Since $\mathcal{E}\left(.\right)$ is strictly concave, then $\mathcal{E}'\left(.\right)$ is strictly decreasing and the quantity between brackets in the above equation is strictly positive. Hence $\mathcal{U}\left(\widetilde{w}_p^{a,x}\right)$ can be viewed, when the expected utility of $\widetilde{w}_p^{a,x}$ is held constant, as a strictly increasing function of p mapping $[0,1]$, on to $[\mathcal{U}(\widetilde{w}_{u(z)}^{a,b}),\mathcal{U}\left(\widetilde{w}_1^{a,z}\right)]$ where $\mathcal{U}\left(\widetilde{w}_1^{a,z}\right) = u\left(z\right)$ and $\mathcal{U}(\widetilde{w}_{u(z)}^{a,b}) = u\left(z\right)+u\left(z\right)\mathcal{E}\left(1-u\left(z\right)\right)+\left(1-u\left(z\right)\right)\mathcal{E}\left(-u\left(z\right)\right)$. Hence, since $\widetilde{w}_{u(z)}^{a,b} \prec \widetilde{w} \prec \delta\left(z\right)$, there exists exactly one binary lottery $\widetilde{w}_p^{a,x}$ that is strongly indifferent to $\widetilde{w}$.

Proof of Proposition 6.

If x_{n+1} were greater than x_n, $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ would exhibit first-order stochastic dominance over $\widetilde{w}_{p_n}^{a,x_n}$. Hence, x_{n+1} is lower than x_n and $\left\{x_n\right\}_{n\in\mathbb{N}}$ is a decreasing sequence. It is also bounded below by a. Consequently, it converges towards a limit $\ell \geq a$. Next, note that the two strongly indifferent lotteries $\widetilde{w}_{p_n}^{a,x_n}$ and $\widetilde{w}_{1-p_{n+1}}^{x_{n+1},b}$ have the same expected utility, that is, we have:

$$p_n u\left(x_n\right) = p_{n+1}u\left(x_{n+1}\right)+\left(1-p_{n+1}\right) \quad for \quad n = 0,1,... \tag{30}$$

and summing the members of the above equalities yields:

$$\pi u\left(w\right) = p_n u\left(x_n\right)+\textstyle\sum_{i=1}^{n}\left(1-p_i\right) \quad for \quad n = 1,2,...$$

The above equality implies $S_n \overset{def}{=} \sum_{i=1}^{n}(1-p_i) \leq \pi u(w)$. Since$\{S_n\}_{n\in\mathbb{N}^*}$ is an increasing sequence, it converges towards a limit $\Sigma \leq \pi u(w)$. As a consequence, $S_n - S_{n-1} = (1-p_{n.}) \to 0$, that is $p_{n.} \to 1$. Moreover, since we have: $\widetilde{w}_{p_{n+1}}^{a,x_{n+1}} \prec \widetilde{w}_{1-p_{n+1}}^{x_{n+1},b} \sim \widetilde{w}_{p_n}^{a,x_n}$, the sequence of binary lotteries $\left\{\widetilde{w}_{p_n}^{a,x_n}\right\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\widetilde{w}_1^{a,l} = \delta(l)$. Similarly, $\{\widetilde{w}_{1-p_n}^{x_n,b}\}_{n\in\mathbb{N}^*}$ converges towards $\widetilde{w}_0^{l,b} = \delta(l)$.

We now show that $\ell = a$. The proof is by contradiction. To see this, assume $\ell > a$. Then, since $\widetilde{w}_{p_n}^{a,x_n} \succ \delta(l)$, there exists a binary lottery $\widetilde{w}_{p_n}^{a,x^*_n}$ such that $l < x^*_n < x_n$, and $\widetilde{w}_{p_n}^{a,x^*_n} \sim \delta(l)$. Let x^*_{n+1} and p^*_{n+1} be defined by $\widetilde{w}_{1-p^*_{n+1}}^{x^*_{n+1},b} \approx \widetilde{w}_{p_n}^{a,x^*_n}$. Since $\{\widetilde{w}_{1-p_n}^{x_n,b}\}_{n\in\mathbb{N}^*}$ converges towards $\delta(l)$, there exists an integer N, such that $m \geq N \Rightarrow l \leq x_m < x^*_{n+1}$ and $p_m \geq p^*_{n+1}$. This implies that $\widetilde{w}_{1-p^*_{n+1}}^{x^*_{n+1},b}$ is preferred to the $\widetilde{w}_{1-p_m}^{x_m,b}$s and, consequently, that $\delta(l)$ is preferred to the $\widetilde{w}_{1-p_m}^{x_m,b}$s, that contradicts the fact that $\{\widetilde{w}_{1-p_n}^{x_n,b}\}_{n\in\mathbb{N}}$ is decreasing and converges towards $\delta(l)$. Hence $\ell = a$ and $\{S_n\}_{n\in\mathbb{N}}$ converges towards $\Sigma = \pi u(w)$. As a consequence, equality (16) is checked.■

6 Figures

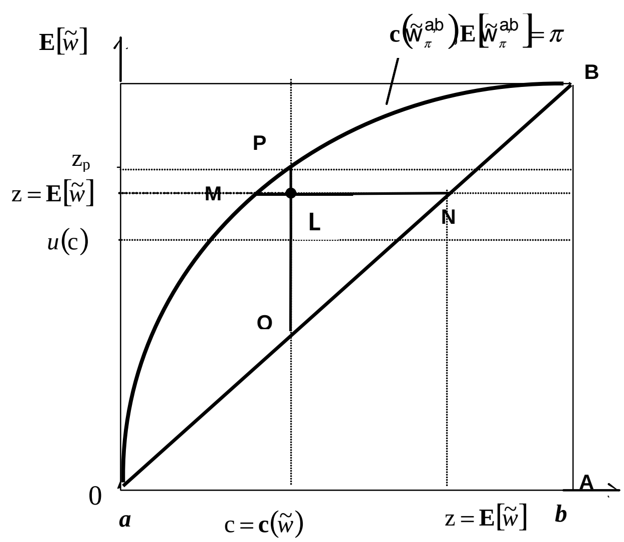

Figure 1