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Equilibrium Data Sets and Compatible Utility Rankings

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Abstract

Sets consisting of finite collections of prices and endowments such that total resources are constant, or collinear, or approximately collinear, can always be viewed as subsets of some equilibrium manifold. The additional requirement that such collections of price-endowment data are compatible with some individual preference rankings is reduced to the existence of solutions to some set of linear inequalities and equalities. This characterization enables us to give simple proofs of the contractibility of the set whose elements are finite equilibrium data collections compatible with given individual preference rankings and the path-connectedness of the set made of finite equilibrium data set.

1. Introduction

A recent development of the theory of general equilibrium deals with the falsifiability issue which, for Brown and Matzkin, is equivalent to the existence of a finite number of endowment vectors and price systems that cannot all belong to some equilibrium manifold associated with preferences satisfying standard assumptions [5]. These authors prove the existence of such sets by way of examples. But, the mere possibility of falsifying a theory as advocated by Popper says little about how easily this result can be achieved. The fact that the set made of the finite collections of data that do not belong to any equilibrium manifold is not empty gives no information about its size assessed by, for example, its Lebesgue measure in bounded subsets. This size may in fact be so small that it would be practically impossible to design tests that could falsify the theory.

We start this paper by showing that finite data sets such that total resources are constant, or collinear, or even only approximately collinear, are always included in some equilibrium manifold. This property is comparable with a recent result by Snyder who shows that, if the number of consumers is larger than or equal to the number of goods, the theory of general equilibrium imposes no restrictions on finite data sets consisting of total resources and equilibrium prices—equilibrium prices being associated here with some individual endowments that are compatible with the total resources [7]. Our result conveys the intuition that finite collections of data that are not included in any equilibrium manifold make up a set that is certainly not large—even if the idea of relative smallness is not rigorously defined and developed in the current paper—since the set does not contain approximately collinear total resources.

Therefore, given a finite data set made of endowment vectors and their associated price system, there is a good chance that there exists an equilibrium manifold that contains these data. In such a

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case, there exists at least one utility function per consumer such that the associated equilibrium manifold contains all the elements of the data set. The indirect utility functions associated with these direct utilities determine the consumers' rankings of the price-income pairs associated with the price-endowment data set, the income being simply the value of the endowments for the associated price system. Therefore, every consumer ranks the price-income data associated with a data set belonging to some equilibrium manifold.

But the equilibrium manifold that contains the data set is, when it exists, rarely unique. Therefore, for each consumer, different rankings may be associated with a given equilibrium data set. Knowing these rankings is often an important issue in applied welfare economics. Therefore, the information that some rankings are incompatible with the equilibrium condition is particularly important. This leads us to strengthen the philosophically oriented problem considered by Brown and Matzkin of determining whether a set of data belongs to some equilibrium manifold into the more restrictive, policy oriented question of whether a given price-endowment data set is compatible with some given individual rankings of the associated price-income data. Note that a set of price-endowment data that would be incompatible with all possible rankings cannot be contained in any equilibrium manifold.

We show in this paper that the problem of determining whether a set of price-endowment data—actually, we express these data as price-income distribution-total resource data—is compatible with given individual rankings of the associated price-income data can be reduced to the existence of solutions to some set of linear inequalities and equalities. A practical consequence of this linearity property is the relatively modest computational complexity of the problem. But linearity has other consequences as well. For example, at the purely theoretical level, it enables us to give simple proofs of global topological properties like the contractibility of data sets compatible with given individual preference rankings and the path-connectedness of finite equilibrium data sets.

We have attempted to make this paper as much self-contained as possible. Several results of this paper depend on properties that come under the general heading of the theory of revealed preferences. These properties are gathered in the Appendix. We either provide proofs or make appropriate references to the literature.

The paper is organized as follows. In Section 2, we recall the main assumptions and definitions, and set the notation. Section 3 is devoted to the property that there are no restrictions on datasets if total resources are constant, or collinear, or almost collinear. Section 4 addresses the equivalence between the compatibility of data sets with specified preference orderings and the existence of solutions to some set of linear equalities and inequalities. Section 5 deals with the contractibility of data sets compatible with some specified preference orderings and with the resulting path-connectedness of equilibrium data sets. An Appendix contains the most technical aspects of some proofs as well as the properties of revealed preference theory used in the paper.

2. Definitions, assumptions and notation

Goods and prices

There is a finite number ℓ of goods. Let $p = (p_1, p_2, \ldots, p_{\ell-1}, p_\ell) \in \mathbb{R}^\ell_{++}$ be the price vector. We normalize the price vector p by picking the ℓ-th commodity as the numeraire, which is equivalent to setting $p_\ell = 1$. Let S denote the set of strictly positive normalized price vectors.

Consumers and their utility functions

There is a finite number $m \geq 2$ of consumers. A consumer is characterized by the consumption set $X = \mathbb{R}^\ell_{++}$, an endowment vector $\omega_i \in X$ and a utility function $u_i : X \to \mathbb{R}$.

We assume that consumer i's utility function u_i belongs to the class $\mathcal{U}$ of smooth maps from X into $\mathbb{R}$ that satisfy the following properties whose mathematical and economic interpretation are standard: 1) $Du_i(x_i) \in X$ for any $x_i \in X$ (smooth monotonicity); 2) the condition $y^T D^2 u_i(x_i)\, y \geq 0$ and $y^T Du_i(x_i) = 0$ with $x_i \in X$ has the only solution $y = 0 \in \mathbb{R}^\ell$ (smooth strict quasi-concavity); 3) $u_i^{-1}(a)$ is closed in $\mathbb{R}^\ell$ for any $a \in \mathbb{R}$ (necessity of strictly positive consumption of every commodity).

We denote by $\Omega = X^m$ the set of endowments of all consumers.

Individual demand functions

Given the utility function $u_i \in \mathcal{U}$, the demand of consumer i for the price vector $p \in S$ and the income $w_i > 0$ is $f_i(p, w_i) = \arg\max u_i(x_i)$ subject to the budget constraint $p \cdot x_i \leq w_i$. The demand function $f_i : S \times \mathbb{R}_{++} \to \mathbb{R}^\ell_{++}$ is not only smooth but satisfies several other properties, one of them being Walras law (namely the identity $p \cdot f_i(p, w_i) = w_i$), another one the symmetry and negative definiteness of the Slutsky matrix. For details, see e.g. [4]. Note that we do not need more than Walras law in this paper.

The price-endowment equilibrium pairs

The price vector $p \in S$ is an equilibrium price vector for the vector of individual endowments $\omega = (\omega_1, \omega_2, \dots, \omega_m) \in \Omega$ and utility profile $\boldsymbol{u} \in \mathcal{U}^m$ if there is equality of aggregate supply and demand for that price vector:

$$\sum_i f_i(p, p \cdot \omega_i) = \sum_i \omega_i. \tag{1}$$

The pair $(p, \omega) \in S \times \Omega$ is then said to be a price-endowment equilibrium or also an equilibrium pair (for the utility profile $\boldsymbol{u} \in \mathcal{U}^m$).

The set $E(\boldsymbol{u})$ consists of the price-endowment equilibria $(p, \omega) \in S \times \Omega$ associated with the utility profile $\boldsymbol{u} = (u_1, u_2, \dots, u_m) \in \mathcal{U}^m$. The set $E(\boldsymbol{u})$ is then a dimension $m\ell$ smooth submanifold of $S \times \Omega$ whose global structure (pathconnectedness, simple connectedness, contractibility, and diffeomorphism with a Euclidean space for example) is investigated in [1], [3] and [2]. It follows from these global properties that not all dimension $m\ell$ smooth manifolds can be identified with equilibrium manifolds of exchange economies.

Equilibrium triples or the price–income distribution–total resource equilibria

Since we want to highlight the role of total resources and income distributions in the properties of equilibrium data sets, we introduce a different parametrization of the set of equilibria by way of prices, income distributions and total resources.

Let $\boldsymbol{w} = (w_1, w_2, \dots, w_m) \in \mathbb{R}^m_{++}$ denote the income distribution between the m consumers making up the economy.

We define a **feasible triple** or a price-income distribution-total resource equilibrium as a triple $(p, \boldsymbol{w}, r) \in S \times \mathbb{R}^m_{++} \times X$ such that the equality

$$w_1 + w_2 + \cdots + w_m = p \cdot r \tag{2}$$

is satisfied. We denote by $\mathcal{B}$ the subset of $S \times \mathbb{R}^m_{++} \times X$ consisting of feasible triples.

We define an **equilibrium triple** for the utility profile $\boldsymbol{u} \in \mathcal{U}^m$ as the triple $(p, \boldsymbol{w}, r) \in S \times \mathbb{R}^m_{++} \times X$ that satisfies the equality

$$\sum_i f_i(p, w_i) = r. \tag{3}$$

Equilibrium triples extend to the case of variable total resources $r \in X$ the price-income equilibria considered in [4]. It follows from Walras law that an equilibrium triple is feasible. We denote by $\mathcal{E}(\boldsymbol{u})^{[T]}$ the set of T two by two distinct equilibrium triples for the utility profile $\boldsymbol{u} \in \mathcal{U}^m$.

Relationship between (price,endowment) equilibria and equilibrium triples

It is obvious that if the pair $(p, \omega) \in S \times \Omega$ where $\omega = (\omega_i)$ is a price-endowment equilibrium in the sense that equation (1) is satisfied, then the triple $b = \big(p, \boldsymbol{w}, r\big)$ where $w_i = p \cdot \omega_i$ for $1 \leq i \leq m$, and $r = \sum_i \omega_i$ is an equilibrium triple. Therefore, this defines a smooth map from the set of price-endowment equilibria into the set of equilibrium triples.

Conversely, given the equilibrium triple $b = (p, \boldsymbol{w}, r)$ for the utility profile $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$, then any pair (p, ω) where $\omega = (\omega_i)$ satisfies the equalities $\sum_{i=1}^m \omega_i = r$ and $p \cdot \omega_i = w_i$ for $i = 1, 2, \dots, m$, is a price-endowment equilibrium for the same utility profile.

The concept of equilibrium triple presents the advantage over the price-endowment equilibrium of making more apparent the associated income distribution and total resources. From now on, we consider only equilibrium triples instead of price-endowment equilibria unless the contrary is explicitely specified.

Configurations of equilibrium data

A *configuration* of T triples is a T-tuple $G = (b^t)_{1 \leq t \leq T}$ where $b^t = (p^t, \boldsymbol{w}^t, r^t) \in S \times \mathbb{R}^m_{++} \times X$. We denote by $\mathcal{B}^{[T]}$ (resp. $\mathcal{E}(\boldsymbol{u})^{[T]}$) the set of configurations G whose components are feasible triples (resp. equilibrium triples for the utility profile $\boldsymbol{u} = (u_1, \dots, u_m) \in \mathcal{U}^m$) that are two by two distinct.

The utility profile $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$ is said to *rationalize* the configuration G—the configuration G is then said to be rationalizable—if the configuration G belongs to $\mathcal{E}(\boldsymbol{u})^{[T]}$.

Rationalizable configurations and associated rankings

Let $G = \big((p^t, \boldsymbol{w}^t, r^t)\big)_{1 \leq t \leq T} \in \mathcal{E}(\boldsymbol{u})^{[T]}$ be a configuration rationalized by the utility profile $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$. Let $x_i^t = f_i(p^t, w_i^t)$ where $f_i : S \times \mathbb{R}_{++} \to X$ is the demand function associated with the utility function $u_i \in \mathcal{U}$. The utility levels $u_i(x_i^t)$ with t varying from 1 to T are not necessarily distinct. Let us sort, for example, these utility levels in ascending order. This enables us to define a preorder $\preceq_i$ of the set $\mathbb{T} = \{1, 2, \dots, T\}$ for consumer i with $1 \leq i \leq m$ by the condition that $t \preceq_i t'$ is equivalent to $u_i(x_i^t) \leq u_i(x_i^{t'})$.

The preorder $\preceq_i$ of the set $\mathbb{T}$ (or of any arbitrary set) is a binary relation that is complete (i.e., $t \preceq_i t'$ or $t' \preceq_i t$ for any t and t' in $\mathbb{T}$), reflexive ($t \preceq_i t$ for any $t \in \mathbb{T}$), transitive ($t \preceq_i t'$ and $t' \preceq_i t''$ imply $t \preceq_i t''$ for t, t' and $t'' \in \mathbb{T}$). An order is a preorder that is antisymmetric (i.e., $t \preceq_i t'$ and $t' \preceq_i t$ imply $t = t'$). We say that the strict order $\prec_i$ is a *refinement* of the preorder $\preceq_i$ if $t \prec_i t'$ implies $t \preceq_i t'$. Note that, for any preorder, there exists at least one order that is a refinement of the preorder.

We define the *ranking profile* of the m consumers for the configuration G by the m-tuple $\preceq = (\preceq_i)$ with $i = 1, 2, \dots, m$. In case the ranking profiles are all strict, we write $\prec = (\prec_i)$.

One of our goals in this paper is to characterize the configurations G that can be rationalized by utility functions $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$ that are compatible with given (pre)order profiles.

3. Variability of total resources and falsifiability

The main result of this section is the following

Theorem 1. *Let $G = (b^t)_{1 \leq t \leq T} \in \mathcal{B}^{[T]}$ be a configuration such that the total resources r^t are collinear for all t. Then, the configuration G is rationalizable by some utility profile $\boldsymbol{u} \in \mathcal{U}^m$.*

Proof. See Section B of the appendix. □

Proposition 2. *The set of configurations in $\mathcal{B}^{[T]}$ that are rationalizable contains in its interior the configurations with collinear total resources.*

Proof. Let $G = (p^t, \boldsymbol{w}^t, r^t)$ be a configuration such that the total resources r^t are collinear for all t. It follow from Theorem 1 that there exists some utility profile $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$ that rationalizes the configuration G. Let $x_i^t = f_i(p^t, w_i^)$ where f_i is the demand function associated with the utility function u_i.

Let us now show that if the configuration $G' = (p'^t, \boldsymbol{w}'^t, r'^t)$ is close enough to G, then G' is also rationalizable. Define for $i \geq 2$ the element $x_i'^t = f_i(p'^t, w_i'^t)$. By construction, the T price-consumption pairs $(p'^t, x_i'^t)$ for every $i \geq 2$ are rationalized by the utility function u_i. Define

$$x_1'^t = r' - \sum_{i \geq 2} x_i'^t.$$

It follows from the continuity of the demand functions f_i that the $x_1'^t$ can be made arbitrarily close to x_1^t by choosing G' close enough to G. It then suffices to apply Proposition 1 of the appendix to G' close enough to G to conclude that the T-tuple $(p'^t, x_1'^t)$ also satisfies SARP and, therefore, can be rationalized by some utility function u_1'.

□

It then follows from Proposition 2 that any configuration where the total resources depart little from being collinear is also rationalizable. In other words, a data set that would not be rationalizable requires that the directions defined by the vectors of total resources vary significantly. Such data cannot be generated by an economy that is undergoing proportional or quasi-proportional growth.

Remark 1. The proof of Proposition 2 actually proves that the set of rationalizable configurations is open.

4. Compatibility of equilibrium configurations and individual rankings

Let $G = (b^t) \in \mathcal{E}^{[T]}$ with $b^t = (p^t, \boldsymbol{w}^t, r^t)$ be a configuration made of T equilibrium triples. The main result of this section is the following characterization by a set of linear equalities and inequalities of the compatibility of the configuration $G \in \mathcal{E}^{[T]}(\boldsymbol{u})$ (where the utility profile $\boldsymbol{u}$ varies in $\mathcal{U}^m$) with the **strict ranking profile** $\prec = (\prec_i)$.

A system of linear equalities and inequalities

Theorem 3. *There exists a utility profile $\boldsymbol{u} = (u_1, u_2, \ldots, u_m) \in \mathcal{U}^m$ compatible with the strict ranking $\prec = (\prec_i)$ of the configuration G if and only if the set of solutions $\{(x_i^t)\}$, with $1 \leq i \leq m$ and $1 \leq t \leq T$, of the following system $LP(G, \prec)$ of linear equalities and inequalities is nonempty:*

$$LP(G,\prec): \quad \begin{cases} p^{t'} \cdot x_i^t > w_i^{t'} & \text{whenever } t' \prec_i t; \quad \text{(L1)} \\ p^t \cdot x_i^t = w_i^t; & \text{(L2)} \\ \sum_i x_i^t = r^t. & \text{(L3)} \end{cases}$$

Proof. The condition is necessary. Let $x_i^t = f_i(p^t, w_i^t)$. Equality *(L3)* follows from the equilibrium condition. Equality *(L2)* follows from Walras law. Inequality *(L1)* then follows from inequality $u_i(x_i^{t'}) < u_i(x_i^t)$.

The condition is sufficient. Pick some arbitrary consumer i. We use equality *(L2)* to substitute $p^t \cdot x_i^t$ to w_i^t in inequality *(L1)*. The collection of inequalities *(L1)* implies that the data (p^t, x_i^t) are well-ranked for the ordered index set $(\mathbb{T}, \prec_i) = \{t_{i_1} \prec_i t_{i_2} \prec_i \cdots \prec_i t_{i_T}\}$ in the sense of Definition A.3 of the Appendix. It then suffices to apply Proposition A.10 of the Appendix to rationalize these data by some utility function $u_i \in \mathcal{U}$ with the property that the strict inequality $u_i(x_i^{t'}) < u_i(x_i^t)$ is equivalent to $t' \prec_i t$. It then suffices to do this for every consumer. Equality *(L3)* implies that the equilibrium condition is satisfied for every t varying from 1 to T.

□

Extension to nonstrict rankings

Theorem 3 can be applied to assess the compatibility of configuration G with the ranking profile $\preceq = (\preceq_i)$ that is not necessarily strict by considering strict ranking profiles $\prec = (\prec_i)$ that are refinements of the ranking profile $\preceq = (\preceq_i)$ as follows from:

Lemma 4. *Let $\preceq = (\preceq_i)$ be some not necessarily strict ranking profile of the configuration G associated with the utility profile $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$. Then, for any strict ranking refinement $\prec = (\prec_i)$ of $\preceq = (\preceq_i)$, there exists a utility profile $\boldsymbol{u}' = (u_i') \in \mathcal{U}^m$ such that the configuration G belongs to $\mathcal{E}^{[T]}(\boldsymbol{u}')$ and the induced ranking profile coincides with $\prec = (\prec_i)$.*

Proof. This is essentially Proposition A.11 of the Appendix. □

Given the ranking profile $\preceq = (\preceq_i)$, it then suffices to apply Theorem 3 to any refinement $\prec = (\prec_i)$ to check the compatibility of the configuration G with the ranking profile $\preceq = (\preceq_i)$.

Remarks on the size of these linear problems

The linear system whose solution set is $LP(G, \prec)$ has $m\,\ell\,T$ real unknowns and $m\,\ell\,T + \ell\,T + mT(T+1)/2$ constraints (including the sign constraints). For a given economy, m and ℓ are constant and the only variable parameter is the number T of equilibrium data. The number of unknowns is linear and the number of constraints quadratic in T. This situation is similar to the one observed by Varian [9] for Afriat's inequalities in the case of one consumer). But, at variance with Afriat's set of inequalities whose solution is practically impossible for T large because of the size of the problem, the set of inequalities and equalities in Theorem (3) decomposes into T smaller linear subproblems which makes finding solutions far more tractable.

The number of unknowns of each subproblem is then equal to $\ell\, m$ while the number of constraints varies from $m\ell + \ell + m$ to $m\,\ell + \ell + mT$ depending on the value of t and of the ranking profile $\prec$. The average value of the number of constraints is therefore equal to $m\ell + \ell + mT/2$. Both average and maximal values are linear in T. This makes each one of the linear subproblems far more tractable than the general problem, an advantage that more than compensates the fact that there exist T such problems.

5. Application: topological properties of sets of equilibrium triples

We apply the characterisation given by Theorem 3 to the proof of the contractibility of subsets made of configurations that are compatible with given rankings, properties that we then apply to proving the pathconnectedness of the set of equilibrium configurations, i.e., configurations that are subsets of some equilibrium manifold.

Contractibility of the set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$

Proposition 5. *The set* $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ *is contractible.*

Proof. See Section C of the Appendix. □

Application to the pathconnectedness of the set of equilibrium triples

Lemma 6. *We have*

$$\mathcal{E}^{[T]} = \bigcup_{\prec} (\mathcal{E}^{[T]}|\prec).$$

Proof. Obvious. □

Lemma 7. *The intersection*

$$\bigcap_{\prec} (\mathcal{E}^{[T]}|\prec)$$

is not empty.

Proof. The idea of the proof is to define a configuration $\tilde{G} \in \mathcal{E}^{[T]}$ that is compatible with all ranking profiles $\prec$.

Step 1. Let $u \in \mathcal{U}$ be some arbitrary utility function. Let $x^1, x^2, \dots, x^t, \dots, x^T$ be T distinct consumption bundles in X yielding the same utility level (i.e., $u(x^1) = u(x^2) = \cdots = u(x^t) = \cdots = u(x^T)$). Let p^t be the supporting price vector of x^t for $t = 1, 2, \dots, T$.

The strict inequality $p^t \cdot x^t < p^t \cdot x^{t'}$ for $t \neq t'$ follows from the strict quasi-concavity of the utility function u combined with $x^t \neq x^{t'}$ (and $u(x^t) = u(x^{t'})$).

Step 2. These inequalities imply as a special case the inequalities

$$p^t \cdot x^t < p^t \cdot x^{t'} \quad \text{for } 1 \le t \prec_i t' \le T$$

for any order $\prec_i$ of $\mathbb{T} = \{1, 2, \dots, T\}$. are satisfied. In other words, the T elements $(p^t, x^t) \in S \times X$ are well-ranked in the sense of Definition A.3 of the Appendix.

Step 3. Let $\tilde{G}$ be the configuration defined by the collection of triples $(\tilde{p}^t, \tilde{\boldsymbol{w}}, \tilde{r}^t)$ where $\tilde{p}^t = p^t$, $\tilde{w}_i^t = \tilde{p}^t \cdot x^t$, and $\tilde{r}^t = mx^t$ for t varying from 1 to T. Let also $\boldsymbol{u} = (u, u, \dots, u)$ denote the utility profile associated with each consumer having $u \in \mathcal{U}$ as utility function.

Let $f : S \times \mathbb{R}_{++} \to X$ be the demand function associated with the utility function u. Then, we have $f(p^t, p^t \cdot x^t) = x^t$ for $t = 1, 2, \dots, T$. Define $\tilde{x}_i^t = x^t$ for all i and t. The following inequalities and equalities

$$\begin{cases} \tilde{p}^{t'} \cdot \tilde{x}_i^t > \tilde{w}_i^{t'} & \text{for all } i \text{ and } t' \neq t; \\ \tilde{p}^t \cdot \tilde{x}_i^t = \tilde{w}_i^t & \text{for all } i; \\ \sum_i x_i^t = \tilde{r}^t; & \end{cases}$$

are then satisfied for all order profiles $\prec = (\prec_i)$. It then follows from Theorem 3 that the configuration $\tilde{G}$ is compatible with the ranking $\prec = (\prec_i)$ for any order profile $\prec$. $\square$

We can now prove:

Proposition 8. *The set $\mathcal{E}^{[T]}$ is pathconnected.*

Proof. The set $\mathcal{E}^{[T]}$ is the union of the path-connected sets $\left(\mathcal{E}^{[T]} \middle| \prec\right)$ for all order profiles $\prec = (\prec_i)$. These sets have a non empty intersection by Lemma 7. Let $\tilde{G}$ be some element of the intersection $\cap_\pi \left(\mathcal{E}^{[T]} \middle| \prec\right)$. It then suffices to join the configurations $G \in \left(\mathcal{E}^{[T]} \middle| \prec\right)$ and $G' \in \left(\mathcal{E}^{[T]} \middle| \prec'\right)$ to the element $\tilde{G}$ by two continuous paths contained in $\left(\mathcal{E}^{T]} \middle| \prec\right)$ and $\left(\mathcal{E}^{T]} \middle| \prec'\right)$ respectively to define a continuous path in $\mathcal{E}^{[T]}$ linking the two configurations G and G'.

$\square$

References

[1] Y. Balasko. Connexité de l'espace des équilibres d'une famille d'économies. *Revue Francaise d'Automatique, Informatique, Recherche Opérationnelle*, 5:121–123, 1973.

[2] Y. Balasko. The graph of the Walras correspondence. *Econometrica*, 43:907–912, 1975.

[3] Y. Balasko. Some results on uniqueness and on stability of equilibrium in general equilibrium theory. *Journal of Mathematical Economics*, 2:95–118, 1975.

[4] Y. Balasko. *Foundations of the Theory of General Equilibrium*. Academic Press, Boston, 1988.

[5] D.J. Brown and R.L. Matzkin. Testable restrictions on the equilibrium manifold. *Econometrica*, 64:1249–1262, 1996.

[6] P.A. Chiappori and J-C. Rochet. Revealed preferences and differentiable demand. *Econometrica*, 55:687–691, 1987.

[7] S. Snyder. Observable implications of equilibrium behavior on finite data. Preprint, Department of Economics, Virginia Polytechnic Institute and State University, 2001.

[8] E.H. Spanier. *Algebraic Topology*. McGraw-Hill, New York, 1966.

[9] H.R. Varian. The nonparametric approach to demand analysis. *Econometrica*, 50:945–973, 1982.

A. Properties of price-consumption data and revealed preference theory

This part is devoted to recalling a few properties of revealed preference theory and to establishing properties of finite sets of price-consumption data $(p^t, x_i^t) \in S \times X$, with t varying from 1 to T, i denoting some arbitrarily chosen consumer, properties that play a crucial role in the proofs of the main results of this paper.

The strong axiom of revealed preferences

The set of data $\{(p^1, x_i^1), (p^2, x_i^2), \dots, (p^T, x_i^T)\}$ satisfies the *strong axiom of revealed preferences* or SARP if, for every integer $n \leq T$ and every subset

$$\{t_1, t_2, \dots, t_n\} \subset \{1, 2, \dots, T\},$$

the inequalities

$$p^{t_1} \cdot (x_i^{t_2} - x_i^{t_1}) \leq 0 \quad , \quad p^{t_2} \cdot (x_i^{t_3} - x_i^{t_2}) \leq 0 \quad , \dots , \quad p^{t_{n-1}} \cdot (x_i^{t_n} - x_i^{t_{n-1}}) \leq 0$$

imply either

$$x_i^{t_1} = x_i^{t_n} \quad \text{or} \quad p^{t_n} \cdot (x_i^{t_1} - x_i^{t_n}) > 0.$$

This set of data satisfies the *strong version* of SARP if, in addition to SARP, we have $x_i^t \neq x_i^{t'}$ whenever $p_t \neq p_{t'}$ [6].

Rationalization of price-consumption data

The utility function $u_i : X \to \mathbb{R}$ *rationalizes* the set of price-consumption data $\{(p^1, x_i^1), (p^2, x_i^2), \dots, (p^T, x_i^T)\}$ if

$$u_i(x_i^t) = \max u_i(x_i) \qquad \text{subject to} \quad p^t \cdot x_i \leq p^t \cdot x_i^t$$

for $t = 1, 2, \dots, T$.

If the set of data $\{(p^1, x_i^1), (p^2, x_i^2), \dots, (p^T, x_i^T)\}$ satisfies the strong version of SARP, one can easily derive from [6] that these data can be rationalized by a utility function $u_i \in \mathcal{U}$. (In [6], the domain of the utility function u_i is restricted to a compact subset of the strictly positive orthant $X = \mathbb{R}^\ell_{++}$ instead of being the full set X; it then suffices to extend the domain of the function u_i to X while satisfying the conditions characterizing the elements of $\mathcal{U}$.)

Openness of the set of price-income data satisfying SARP

First, let us identify the Cartesian product $S^T \times X^{[T]}$ with the set of T-tuples

$$\big((p^1, x_i^1), (p^2, x_i^2), \dots, (p^T, x_i^T)\big)$$

with distinct coordinates $x_i^1, x_i^2, \dots, x_i^T$, and let $\mathcal{S}$ denote the subset of $S^T \times X^{[T]}$ made of the T-tuples that satisfy SARP.

We reformulate SARP as follows. Let σ be some ordered subset $\{j_1, j_2, \dots, j_n\}$ of $\mathbb{T} = \{1, 2, \dots, T\}$. We define F_σ as the subset of $S^T \times X^{[T]}$ consisting of the T-tuples

$$\big((p^1, x_i^1), \dots, (p^t, x_i^t), \dots, (p^T, x_i^T)\big)$$

that satisfy the following inequalities where $n \leq T$:

$$\begin{aligned} p^{j_1} \cdot x_i^{j_1} &\leq p^{j_1} \cdot x_i^{j_2}, \\ p^{j_2} \cdot x_i^{j_2} &\leq p^{j_2} \cdot x_i^{j_3}, \\ \dots &\leq \dots, \\ p^{j_{n-1}} \cdot x_i^{j_{n-1}} &\leq p^{j_{n-1}} \cdot x_i^{j_n}, \\ p^{j_n} \cdot x_i^{j_n} &\leq p^{j_n} \cdot x_i^{j_1}. \end{aligned}$$

Define $\mathfrak{S}$ as the set of ordered subsets of $\{1, 2, \dots, n\}$ with at least two elements. Define $\mathcal{F} = \bigcup_{\sigma \in \mathfrak{S}} F_\sigma$. Then, the set $\mathcal{S}$ of T-tuples of $S^T \times X^{[T]}$ that satisfy SARP is the complement of the set $\mathcal{F}$:

$$\mathcal{S} = S^T \times X^{[T]} \setminus \mathcal{F}.$$

This reformulation of SARP enables us to give a very short proof of the following:

Proposition A. 1. *The set $\mathcal{S}$ is open in $S^T \times X^{[T]}$.*

Proof. The number of ordered subsets σ of the finite set $\mathbb{T} = \{1, 2, \dots, T\}$ is finite. The set F_σ for a given ordered σ is defined by weak inequalities; it is therefore closed in $S^T \times X^{[T]}$. The set $\mathcal{F}$ is then closed as a finite union of closed sets, and $\mathcal{S}$ is open as the complement of a closed set.

□

Corollary A. 2. *With $(p^1, p^2, \dots, p^T) \in S^T$ arbitrarily given, the set of T-tuples $(x_i^1, x_i^2, \dots, x_i^T) \in X^m$ for which the T-tuple $\big((p^1, x_i^1), (p^2, x_i^2), \dots, (p^T, x_i^T)\big)$ satisfies SARP is open in X^m.*

Proof. The intersection

$$\mathcal{S} \cap (p^1, p^2, \dots, p^T) \times X^{[T]}$$

is open in $\{(p^1, p^2, \dots, p^T)\} \times X^{[T]}$ because $\mathcal{S}$ is open. The projection

$$\{(p^1, p^2, \dots, p^T)\} \times X^{[T]} \to X^{[T]}$$

is a homeomorphism, and the image of $\mathcal{S} \cap (p^1, p^2, \dots, p^T) \times X^{[T]}$ is therefore open in $X^{[T]}$. We conclude by observing that this image coincides with the set of T-tuples $(x_i^1, x_i^2, \dots, x_i^T) \in X^{[T]}$ for which the T-tuple

$$\big((p^1, x_i^1), (p^2, x_i^2), \dots, (p^T, x_i^T)\big) \in S^T \times X^{[T]}$$

satisfies SARP. □

Remark 2. Proposition A.1 and its corollary seem to be new.

Well-ranked data set for a specified ordered index set

Definition A. 3. *The data (p^t, x_i^t) indexed by the ordered set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$ are well-ranked if the inequalities*

$$p^t \cdot x_i^t < p^t \cdot x_i^{t'} \tag{4}$$

are satisfied for $1 \leq t < t' \leq T$.

Note that the property of being well-ranked is a property of the data set *and* of its indexation by the *ordered* set $(\mathbb{T}, <)$. Definition A.3 obviously extends to any finite "abstract" set $(I, \prec)$ equipped with the order relation $\prec$. Note also that well-ranked data are necessarily two by two different.

In the following proposition, we consider two ordered sets $(I, \prec)$ and $(I', \prec')$. The ordered set $(I', \prec')$ is an *ordered subset* of the ordered set $(I, \prec)$ if $I' \subset I$ and the restriction of the order $\prec$ of I to I' coincide with the order $\prec'$ of I'. In such a case, we simplify the notation by simply writing $(I', \prec)$ for the ordered subset.

Proposition A. 4. *Let (p^t, x_i^t) be a set of data that are well-ranked for the ordered index set $(I, \prec)$. Then, the subset of data (p^t, x_i^t) indexed by the ordered subset $(I', \prec)$ is also well-ranked.*

Proof. Obvious. □

An important application of the property of being well-ranked is the following:

Proposition A. 5. *Any set of well-ranked data for the ordered index set $(I, \prec)$ can be rationalized by some utility function $u_i \in \mathcal{U}$.*

Proof. It suffices that we prove that the set of well-ranked data satisfies the *strong version* of SARP. There is no loss of generality in assuming $(I, \prec) = (\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$.

First, let us show that $p^t \neq p^{t'}$ implies $x_i^t \neq x_i^{t'}$. The inequality $p^t \neq p^{t'}$ implies $t \neq t'$. Without loss of generality, we can assume $t < t'$. It then follows from the data being well-ranked that the strict inequality $p^t \cdot x_i^t < p^t \cdot x_i^{t'}$ is satisfied, which implies that x_i^t and $x_i^{t'}$ must be different.

Let us now prove that the set of well-ranked data satisfies SARP. Assume that we have

$$p^{t_1} \cdot x_i^{t_1} \geq p^{t_1} \cdot x_i^{t_2} \quad , \quad p^{t_2} \cdot x_i^{t_2} \geq p^{t_2} \cdot x_i^{t_3} \quad , \dots , \quad p^{t_{n-1}} \cdot x_i^{t_{n-1}} \geq p^{t_{n-1}} \cdot x_i^{t_n}$$

and let us show that, if $x_i^{t_1} \neq x_i^{t_n}$, the strict inequality

$$p^{t_n} \cdot x_i^{t_n} < p^{t_n} \cdot x_i^{t_1}$$

is satisfied.

The data being well-ranked for the ordered index set $(\mathbb{T},<) = \{1 < 2 < \cdots < T\}$, inequality $p^{t_1} \cdot x_i^{t_1} \geq p^{t_1} \cdot x_i^{t_2}$ is possible only if $t_2 \leq t_1$ and, therefore, implies the inequality $t_2 \leq t_1$. Similarly, inequality $p^{t_2} \cdot x_i^{t_2} \geq p^{t_2} \cdot x_i^{t_3}$ implies $t_3 \leq t_2$, and this goes on up to the last inequality, which implies $t_n \leq t_{n-1}$. By the transitivity of the order relation $\leq$, the sequence of inequalities

$$t_n \leq t_{n-1} \leq \cdots \leq t_2 \leq t_1$$

yields the inequality $t_n \leq t_1$. We cannot have $t_n = t_1$ because $x_i^{t_n}$ and $x_i^{t_1}$ are different. We therefore have $t_n < t_1$; the strict inequality

$$p^{t_n} \cdot x_i^{t_n} < p^{t_n} \cdot x_i^{t_1}$$

follows again from the data being well-ranked.

□

Proposition A. 6. *Let (p^t, x_i^t) with $t = 1, 2, \ldots, T$ be a set of data two by two different (i.e., $(p^t, x_i^t) \neq (p^{t'}, x_i^{t'})$ for $t \neq t'$) rationalized by a utility function $u_i \in \mathcal{U}$ and such that the inequalities*

$$u_i(x_i^1) \leq u_i(x_i^2) \leq \cdots \leq u_i(x_i^T)$$

are satisfied. Then these data are well-ranked for the ordered index set $(\mathbb{T},<) = \{1 < 2 < \cdots < T\}$.

Proof. Let t be arbitrary between 1 and T. For any $t' > t$, we have $u_i(x_i^t) \leq u_i(x_i^{t'})$. This implies the inequality $p^t \cdot x_i^t \leq p^t \cdot x_i^{t'}$. This inequality is strict for $t \neq t'$ because the utility function u_i is strictly quasi-concave and $x_i^t \neq x_i^{t'}$. □

Remark A. 1. The concept of well-ranked data is due to Varian [9].

Strongly ranked data set for a specified ordered index set

The utility function whose existence is established in Proposition A.5 does not guarantee us that the utility levels $u_i(x_i^t)$ are ranked in increasing orders, i.e., that the inequalities

$$u_i(x_i^1) \leq u_i(x_i^2) \leq \cdots \leq u_i(x_i^T) \tag{5}$$

are satisfied even for data that are well-ranked for the ordered index set $(\mathbb{T},<)$. This leads us to strengthen the concept of well-ranked data set with respect to an ordered index set as follows:

Definition A. 7. *The data (p^t, x_i^t) indexed by the ordered set $(\mathbb{T},<) = \{1 < 2 < \cdots < T\}$ are* strongly ranked *if, in addition to being well-ranked, they satisfy the inequality*

$$p^{t+1} \cdot x_i^t \leq p^{t+1} \cdot x_i^{t+1}$$

for every $t \neq T$.

A set of indexed data can be well-ranked without being strongly ranked. The following proposition reflects the additional information associated with strongly ranked data.

Proposition A. 8. *Let (p^t, x_i^t) be strongly ranked data for the ordered index set $(\mathbb{T},<) = \{1 < 2 < \cdots < T\}$. Then, the strict inequalities*

$$u_i(x_i^1) < u_i(x_i^2) < \cdots < u_i(x_i^T) \tag{6}$$

are satisfied for any utility function $u_i \in \mathcal{U}$ that rationalizes these data.

Proof. It follows from the definition of strongly ranked data sets that the inequality

$$p^t \cdot x_i^{t-1} \leq p^t \cdot x_i^t$$

is satisfied for $t \neq 1$. It then follows from $x_i^{t-1} \neq x_i^t$ that we have

$$u_i(x_i^{t-1}) < u_i(x_i^t)$$

for $t \neq 1$. □

By Proposition A.5, any well-ranked data set is rationalizable by some utility function $u_i \in \mathcal{U}$. The ranking of the commodity bundles x_i^t is then independent of the utility function $u_i \in \mathcal{U}$. In the next proposition, we see that, if well-ranked data are not always strongly ranked, it is nevertheless possible to embed these well-ranked data into a larger set of strongly ranked data.

Proposition A. 9. *Any set of well-ranked data for the ordered index set* $(\mathbb{T}, <) = \{1, 2, \dots, T\}$ *can be embedded into some larger set of strongly ranked data for some ordered index set* $(J, \prec)$ *(with* $\mathbb{T} \subset J$*), with* 1 *being the smallest element of* $(J, \prec)$*.*

Proof. The proof works by induction on T, the number of well-ranked data. For $T = 1$, there is nothing to prove because any data set of one element is well-ranked and strongly ranked.

Induction argument for T arbitrary. The induction hypothesis can be stated as follows: any set of T well-ranked data (p^t, x_i^t) for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$ can be embedded into a larger set of strongly ranked data for the ordered index set $(J, \prec)$ whose smallest element is the element 1, the smallest element of the set I. This property is assumed to be satisfied for $T - 1$, and we establish that it is then true for T.

Let therefore T well-ranked data (p^t, x_i^t) for the ordered set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$. It follows from Proposition 4 that the $T - 1$ data (p^t, x_i^t) indexed by the ordered subset $(I', <) = \{2 < 3 < \cdots < T\}$ are also well-ranked. Therefore, by the induction assumption, we can embed these $T - 1$ data into a larger set of strongly ordered data with an ordered index set $(J', \prec)$ whose smallest element is the element $2 \in (I', <)$.

One checks readily that these strongly ranked data and the pair (p^1, x_i^1) of the original data set define a set of data that are well-ranked for the ordered set $(J'', \prec) = \{1\} \cup (J', \prec)$ with $1 \prec t$ for $t \in J'$.

Let us show that these data are either strongly ranked for the ordered index set $(J'', \prec)$ or that we can find an additional pair $(p^{A_{12}}, x_i^{A_{12}})$ such that the data set consisting of (p^1, x_i^1), $(p^{A_{12}}, x_i^{A_{12}})$, and the subsequent (p^t, x_i^t) for $t \in J'$ are strongly ranked for the ordered index set $(J, \prec) = \{1 \prec A_{12} \prec 2\} \cup (J', \prec)$.

If the inequality $p^2 \cdot x_i^1 \leq p^2 \cdot x_i^2$ is satisfied, then the full data set is clearly strongly ranked for the ordered index set $(J'', \prec)$ and there is nothing more to prove. (It suffices to take $(J, \prec) = (J'', \prec)$, and to observe that 1 is the smallest element of both index set $(\mathbb{T}, <)$ and $(J, \prec)$.)

Assume now that the inequality $p^2 \cdot x_i^1 > p^2 \cdot x_i^2$ is satisfied. Define $\epsilon = \inf_{2 \leq t \leq T} p^1 \cdot (x_i^t - x_i^1)$. Let us show that there exists some $x_i^{A_{12}} \in \mathbb{R}_{++}^{\ell}$ that satisfies the following equalities:

$$p^1 \cdot x_i^{A_{12}} = p^1 \cdot x_i^1 + \epsilon/2 \quad , \quad p^2 \cdot x_i^{A_{12}} \leq p^2 \cdot x_i^2 \tag{7}$$

The set

$$\{x_i \in X \mid p^2 \cdot x_i \leq p^2 \cdot x_i^2\}$$

is convex and bounded from below by 0, contains elements arbitrarily close to 0, and also contains the point x_i^2. The image of this set by the linear map $x_i \to p^1 \cdot x_i$ is therefore an interval of the form $(0, A]$. Now $p^1 \cdot x_i^2$ belongs to this interval as the image of the point x_i^2. This implies that $p^1 \cdot x_i^1 + \epsilon/2$ also belongs to this interval by the definition of ϵ, which proves the existence of some point $x_i^{A_{12}}$ satisfying the above equalities and inequalities.

Set $p^{A_{12}} = p^1$. Let us now show that the set consisting of (p^1, x_i^1), $(p^{A_{12}}, x_i^{A_{12}})$, and of the data (p^t, x_i^t) for $t \in J'$ is strongly ranked with the respect to the ordered index set $(J, \prec) = \{1 < A_{12} < 2\} \cup (J', \prec)$. By construction, we have $p^1 \cdot x_i^1 \leq p^1 \cdot x_i^{A_{12}} = p^1 \cdot x_i^1 + \epsilon/2$ and $p^{A_{12}} \cdot x_i^{A_{12}} = p^1 \cdot x_i^{A_{12}} = p_1 \cdot x_i^1 + \epsilon/2 < p^{A_{12}} \cdot x_i^t$ for $t \geq 2$, which proves that these data are indeed well-ranked. The inequality

$$p^{A_{12}} \cdot x_i^1 = p^1 \cdot x_i^1 \leq p^1 \cdot x_i^1 + \epsilon/2 = p^1 \cdot x_i^{A_{12}} = p^{A_{12}} \cdot x_i^{A_{12}}$$

and the inequality

$$p^2 \cdot x_i^{A_{12}} \leq p^2 \cdot x_i^2$$

are satisfied by construction. These inequalities combined with the strong ranking of the data for $t \geq 2$ imply that these data are strongly ranked for the ordered index set $(J, \prec) = \{1 < A_{12} < 2 < \cdots < T\}$. Note that the element 1 is the smallest element of the ordered set $(J, \prec)$. □

Proposition A. 10. *Any set of well-ranked data* (p^t, x_i^t) *for the ordered index set* $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$ *is rationalizable by a utility function* $u_i \in \mathfrak{U}$ *for which the strict inequalities*

$$u_i(x_i^1) < u_i(x_i^2) < \cdots < u_i(x_i^T)$$

are satisfied.

Proof. These data being well-ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$, it follows from Proposition A.9 that they can be embedded into a larger set of strongly ranked data indexed by some ordered indexed set $(J, <)$, with $(\mathbb{T}, <) \subset (J, \prec)$. The strict inequalities of the Proposition then follow from the fact that the order of $\mathbb{T}$ is the restriction of the order of J. □

Proposition A. 11. *Let (p^t, x_i^t) with $t = 1, 2, \dots, T$ a set of data two by two different (i.e., $(p^t, x_i^t) \neq (p^{t'}, x_i^{t'})$ for $t \neq t'$) rationalized by some utility function $u_i \in \mathcal{U}$ and such that the weak inequalities*

$$u_i(x_i^1) \leq u_i(x_i^2) \leq \cdots \leq u_i(x_i^T)$$

are satisfied. Then these data can be rationalized by a utility function $\tilde{u}_i \in \mathcal{U}$ such that the strict inequalities

$$\tilde{u}_i(x_i^1) < \tilde{u}_i(x_i^2) < \cdots < \tilde{u}_i(x_i^T)$$

are satisfied.

Proof. By Proposition A.6, the data are well-ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$. It then follows from Proposition A.10 that there exists a utility function $\tilde{u}_i \in \mathcal{U}$ that rationalizes the T pairs (p^t, x_i^t) (for $t = 1, \dots, T$) and such that the strict inequalities

$$\tilde{u}_i(x_i^1) < \tilde{u}_i(x_i^2) < \cdots < \tilde{u}_i(x_i^T)$$

are satisfied. □

Application to increasing sequences of commodity bundles

The following proposition and its corollary can easily be proved directly. We prove them here as straightforward consequences of the concept of strongly ordered data.

Proposition A. 12. *Let $x_i^1, x_i^2, \dots, x_i^T$ be a strictly increasing sequence of elements in $X = \mathbb{R}^\ell_{++}$. Let $p^1, p^2, \dots, p^T$ be arbitrarily chosen price vectors. The data (p^t, x_i^t) are strongly ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$.*

Proof. The (vector) inequality $x_i^t < x_i^{t'}$ for $t < t'$ implies the strict inequality $p^t \cdot x_i^t < p^t \cdot x_i^{t'}$, which proves that these data are well-ranked for the ordered index set $(\mathbb{T}, <) = \{1 < 2 < \cdots < T\}$. The inner product of the inequality $x_i^t < x_t^{t+1}$ with the price vector p^{t+1} yields the strict inequality $p^{t+1} \cdot x_i^t < p^{t+1} \cdot x_i^{t+1}$, which proves that these data are strongly ranked. □

Corollary A. 13. *Let $x_i^1, x_i^2, \dots, x_i^T$ be a strictly increasing sequence of elements in $X = \mathbb{R}^\ell_{++}$. Let $p^1, p^2, \dots, p^T$ be arbitrarily chosen price vectors. There exists a utility function $u_i \in \mathcal{U}$ that rationalizes the data (p^t, x_i^t) with $t = 1, 2, \dots, T$. In addition, the inequality $u_i(x_i^1) < u_i(x_i^2) < \cdots < u_i(x_i^T)$ is always satisfied.*

Proof. Follows directly from Proposition A.12 combined with Proposition A.8.

□

B. Proofs of rationalisability of configurations with collinear and quasi-collinear total resources

Proof of Theorem 1

Let $(p^t, \boldsymbol{w}^t, r^t)$ be a collection of T two by two distinct price-income distribution-total resource triples. The idea is to show that there exist commodity bundles $x_i^t \in X$ and utility functions $u_i \in \mathcal{U}$ for i varying from 1 to m and t from 1 to T such that we have $x_i^t = f_i(p^t, w_i^t)$ (where f_i is the demand function associated with the utility function u_i) and $\sum_i x_i^t = r^t$.

Step 1: Constant total resources and some strict inequalities (generic case)
We first assume that the following additional property is satisfied:

$$\frac{w_i^t}{p^t \cdot r} \neq \frac{w_i^{t'}}{p^{t'} \cdot r} \tag{8}$$

for all t, t', and i. Define

$$x_i^t = \frac{w_i^t}{p^t \cdot r} r.$$

By construction, all the vectors x_i^t for a given i are collinear with the positive vector $r \in \mathbb{R}^\ell_{++}$. It follows from inequalities (8) that the x_i^t's are all distinct for any given i. It then suffices to apply Lemma 13 to conclude that, for every i, the T pairs

(p^t, x_i^t) satisfy SARP. This implies for every i between 1 and m the existence of a utility function u_i such that the equality $x_i^t = f_i(p^t, w_i^t)$ is satisfied. In addition, it follows from the formula defining x_i^t that we have

$$\sum_i x_i^t = \frac{\sum_i w_i^t}{p^t \cdot r} r = r,$$

from which follows that the equality

$$\sum_i f_i(p^t, w_i^t) = r = r^t$$

is satisfied for t varying from 1 to T. This proves that such price-income distribution-total resource triples are indeed equilibrium triples for suitably defined utility functions.

Step 2: Constant total resources with no restrictions

The following step is to deal with situations where, for some consumer i, there exist t and t' such that the equality

$$\frac{w_i^t}{p^t \cdot r} = \frac{w_i^{t'}}{p^{t'} \cdot r}$$

is satisfied with $p^t \neq p^{t'}$.

Because of this equality, the commodity bundles $x_i^t = \dfrac{w_i^t}{p^t \cdot r} r$ and $x_i^{t'} = \dfrac{w_i^{t'}}{p^{t'} \cdot r} r$ are not good candidates for our construction because they are equal while the candidate "supporting" price vectors p^t and $p^{t'}$ are different. The idea is therefore to perturb x_i^t and $x_i^{t'}$ in such a way that the perturbed sequence $x_i^1, x_i^2, \ldots, x_i^T$ remains totally ordered, and the equalities $\sum_i x_i^t = r^t$ are satisfied for all t's.

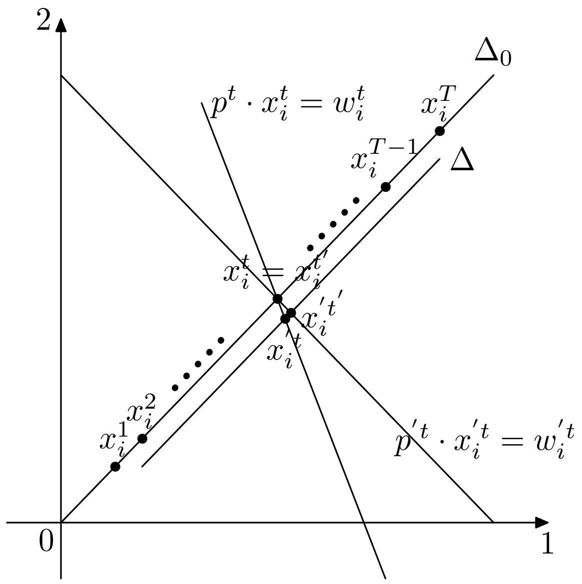

Figure 1: Perturbation of the collection $\{x_i^t\}$ for $1 \leq t \leq T$

In order to do that, consider the line Δ_0 that passes through the origin and that is collinear with the vector $r \in X$. Let Δ be a line parallel to the line Δ_0 and sufficiently close to Δ_0 for the following property to be satisfied. The intersection points $x_i'^t$ and $x_i'^{t'}$ of Δ with the budget hyperplanes $p^t \cdot x_i^t = w_i^t$ and $p^{t'} \cdot x_i^{t'} = w_i^{t'}$ are distinct and, if we define $x_i'^{t''} = x_i^{t''}$ for $t'' \neq t, t'$, the sequence $x_i'^1, \ldots, x_i'^T$ is completely ordered. That this property is satisfied follows from the fact that the sequence $x_i^1, x_i^2, \ldots, x_i^T$ is already fully ordered, with the elements x_i^t and $x_i^{t'}$ being identical, and the new sequence is obtained by just perturbing those two identical elements; the elements $x_i'^t$ and $x_i'^{t'}$ can therefore be compared to all the other elements of the sequence provided the perturbation is small enough. In addition, thanks to the choice of the direction of the line Δ, these two elements are themselves ordered.

The next step is to find another consumer j and to perturb the corresponding sequence $x_j^1, x_j^2, \ldots, x_j^T$, so that the total resources remain constant. Therefore, we pick arbitrarily some consumer j, and define the new sequence $x_j^{'1}, x_j^{'2}, \ldots, x_j^{'T}$ by

$$x_j^{'t} = x_j^t + (x_i^t - x_i^{'t}) \quad , \quad x_j^{'t'} = x_j^{t'} + (x_i^{t'} - x_i^{'t'}) \quad , \quad x_j^{'t''} = x_j^{t''}$$

with $t'' \neq t, t'$.

Using the same line of reasoning as above, we observe that the perturbation that defines the consumption bundles of consumer i can be made small enough for the new sequence to be also fully ordered and the already distinct elements to remain distinct through the perturbation. In addition, the total resources are then, by construction, equal to the vector r . Overall, this construction reduces by at least one unit the number of non distinct commodity bundles. It then suffices to iterate this construction for every consumer i and pairs (t, t') such that $x_i^t = x_i^{t'}$. Eventually, one gets for each consumer a collection of ordered sequences of T elements that sum up to the vector of total resources $r \in X$. One then concludes with the application of Lemma 13.

Step 3: Collinear total resources
Assume now that total resources are collinear instead of being constant. It then suffices to reproduce the same line of reasoning as in the case of constant total resources.

C. Proof of Proposition 5

The topological space Z is said to be *contractible* if there exists a continuous map $h : Z \times [0,1] \to Z$ that satisfies the following properties: 1) $h(.,1)$ is the identity map of Z, i.e., $h(.,1) = \mathrm{id}_Z$; 2) $h(z,0) = z_0 \in Z$ for $z \in Z$ and some $z_0 \in Z$. The intuition behind this definition is that a contractible space can be continuously deformed into a point, here the point z_0.

A related idea is the one of a *deformation retract* subset Z_0 of Z. By definition, the topological subspace Z_0 is a deformation retract of Z if there exists a continuous map $h : Z \times [0,1] \to Z$ that satisfies the following properties: 1) $h(.,1)$ is the identity map of Z, i.e., $h(.,1) = \mathrm{id}_Z$; 2) $h(z,0) \in Z_0$ for $z \in Z$; 3) $h(z,0) = z$ for all $z \in Z_0$. If the subspace Z_0 is a deformation retract of Z, the spaces Z and Z_0 are said have the same homotopy type. If Z_0 is contractible, one easily shows that Z is also contractible. (First, contract Z to Z_0, and then Z_0 to a point.)

It is intuitively clear (and this can evidently be established rigorously) that a contractible space is pathconnected and simply connected (i.e., every closed simple path can be deformed continuously into a point). For details, see [8].

Lemma 9. *Let $G = \big((p^t, \boldsymbol{w}^t, r^t)\big) \in \big(\mathcal{E}^{[T]}\big|\prec\big)$ and $G^* = \big((p^t, \boldsymbol{w}^{*t}, r^{*t})\big) \in \big(\mathcal{E}^{[T]}\big|\prec\big)$. Then, for any real number $\lambda \in [0,1]$, the configuration $G(\lambda) = \lambda G + (1-\lambda)G^*$ belongs to $\big(\mathcal{E}^{[T]}\big|\prec\big)$.*

Proof. It suffices to observe that the equalities and inequalities in Theorem 3 that characterize elements of the set $\big(\mathcal{E}^{[T]}\big|\prec\big)$ are linear with respect to x_i^t and w_i^t. □

We now define special configurations that belong to the set $\big(\mathcal{E}^{[T]}\big|\prec\big)$ for a given ranking profile $\prec = (\prec_i)$. Let

$$t_{i_1} \prec_i t_{i_2} \prec_i \cdots \prec_i t_{i_T}.$$

Let $(\lambda_{t_{i_1}}, \lambda_{t_{i_2}}, \ldots, \lambda_{t_{i_T}})$ be a strictly increasing sequence of strictly positive (real) numbers:

$$0 < \lambda_{t_{i_1}} < \lambda_{t_{i_2}} < \cdots < \lambda_{t_{i_T}}.$$

Let $\tau_i \in X$ be some strictly positive vector. Define the vector $x_i^{*t} = \lambda_t \tau_i \in X$. The sequence x_i^{*t} satisfies the strict (vector) inequalities

$$x_i^{*t_{i_1}} < x_i^{*t_{i_2}} < \cdots < x_i^{*t_{i_T}}.$$

Let $\boldsymbol{p} = (p^t)$, with $p^t \in S$ for $t = 1, 2, \ldots, T$. Define $w_i^{*t} = p^t \cdot x_i^{*t}$ for $1 \le i \le m$, $\boldsymbol{w}^{*t} = (w_i^{*t})$, and $r^{*t} = \sum_i x_i^{*t}$. We denote by $G^*(\boldsymbol{p})$ the configuration $(p^t, \boldsymbol{w}^{*t}, r^{*t})$.

Lemma 10. *The configuration $G^*(\boldsymbol{p}) = (p^t, \boldsymbol{w}^*, r^*)$ belongs to $\big(\mathcal{E}^{[T]}\big|\prec\big)$ for any $\boldsymbol{p} = (p^t) \in S^T$.*

Proof. It follows from Corollary A.13 of the Appendix that the pairs (p^t, x_i^{*t}) for i fixed and t varying from 1 to T can be rationalized by some utility function $u_i \in \mathcal{U}$ for any price vectors p^t with $t = 1, 2, \ldots, T$. Let $\boldsymbol{u} = (u_i) \in \mathcal{U}^m$ be the utility profile defined by varying i from 1 to m. The configuration $G^*(\boldsymbol{p}) = (p^t, \boldsymbol{w}^{*t}, r^{*t})$ then belongs to $\big(\mathcal{E}^{[T]}\big|\prec\big)$. □

Remark 3. A consequence of Lemma 10 is that the set $\big(\mathcal{E}^{[T]}\big|\prec\big)$ is not empty.

Proof of the contractibility property

Proof. Let $\mathcal{X}_0$ be the subset of $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ consisting of the configurations $G^*(\boldsymbol{p})$ where the price sequence $\boldsymbol{p} = (p^1, p^2, \ldots, p^T)$ is varied in S^T. The map $\boldsymbol{p} \to G^*(\boldsymbol{p})$ is continuous. The inverse map is the projection $(p^t, \boldsymbol{w}^t, r^t) \to \boldsymbol{p} = (p^t)$. These two maps are continuous, which proves that $\mathcal{X}_0$ is homeomorphic to S^m and is therefore contractible as a Cartesian product of contractible spaces.

We now define the map $h : \left(\mathcal{E}^{[T]}\middle|\prec\right) \times [0,1] \to \left(\mathcal{E}^{[T]}\middle|\prec\right)$ by

$$h(G, \lambda) = \lambda G + (1-\lambda) G^*(\boldsymbol{p})$$

where $\boldsymbol{p} = (p^t)$ is fixed. This map is clearly continuous. In addition, $h(G,1) = G$, $h(G,0) = G^*(\boldsymbol{p}) \in \mathcal{X}_0$, and for $G = G^*(\boldsymbol{p})$, it comes $h(G^*(\boldsymbol{p}), \lambda) = G^*(\boldsymbol{p})$. The set $\mathcal{X}_0$ is therefore a deformation retract of $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ by the map h. The set $\mathcal{X}_0$ being contractible, the set $\left(\mathcal{E}^{[T]}\middle|\prec\right)$ itself is contractible. $\square$