

SPLINE FUNCTIONS: AN ALTERNATIVE TO ESTIMATING INCOME-EXPENDITURE RELATIONSHIPS FOR BEEF

Chung-Liang Huang and Robert Raunika

Income-expenditure relationships are important components in many economic models used to project food expenditure and to understand food-expenditure behavior. The empirical estimation of income-expenditure relations has concentrated on the effects of income in explaining the variations of the household food expenditure. However, the problem of structural or parametric homogeneity for Engel curves in the analysis of household food expenditure behavior has received less attention in the applied demand literature.

Agarwala and Drinkwater argue that the familiar Engel curve results require modification when applied in situations in which the structure of the population and economy is diverse and changing. When economic and socioeconomic characteristics change, policies predicated on forecasts of such change cannot be based on parameter estimates from models that implicitly or explicitly assume that such variations cannot occur. Therefore, meaningful applications of even the simplest income-expenditure parameters to policy analysis should be conditioned on evidence of structural or parametric homogeneity.

The traditional approach to test the assumption of structural homogeneity for Engel curves is based on sample partitions. Forsyth studied the income-expenditure relationships by stratifying the sample according to numbers of persons in the household. Hassan and Johnson examined the parametric homogeneity for Engel curves in Canada across sample partitions based on cities, family income, life cycles, age of family head, tenure in home, and education of family head. With few exceptions, their results show a lack of homogeneity of the Engel curve coefficients across sample partitions. Stratifying the sample by socioeconomic characteristics is cumbersome because it can result in many estimated relationships. Moreover, partitioning the sample into different socioeconomic groups substantially reduces the degrees of freedom for the estimated relationships fitted to the subsamples and, hence, reduces the estimates reliability.

This study develops an alternative approach to account for the effect of socioeconomic characteristics upon food expenditures. Specifically,

spline functions were developed to reflect differences in income-expenditure relationships by allowing different functional forms within the various subintervals of income and household size variables. The authors demonstrate how spline functions capture various empirical economic relationships and test the hypothesis that consumers react differently at different income levels.

THE STATISTICAL MODEL

Adopted from the engineering discipline, spline functions have been applied to several economic problems in recent years (Barth, Kraft, and Kraft). The development of spline theory and piecewise regression models are well known and discussed elsewhere (Poirier; Smith; Wold). Recently, Buse and Lim have shown that spline functions can be regarded as a special case of restricted least squares. They demonstrate how the continuity restrictions and the validity of the restrictions can be tested using restricted least squares; and prove that under a common set of restrictions, the two procedures are equivalent.

An alternative way of handling the restricted least squares problem is to incorporate the restrictions in the fitting process so that the estimated coefficients satisfy the restrictions exactly. This can be done by working out directly the special form of the estimating equations, the approach employed by Suits, Mason, and Chan which related interest rates to money supply and inflation. By using appropriately defined composite variables, they demonstrated that the multivariate spline functions can be treated as a least squares regression model and fitted by standard ordinary least squares (OLS) procedures.

The development and formulation of spline functions for estimating income-expenditure relationships are briefly discussed to show how this procedure is used for investigating the structural homogeneity of household expenditure behavior with respect to household income and size. For simplicity, household income is employed to introduce the procedure.

To begin with, one may choose to fit a piecewise linear regression; that is, one linear segment

Authors are Assistant Professor and Professor, respectively, in the Department of Agricultural Economics, University of Georgia, College of Agriculture, Georgia Experiment Station, Experiment, Georgia.

The authors are grateful to Stanley M. Fletcher and to the anonymous reviewers for their helpful comments and suggestions.

to each specific income group. The relation can be represented as:

$$(1) E = \sum_{i=1}^n [a_i + b_i(Y - Y_i)]S_i + U, \\ i = 1, 2, \dots, n$$

where E and Y represent household food expenditure and income, respectively. Y is to be divided into n segments, where Y_1, \dots, Y_{n+1} defines the n+1 points, called knots. S_i is a dummy variable whose value is 1 for all observations, such that $Y_i \leq Y < Y_{i+1}$, and is 0, otherwise. U is a random disturbance associated with E.

In general, equation (1) allows discontinuity at each Y_i . In addition, a curvilinear relationship is generally considered more appropriate for the income-expenditure relation than the linear approximation. Spline functions overcome these limitations by replacing the linear formulation of equation (1) with polynomial approximations. However, the number and position of knots and the degrees of the polynomial pieces may vary in different situations and are the major difficulties confronted in estimating spline functions.

If each knot is defined as a variable, its position must be estimated and entered into the regression problem in a nonlinear fashion, and all the problems arising in nonlinear regression are present.¹ Although some research in this direction has been done (Bellman and Roth; Gallant and Fuller; McGee and Carleton), the use of variable-knot splines requires very large amounts of computation to find knot locations that give an absolute minimum for the residual sum of squares, and the testing of hypotheses is virtually impossible (Smith). However, spline function estimation with fixed knots is straight forward, using standard regression procedures (e.g., Barth, Kraft and Kraft; Suits, Mason and Chan; Poirier; Smith).

With respect to the degrees of the polynomial pieces, there is no *a priori* basis for the determination of a specific degree. However, a spline function with polynomials of degree three; that is, cubic splines, is the most common form used in practice. In general, cubic splines are used because they are of low degree; fairly smooth, assuming continuity restrictions up to the second derivative; and yet have the power to improve significantly the fit, as well as a higher degree of polynomial.

In this study, the cubic spline function with fixed knots is assumed and the range of household income was divided into n segments. Equation (1) now becomes

$$(2) E = \sum_{i=1}^n [a_i + b_i(Y - Y_i) + c_i(Y - Y_i)^2 + d_i(Y - Y_i)^3]S_i + U.$$

To ensure that equation (2) is continuous at each knot, constraints on the coefficients are required. These constraints make the function continuous and guarantee continuity of the first and second derivatives. Thus,

$$(3) a_i = a_{i-1} + b_{i-1}(Y_i - Y_{i-1}) + c_{i-1}(Y_i - Y_{i-1})^2 + d_{i-1}(Y_i - Y_{i-1})^3, \\ b_i = b_{i-1} + 2c_{i-1}(Y_i - Y_{i-1}) + 3d_{i-1}(Y_i - Y_{i-1})^2, \\ c_i = c_{i-1} + 3d_{i-1}(Y_i - Y_{i-1}), \quad i = 2, 3, \dots, n.$$

By substituting (3) into equation (2), and collecting terms with the same coefficient, equation (2) can be expressed as

$$E = a_1 \sum_{i=1}^n S_i + b_1(Y - Y_1) \sum_{i=1}^n S_i + c_1(Y - Y_1)^2 \sum_{i=1}^n S_i + d_1(Y - Y_1)^3 \sum_{i=1}^n S_i + \sum_{i=2}^n [(d_i - d_{i-1})(Y - Y_i)^3 \sum_{j=i}^n S_j] + U,$$

or

$$(4) E = a_1 + b_1(Y - Y_1) + c_1(Y - Y_1)^2 + d_1(Y - Y_1)^3 + \sum_{i=2}^n (d_i - d_{i-1})(Y - Y_i)^3 S_{i-1} + U,$$

where S_{i-1} is a new set of dummy variables, such that $S_{i-1} = 1$, if and only if $Y \geq Y_i$, otherwise $S_{i-1} = 0$.

Given the basic formulation of equation (4), the model can be generalized to fit a spline function that involves more than one independent variable (Suits, Mason, and Chan). This analysis incorporates the additional variable of household size in the same manner as the income variable in the regression.² Hence, m segments of household size within the sample range were established and added to equation (4).³ The final estimating

¹ Wold argues that the choice of knot positions in a spline function can be viewed as analogous to the specification of functional form in a traditional curve fitting problem. Hence, the knots should be chosen to correspond to the overall behavior of the data than be considered as parameters.

² A potential difficulty with this formulation may arise because household size is a discrete variable. This suggests that the scatter of observations is distributed as isolated groups, with gaps between each household size instead of scattered throughout the observed range. Thus, a spline function for a discrete variable is less restrictive because it is freer to move through the sparse parts of the data, as compared with a continuous variable. Consequently, it may lead to spurious curvature. However, judging from the results obtained in the study, this does not seem to be the case. The potential pitfall of creating spurious curvature in the case of a discrete variable may be reduced if the knots are kept at a minimum number, or if the entire observed range is used, so that the scatter of observations can still exert discipline over the curvature of the function.

³ If one expects that a change in household income affects household size and/or vice versa, then it would be appropriate to include an additional variable in the model to account for possible interaction effect between household income and size. Preliminary investigations of the sample data suggest that little relationship exists between household income and size ($r=0.07$). Therefore, it seems reasonable to assume that household income and size are independent in the formulation of the model. Furthermore, the data indicate that household income and size are significantly correlated with the income-size interaction ($r=0.80$ and 0.56 , respectively). The addition of an interaction variable would likely introduce problems of multicollinearity to the statistical model, and, hence, reduce the reliability of the results.

equation is represented by additive splines in household size and income. That is⁴

$$(5) E = \alpha + \sum_{k=1}^3 \beta_k(H-H_1)^k + \sum_{j=2}^m (\beta_{j+2}-\beta_{j+1}) (H-H_j)^3 D_{j-1} + \sum_{k=1}^3 \gamma_k(Y-Y_1)^k + \sum_{i=2}^n (\gamma_{i+2}-\gamma_{i+1}) (Y-Y_i)^3 S_{i-1} + U,$$

where H is the number of persons in the household. E and Y represent household food expenditure and income, respectively, as previously defined. H_1 and H_j , $j=2,3,\dots,m$, define the knots where household size is divided into m segments. Y_1 and Y_i , $i=2,3,\dots,n$, define the knots for household income. D_{j-1} is a dummy variable with $D_{j-1}=1$, if $H \geq H_j$, and 0, otherwise; S_{i-1} represents another set of dummy variables, with $S_{i-1}=1$, if $Y \geq Y_i$, and 0, otherwise.

Thus, equation (5) represents a multiple regression of E on a set of composite variables. Estimates of coefficients in equation (5) are obtained directly from the regression analysis. With this formulation, the analogy of the spline method to the adaptive regression model suggested by Cooley and Prescott becomes evident.⁵ Cooley and Prescott argue that the parameters in most economic models cannot, in general, be expected to be constant over all the observations. In time-series studies, there can be variation over time in the parameters. In cross-section studies, there can be heterogeneity in the parameters across different cross-section units. Since structural relationships of household food expenditure were postulated to change as the level of household income and size change, equation (5) can be regarded as an alternative to varying-parameter models.⁶ This analogy implies that the use of spline functions is an appropriate procedure for application in the present study.

THE DATA AND ESTIMATION PROCEDURE

Household food purchase data from a consumer panel consisting of approximately 120 reporting households in Griffin, Georgia, during the 1975-77 period were used for this analysis. Four beef expenditure categories were examined with separate regression equations: (a) fresh beef (includes all types of beef that were purchased in fresh form, such as ground beef, beef roasts,

steaks, stew beef, short ribs and other beef); (b) ground beef (includes all types, e.g., hamburger, ground chuck, extra lean); (c) beef roasts (includes chuck roast, rib roast and other roasts); and (d) beef steaks (includes round steak, sirloin steak, T-bone steak and other steaks).

To estimate equation (5) statistically, knot locations were specified, using an empirical approach to determine the appropriate position of the knots. Therefore, the knots are located at points separating selected intervals within which the scatter of observations is distributed in similar patterns. In addition, since each additional interval used to fit the function involves an additional variable in the regression equation and loss of an additional degree of freedom in the residual, it is also desirable to keep the number of knots as small as possible. For convenience and simplicity, the same number and position of knots were chosen for each beef expenditure category, although the number and location of the knots may vary among different equations. Based on these considerations, equation (5) was fitted to the sample data of each beef expenditure category with household income divided into three segments, such that $\$1,285 \leq Y < \$10,000$, $\$10,000 \leq Y < \$25,000$, and $Y \geq \$25,000$; and household size was divided into two intervals of $1 \leq H \leq 3$ and $H > 3$.

A spline function of equation (5) was specified and estimated by OLS for each beef expenditure category. Within the framework of least squares, the existence of significantly different fit between two spline models of different degrees in polynomials can be tested. The test procedure involves the F-test, which compares the difference in error sum of squares between the two models. The coefficient of partial determination, partial R^2 , associated with additive splines in income and household size, respectively, can also be calculated and their significance tested by using the F-statistic. In addition, the significance of an individual coefficient can be determined by testing the validity of the occurrence of a structural change at the endpoints of the polynomial segment in a particular interval. For example, in equation (5), the null hypothesis tested is whether $\beta_{j+2} = \beta_{j+1}$, or $\gamma_{i+2} = \gamma_{i+1}$. Because this is a linear restriction, the standard test using the t-statistic is appropriate.

RESULTS AND DISCUSSION

Results obtained by applying spline functions to household beef expenditures are presented in Table 1. The F-test was used to determine whether the additive cubic splines in household income and household size were significantly dif-

⁴ Although the cubic splines were specified both for household income and size, there is nothing about either the theory or the practice that requires all individual segments to be fitted by polynomials of the same degree. Equation (5) can be reduced to quadratic or linear splines simply by adding and deleting the appropriate terms.

⁵ The adaptive regression suggested by Cooley and Prescott allows the constant term to vary in an autoregressive fashion to account for structural change. They argue that for most economic time-series, their model gives better results for economic forecasting in practice.

⁶ Several models for tackling the problem of variational parameters in addition to the adaptive regression model are discussed in Maddala (Chap. 17).

TABLE 1. Partial Regression Coefficients and Standard Errors for Different Beef Expenditure Categories, Griffin Consumer Panel, 1975-1977

Variable ^a	Category			
	Fresh beef	Ground beef	Beef roasts	Beef steaks
Constant	26.6629	-4.4930	8.9631	4.0368
(H-H ₁)	16.6308 (28.6340)	1.4823 (10.1071)	3.9098 (11.2154)	4.4815 (13.4978)
(H-H ₁) ²	9.8929 (8.9669)	5.5605* (3.1610)	2.2989 (3.5122)	1.6977 (4.2269)
(H-H ₂) ² D ₁	-19.1323* (10.4256)	-9.1695** (3.6747)	-4.4585 (4.0835)	-4.1495 (4.9145)
(Y-Y ₁)	0.0068** (0.0029)	0.0097** (0.0031)	0.0020* (0.0011)	0.0031** (0.0014)
(Y-Y ₂)S ₁	-0.0068* (0.0040)		-0.0022 (0.0016)	-0.0022 (0.0019)
(Y-Y ₃)S ₂	0.0096** (0.0037)		0.0043** (0.0015)	0.0038** (0.0018)
(Y-Y ₁) ²		-0.7E-06** (0.22E-06)		
(Y-Y ₂) ² S ₁		0.85E-06** (0.28E-06)		
(Y-Y ₃) ² S ₂		-0.19E-06 (0.17E-06)		
R ²	0.339	0.354	0.202	0.218
F-value	21.874	19.921	10.804	11.949

Note: Numbers in parentheses are estimated standard errors.

* Significant at the 0.10 significance level.

** Significant at the 0.05 significance level.

^a Variable identifications are: H = household size; H₁ = 1; H₂ = 3; D₁ = 1, if H > H₂, and = 0, otherwise; Y = household income; Y₁ = 1,285; Y₂ = 10,000; Y₃ = 25,000; S₁ = 1, if Y ≥ Y₂, and = 0, otherwise; S₂ = 1, if Y ≥ Y₃, and = 0, otherwise.

ferent from quadratic or linear splines for each beef expenditure category. The results indicate that none of the cubic segments is statistically significant at the 0.10 significance level. Except for ground beef, the results also suggest that the quadratic segments for income are not significantly different from the linear segments. Thus, the additive quadratic splines in both income and household size were selected as the statistically appropriate model for ground beef expenditure. For the other beef expenditures categories (i.e., fresh beef, beef roasts, and beef steaks), the statistically appropriate model incorporates a quadratic spline in household size and a linear spline in household income.⁷

The coefficient of multiple determination, R², indicates that the spline function fits the data reasonably well (Table 1). The F-statistic of each regression suggests that variations in household expenditure for beef accounted for by level of income and household size were significant at the 0.001 significance level. Partial R²s calculated for income and household size for each expenditure equation were all highly significant; however, the

TABLE 2. Coefficients of Partial Determination, Partial R², for Different Beef Expenditure Categories by Household Income and Size, Griffin Consumer Panel, 1975-1977

Variable	Category			
	Fresh beef	Ground beef	Beef roasts	Beef steaks
Household income	0.098 (9.230)	0.044 (2.959)	0.082 (7.688)	0.147 (14.656)
Household size	0.228 (25.172)	0.291 (34.828)	0.095 (8.913)	0.046 (4.141)

Note: Numbers in parentheses are calculated F-values. All the partial R²s are significant at the 0.05 significance level.

relative contributions of household income and size in explaining household beef expenditures vary among different equations (Table 2).⁸ The results show that household size is of major importance in determining the levels of household beef expenditures, with the exception of beef steaks. This observation is consistent with Rogers and Green's findings based on the 1972-73 BLS expenditure survey. Comparing the 1972-73 and 1960-61 BLS survey data, Rogers and Green observe that income has become less important in explaining the level of expenditures for food consumed at home.

Perhaps the most revealing results were the contrasts in expenditure patterns for the different types of beef purchased. The flexibility of the spline functions facilitates the examination of the structural differences for different income and household size groups. For a given income level, expenditures for beef generally increase as the size of the household increases, suggesting the prevalence of economies of scale. Since all the quadratic terms in the interval relating to households with more than three persons have the negative sign, economies of scale in beef expenditures as household size increases above three is indicated.⁹ These changes are statistically significant for fresh beef and ground beef. Moreover, the number of households with 3 persons or less is approximately equal to the number with 4 persons or more in the sample. However, larger households (i.e., H > 3), on average, consist of 3.6 adults and teenagers, and 1.6 younger children (under 10 years of age). In contrast, households with 3 or less persons, on average, consist of 2 adults and teenagers, and 0.2 younger children. Thus, the fact that younger children eat less than teenagers and adults may also contribute to the decreasing rate of increase in beef expenditures as household size increases (Huang and Raunikar). Also, this may be attributed in part to the flexibility of serving ground beef and beef roasts in family meals as compared with serving beef steaks.

⁷ Although the polynomial pieces in income were found to be of linear form, the improvement in goodness of fit of the present formulation was found to be statistically significant over the form in which income is treated as an additive linear variable.

⁸ Instead of examining the significance of individual segments of income and household size, partial R²s that compare residual sum of squares associated with spline functions in income and household size, respectively, are appropriate measures for determining the relative contributions of income and household size in explaining household beef expenditure variations.

⁹ Previous research suggests that the impact of an additional member on household food expenditures decreases with an increase in household size (Buse and Salata; Prais and Houthakker; Price). This effect is generally referred to as economies of scale in household food expenditure. Economies of scale in food expenditure may arise in the purchasing, storage, and preparation of foods, and the effect is approximated by the square of the number of persons in the household. Thus, the negative quadratic terms of household size greater than three suggest that for larger households, beef expenditures increased at a decreasing rate with the addition of household members.

In terms of income-expenditure relationships for fresh beef and selected types of beef, the estimated relations are depicted in Figure 1. For a household size of three, Figure 1 indicates that the patterns of fresh beef expenditure in response to income differ among income levels. For example, household expenditures for fresh beef increases rather rapidly as income increases from \$2,000 to \$10,000, remains quite constant between the range of \$10,000 and \$24,000, and again increases as household income increases above \$24,000 (Figure 1). Furthermore, the t-test indicates that, for fresh beef expenditure, the linear segments are statistically significant at either the 0.05- or the 0.10-significance levels, suggesting that the slopes are different among the various income levels (Table 1). This implies that the marginal propensity to consume is much higher for the low- and high-income households, as compared with the middle-income households in the case of fresh beef.

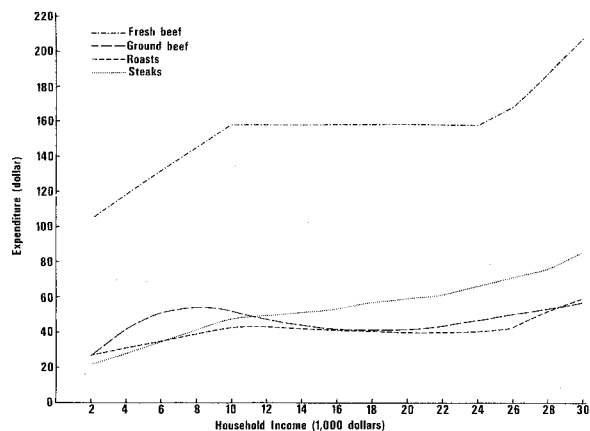


FIGURE 1. Beef Expenditures as a Function of Household Income (Household Size = 3)

The estimated income-expenditure relationships also reveal a sharp contrast in the expenditure patterns among ground beef, beef roasts, and beef steaks (Figure 1). Household food expenditure for ground beef reaches a maximum approximately at the income level of \$8,000 and then gradually declines as income further increases. Even though the ground beef expenditure curve tends to rise slightly toward the higher income levels, this pattern does not seem to be significant. In general, expenditure for beef roasts resembles the ground beef expenditure pattern except for absolute magnitude differences. Nevertheless, a significant structural change, unlike that of ground beef, is found at the higher level of household income. In contrast to ground beef and beef roasts, the expenditure curve for beef steaks shows a steadily increasing pattern as household income increases. Similar to beef roasts, expenditure for beef steaks re-

flects a significant structural change when household income approaches the \$25,000 level.

In summation, the results clearly suggest that beef purchasing behavior changes as household income increases. Households with lower income tend to spend more of their food dollars for ground beef, with no appreciable difference between beef roasts and beef steaks. As income increases, expenditures for beef steaks increase over the income range, with some evidence that expenditures for ground beef decline over the middle-income levels. Hence, for the higher income families, a greater proportion of beef expenditures was spent for beef steaks, with no apparent difference between ground beef and beef roasts. More specifically, the results suggest that different beef expenditure patterns emerge as household income changes. This implies that over the range of the lower incomes (\$2,000–\$10,000), household food expenditures for beef steaks and beef roasts are of about the same magnitudes at each income level. However, as income increases above the low-income levels, household food expenditures for beef steaks are greater than for beef roasts and ground beef, which are of similar magnitudes at income levels above \$10,000.

CONCLUSION

This paper demonstrates the application of spline functions to income-expenditure relationships, using household food purchase data from a consumer panel of approximately 120 families. The use of standard regression procedures provides flexibility and convenience in the estimation of spline functions. More important, the use of spline functions to approximate behaviorally determined income-expenditure relations illustrates that various beef expenditure patterns of structural differences can be investigated without sample stratifications.

The results of this analysis indicate a unique expenditure pattern for each type of beef. Spline functions, as an approximation for estimating income-expenditure relations, provided a procedure that showed that consumers react differently to an income increase at the low-income level than to an income increase at the higher income level. The analysis indicates that, as expenditures for beef change with increased income, the mix of the household's beef expenditures also changes. Thus, expenditure for ground beef was found to be predominant in the low-income households, increasing rapidly as income increased to about the \$8,000 level. In contrast to ground beef, expenditure for beef steaks was more responsive and predominant in the high-income households. Moreover, the results of this study suggest that the relative importance of in-

come and household size in affecting household beef expenditures may vary substantially among different types of beef. Judging from the partial R^2 s, household size was found to be a more significant factor than income in determining expenditures for ground beef and beef roasts, but the opposite was true on expenditure for beef steaks.

Although the analysis has been limited to beef expenditures, the same procedure and principles are applicable to other commodities. It would be desirable to extend the investigation with a national data base and expand the model to incorporate not only the size of household, but also the household composition and other related socioeconomic variables.

REFERENCES

- Agarwala, R. and J. Drinkwater. "Consumption Functions with Shifting Parameters Due to Socio-economic Factors." *Rev. Econ. Statist.* 64(1972):89-96.
- Barth, J., A. Kraft, and J. Kraft. "Estimation of the Liquidity Trap Using Spline Functions." *Rev. Econ. Statist.* 58(1976):218-22.
- Bellman, R. and R. Roth. "Curve Fitting by Segmented Straight Lines." *J. Amer. Statist. Assoc.* 64(1969):1079-84.
- Buse, A. and L. Lim. "Cubic Splines as a Special Case of Restricted Least Squares." *J. Amer. Statist. Assoc.* 72(1977):64-68.
- Buse, R. C. and L. E. Salathe. "Adult Equivalent Scales: An Alternative Approach." *Amer. J. Agr. Econ.* 60(1978):460-68.
- Cooley, T. and E. C. Prescott. "An Adaptive Regression Model." *Int. Econ. Rev.* 14(1973):364-71.
- Forsyth, F. G. "The Relationship Between Family Size and Family Expenditure." *J. Royal Statist. Soc. A* 123(1960):367-93.
- Gallant, A. R. and W. A. Fuller. "Fitting Segmented Polynomial Regression Models Whose Join Points Have to be Estimated." *J. Amer. Statist. Assoc.* 68(1973):144-47.
- Hassan, Z. A. and S. R. Johnson. *Family Expenditure Patterns in Canada*. Agriculture Canada, Ottawa, Econ. Br. Pub. 76/3, 1976.
- Huang, C. L. and R. Raunika. "Estimating the Effect of Household Age-Sex Composition on Food Expenditures." *S. J. Agri. Econ.* 10(1978):151-55.
- Maddala, G. S. *Econometrics*. New York: McGraw-Hill Book Co., 1977.
- McGee, V. and W. T. Carleton. "Piecewise Regression." *J. Amer. Statist. Assoc.* 65(1970):1109-24.
- Poirier, D. J. "Piecewise Regression Using Cubic Splines." *J. Amer. Statist. Assoc.* 68(1973):515-24.
- Prais, S. J. and H. S. Houthakker. *The Analysis of Family Budgets*. Cambridge: Cambridge University Press, 1955.
- Price, D. W. "Unit Equivalent Scales for Specific Food Commodities." *Amer. J. Agr. Econ.* 52(1970):224-33.
- Rogers, D. S. and H. L. Green. "Changes in Consumer Food Expenditure Patterns." *J. Mkt.* 42(1978):14-19.
- Smith, P. L. "Splines as a Useful and Convenient Statistical Tool." *Amer. Statistician* 33(1979):57-62.
- Suits, D. B., A. Mason, and L. Chan. "Spline Functions Fitted by Standard Regression Methods." *Rev. Econ. Statist.* 60(1978):132-39.
- Wold, S. "Spline Functions in Data Analysis." *Technometrics* 16(1974):1-11.