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Discussion Paper No. 11-01

May 2011

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Prizes versus Wages with Envy and Pride*

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May 2011

Abstract

We show that if agents are risk neutral, prizes outperform wages if and only if there is sufficient pride and envy relative to the noisiness of performance. If agents are risk averse, prizes are a necessary supplement to wages (as bonuses).

Keywords: Envy, Pride, Wages, Prizes, Bonus

JEL Classification: C72, D01, D23, L14.

1 Introduction

Prizes are the simplest among contracts that reward agents based on their relative performance : agents' outputs are ranked, and the highest output is given a predetermined prize. On the other hand, *wages* are purely individual contracts, paid to an agent based on his output alone without regard to what others are doing. The motivating power of prizes versus wages has been most famously considered in Lazear and Rosen (1981), who showed that wages can do at least as well as prizes from the principal's point-of-view. In a follow-up paper, Green and Stokey (1983) argued that, if agents are risk averse and if their productivities are sufficiently correlated via a common random shock, then prizes outperform wages. The reason is that the incentives provided by wages are reduced on account of the shock and the risk aversion, while the incentives generated by prizes are invariant of the shock because it is common.

But, without correlation, can prizes still outperform wages?

It turns out that they can, provided agents have "other-regarding" preferences over their rewards. In a pioneering paper, Itoh (2004) characterized optimal contracts in

*First version: October 2005.

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a binary framework with two possible effort levels of the agents (work and shirk) and two possible output levels (success and failure) that are independent across agents. In particular Itoh showed that a prize contract is optimal if agents care about their *status* vis-a-vis others, feeling *envy* (loss in utility) when their reward is lower, and *pride* (gain in utility) when it is higher.[1] Such concern for status seems to be prevalent. Indeed there is a large empirical literature, starting from Easterlin (1975), who argued that happiness depends not just on absolute, but also on relative, consumption .[2]

Itoh's (2004) specification of agents' utilities follows the simple functional form proposed in Fehr and Schmidt (1999), with one important difference. Fehr and Schmidt (1999) postulate "inequity aversion", i.e., any deviation of an agent's reward from another's results in a loss of utility: he feels "compassion" if he is ahead of his rival, and envy if he is behind, losing utility in either case. In Itoh's (2004) framework, compassion is permitted, but not required. Itoh considers envy in conjunction with either pride or compassion.With envy and pride, prizes outperform wages in Itoh's model; when compassion replaces pride, prizes can still be effective, but only when they take the form of a "team prize", shared equally by everyone if, and only if, all of them achieve success simultaneously (see Itoh 2004 for details). Our focus here is on the standard prize, and on the delineation of regimes when such prizes outperform wages or vice-versa, once we step outside the world of binary outputs and allow for noise.

Itoh's result is very sharp: the optimal contract consists of a prize[3] provided there is *any* positive degree of envy and pride (henceforth, E&P), no matter how small. However, this does not seem to be consistent with what is observed in labor markets: even though arguably most people are not immune to E&P, it is not often that they work *purely* for prizes.

Our analysis shows that Itoh's conclusion needs to be modified if his binary framework is replaced by one with a continuum of output levels.(Such a continuum is better suited for many applications, and also permits the modeling of random unbiased noise in output, that is independent of effort.) First suppose there is no E&P. When there is also no noise, wages induce the same incentives as prizes for risk neutral agents, and outperform prizes for risk averse agents. However, with the introduction of noise, wages strictly dominate prizes not just with risk aversion, but even risk neutrality:

[1]See Proposition 4, case (2a), in Itoh (2004). Note also that Itoh his does not use the words "envy" and "pride" — this terminology is ours.

[2]This externality, stemming from status concerns, has been formally modeled along two different lines. The cardinal approach makes utility depend on the *difference* between an individual's consumption and others' consumption (see, e.g., Duesenberry (1949), Pollak (1976)). The ordinal approach makes utility depend on the *individual's rank* in the distribution of consumption (see, e.g., Frank (1985), Direr (2001), and Hopkins and Kornienko (2004)). The model of Itoh (2004) and the one presented in this paper are in the cardinal tradition. The ordinal approach is examined in Dubey and Geanakoplos (2005).

[3]The prize in the optimal contract is given only in the case of clear victory: its recipient must succeed and all his rivals must fail.

since outputs are independent, compensating a worker on the basis of relative performance only distorts his incentives (the shirker wins the prize with positive probability just because of luck).

By a continuity argument, wages also dominate prizes for *small* E&P. The precise analysis is carried out in Section 2, which examines the case of two risk neutral agents.We show that for any level of noise below a certain bound, there is a threshold of E&P such that prizes outperform wages for E&P above the threshold, but wages outperform prizes for E&P below the threshold. Furthermore there is a second, larger bound such that when the noise level exceeds it, no amount of E&P can restore the superiority of prizes. Thus Itoh's conclusion regarding the superiority of prizes in the case of risk neutral agents remains valid, but under two conditions: the level of random noise must be sufficiently low, and E&P must be sufficiently high. In the regime where noise is high, wages are always better than prizes.

Now consider risk averse agents. Even when there is no noise in output, agents with the same skill that work for a prize and exert identical effort still face a 50% uncertainty about who will get it, which is not the case with wages. Thus risk aversion will cause wages to outperform prizes. However Itoh's (2004) intuition, that relative performance should not be ignored, still holds. We present robust conditions in Section 3 under which wages, *supplemented* by prizes (bonuses), constitute an improvement on wages alone. Bonuses are quite common in practice in labor markets.

In the basic verson of our model, we confine ourselves to just two agents, and assume a linear (piece-rate) wage structure. This assumption only strenthens our conclusions on the superiority of wages. Even without this assumption, we find in Section 4 that, in many instances, prizes outperform non-linear wages when noise is small. When there are more than two agents, the main message of Itoh is further reinforced. We show in Section 5 that, no matter how large the noise and how small the E-P, the superiority of prizes is restored when the group of competing agents is big enough, since a shirker will very rarely be lucky enough to pass so many hard-working rivals.

2 Pride and Envy

2.1 The Basic Model

We consider two identical agents with utility

$$u(A, B, e) = A + \beta \max(A - B, 0) - \alpha \max(B - A, 0) - ce,$$

where A is the money the agent gets, B is the money his rival gets, and e is the effort he exerts.[4] The parameters $\beta \geq 0$ and $\alpha \geq 0$ correspond to pride and envy, and $c > 0$ is the marginal disutility of effort.

[4]Utility functions of this form were considered, e.g., in Kirchsteiger (1994), Bolle (2000), Fershtman et al (2003), and Itoh (2004). Fehr and Schmidt (1999) considered this particular utility

Let a finite $\mathcal{E} \subset [0,1]$ be the set of effort levels available to each agent, with $0 \in \mathcal{E}$ and $1 \in \mathcal{E}$. (Thus, we require is that it contain two special levels: $0 \equiv$ "shirking", and $1 \equiv$ "working at full capacity".) If agent $i \in \{1,2\}$ chooses effort level $e_i \in \mathcal{E}$, he produces $e_i + \varepsilon_i^\sigma$ units of output, where ε_1^σ and ε_2^σ are random noises (i.i.d. nonatomic random variables with mean zero), parameterized by a scalar $\sigma > 0$ measuring their noisiness[5]. We denote by G^σ the cumulative distribution function of the random variable $\varepsilon_1^\sigma - \varepsilon_2^\sigma$. Clearly, since ε_1^σ and ε_2^σ have positive variance and are nonatomic i.i.d. random variables, we have $G^\sigma(0) = 1/2$. We suppose that as noise disappears, $\lim_{\sigma\to 0} G^\sigma(t) = 0$ for every $t < 0$, and as noise goes to infinity, $\lim_{\sigma\to\infty} G^\sigma(t) = 1/2$ for every t. We also assume that G^σ is continuous and convex on $[-1,0]$ (i.e., G^σ possesses a density function which is nondecreasing on $[-1,0]$).

To include deterministic output in our analysis, we also allow for $\sigma = 0$, in which case both ε_1^0 and ε_2^0 are fixed at zero.

If each ε_i^σ is *normally distributed*, with mean zero and standard deviation σ, then $\varepsilon_1^\sigma - \varepsilon_2^\sigma$ is also normally distributed, with mean zero and standard deviation $\sqrt{2}\sigma$; thus, $G^\sigma(x) = \frac{1}{2\sigma\sqrt{\pi}} \int_{-\infty}^{x} e^{-\frac{t^2}{4\sigma^2}} dt$.

If the ε_i^σ are *uniformly distributed* on $[-\sigma,\sigma]$, then

$$G^\sigma(x) = \begin{cases} 0, & \text{if } x \le -2\sigma, \\ \frac{1}{8\sigma^2}(x+2\sigma)^2, & \text{if } -2\sigma \le x \le 0, \\ 1 - \frac{1}{8\sigma^2}(-x+2\sigma)^2, & \text{if } 0 \le x \le 2\sigma, \\ 1, & \text{if } x \ge 2\sigma. \end{cases}$$

It is easy to check that all our hypotheses are satisfied for the normal and uniform noise terms.

2.2 The Wage and Prize Games

We will compare two types of contracts that the principal may write. The first is a piece-rate wage contract: each agent is paid rq, when the piece-rate is r and his output is q. In the second contract, a prize P is awarded to the agent with the highest output; in case of ties, a fair coin is tossed to decide who gets the prize. There is always one winner.

Each of these contracts induces, in an obvious manner, a non-cooperative game in which agents' strategies are to choose effort levels. Denote these games with wages, prizes by $\Gamma^\sigma_{\alpha,\beta}(r)$, $\tilde{\Gamma}^\sigma_{\alpha,\beta}(P)$.

function but took β negative, implying that people feel compassion when they are ahead. In conjunction with envy from being behind, their formulation amounts to "inequity aversion".

[5] Our model allows for negative outputs of the agents. This might make sense in certain contexts (think of money managers who make losses). But the case of exclusively non-negative outputs can be incorporated by putting a positive lower bound on effort levels and a suitably small upper bound on the support of the random noise.

The principal wishes to elicit maximal effort from the agents (i.e., $e_1 = e_2 = 1$) at minimal expected cost to himself. Let

$$M^{\sigma}_{\alpha,\beta} = 2\min\left\{r \mid (e_1 = 1, e_2 = 1) \text{ is a Nash equilibrium of } \Gamma^{\sigma}_{\alpha,\beta}(r)\right\},$$

$$\tilde{M}^{\sigma}_{\alpha,\beta} = \min\left\{P \mid (e_1 = 1, e_2 = 1) \text{ is a Nash equilibrium of } \tilde{\Gamma}^{\sigma}_{\alpha,\beta}(P)\right\}.$$

Clearly $M^{\sigma}_{\alpha,\beta}, \tilde{M}^{\sigma}_{\alpha,\beta}$ is the minimal expected payment by the principal needed to elicit maximal effort via wages, prizes[6].

Our first proposition establishes explicit formulae for $M^{\sigma}_{\alpha,\beta}$ and $\tilde{M}^{\sigma}_{\alpha,\beta}$.

Proposition 1. *Let*

$$\psi^{e}_{\sigma} \equiv E\left[\max\{e + \varepsilon^{\sigma}_1 - 1 - \varepsilon^{\sigma}_2, 0\}\right]$$

for every $e \in \mathcal{E}\backslash\{1\}$*, and let*

$$\Delta^{e}_{\sigma} \equiv \psi^{1}_{\sigma} - \psi^{e}_{\sigma}. \tag{1}$$

(It is easy to see that $0 \leq \frac{\Delta^{e}_{\sigma}}{1-e} \leq \frac{1}{2}$*). Denote*

$$\Delta_{\sigma} \equiv \begin{cases} \max_{e\in\mathcal{E}\backslash\{1\}} \frac{\Delta^{e}_{\sigma}}{1-e}, & \text{if } \beta \leq \alpha, \\ \min_{e\in\mathcal{E}\backslash\{1\}} \frac{\Delta^{e}_{\sigma}}{1-e}, & \text{if } \beta > \alpha. \end{cases} \tag{2}$$

Then

$$M^{\sigma}_{\alpha,\beta} = \frac{2c}{1 + \alpha + (\beta - \alpha)\Delta_{\sigma}}, \tag{3}$$

and

$$\tilde{M}^{\sigma}_{\alpha,\beta} = \frac{c}{\frac{1}{2} - G_{\sigma}(-1)} \cdot \frac{1}{1 + \alpha + \beta}. \tag{4}$$

Proof. In the game $\Gamma^{\sigma}_{\alpha,\beta}(r)$ the expected utility of agent i, when he chooses effort level e_i and his rival j chooses effort level e_j, is

$$re_i + \beta rE\left[\max\{e_i + \varepsilon^{\sigma}_i - e_j - \varepsilon^{\sigma}_j, 0\}\right] - \alpha rE\left[\max\{e_j + \varepsilon^{\sigma}_j - e_i - \varepsilon^{\sigma}_i, 0\}\right] - ce_i.$$

In order for $(e_1 = 1, e_2 = 1)$ to be a Nash equilibrium of $\Gamma^{\sigma}_{\alpha,\beta}(r)$, it is necessary and sufficient that (under the piece-rate r) effort level 1 is not less attractive to an agent than any $e \in \mathcal{E}\backslash\{1\}$, given that his rival chooses effort level 1. Thus, we must have

$$r + \beta rE\left[\max\{1 + \varepsilon^{\sigma}_i - 1 - \varepsilon^{\sigma}_j, 0\}\right] - \alpha rE\left[\max\{1 + \varepsilon^{\sigma}_j - 1 - \varepsilon^{\sigma}_i, 0\}\right] - c$$

[6] We have assumed a *single* prize for the best-performing agent. If the loser were also awarded, incentives to exert maximal effort would become smaller. Thus a single prize will, in fact, always be preferred by the principal.

$$\geq re + \beta r E\left[\max\{e+\varepsilon_i^\sigma - 1 - \varepsilon_j^\sigma, 0\}\right] - \alpha r\left[\max\{1+\varepsilon_j^\sigma - e - \varepsilon_i^\sigma, 0\}\right] - ce,$$

i.e.,

$$r(1-\alpha\psi_\sigma^1 + \beta\psi_\sigma^1) - c \geq r(e + \alpha\left(e-1-\psi_\sigma^e\right) + \beta\psi_\sigma^e) - ce,$$

for every $e \in \mathcal{E}\backslash\{1\}$. (Here we use the obvious fact that $-E\left[\max\{1+\varepsilon_j^\sigma - e - \varepsilon_i^\sigma, 0\}\right] + \psi_\sigma^e = e-1$.) Thus, in order to implement $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium of $\Gamma_{\alpha,\beta}^\sigma(r)$ is is necessary and sufficient that r satisfy

$$r \geq \frac{c\,(1-e)}{1-e+\alpha\left(1-e-\Delta_\sigma^e\right)+\beta\Delta_\sigma^e} = \frac{c}{1+\alpha+(\beta-\alpha)\,\frac{\Delta_\sigma^e}{1-\varepsilon}},$$

for every $e \in \mathcal{E}\backslash\{1\}$, and (3) follows.

Next consider the prize game $\tilde{\Gamma}_{\alpha,\beta}^\sigma(P)$. Here the expected utility of agent i, when he chooses effort level e_i and his rival j chooses effort level e_j, is $G^\sigma(e_i - e_j)(1+\beta)P - \left[1-G^\sigma(e_i-e_j)\right]\alpha P - ce_i$ (and, if $e_i = e_j$ and $\sigma = 0$, replace $G^\sigma(e_i - e_j)$ by $\frac{1}{2}$). Thus, in order to implement $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium of $\tilde{\Gamma}_{\alpha,\beta}^\sigma(P)$, it is necessary and sufficient that P satisfy

$$\frac{1}{2}(1+\beta)P - \frac{1}{2}\alpha P - c \geq G^\sigma(e-1)(1+\beta)P - \left[1-G^\sigma(e-1)\right]\alpha P - ce \qquad (5)$$

for every $e \in \mathcal{E}\backslash\{1\}$. The minimal P that satisfies (5) for every $e_i \in \mathcal{E}$ is thus:

$$\frac{c\max_{e\in\mathcal{E}\backslash\{1\}}\frac{1-e}{\frac{1}{2}-G^\sigma(e-1)}}{1+\alpha+\beta}.$$

Since G^σ is convex on $[-1,0]$ and $0 \in \mathcal{E}$, the maximum in this expression is attained for $e = 0$ (i.e., (5) only needs to holds for $e = 0$), and this leads to (4).■

Theorem 1 (**If there is no Envy or Pride, then Wages are never worse than Prizes, and are better than Prizes with sufficient Noise**). *If there is no envy or pride, then wages are never worse than prizes:* $M_{0,0}^\sigma \leq \tilde{M}_{0,0}^\sigma$ *for any* σ. *If* $G^\sigma(-1) > 0$ *then wages outperform prizes:* $M_{0,0}^\sigma < \tilde{M}_{0,0}^\sigma$.

Proof. Immediate from Proposition 1, since $M_{0,0}^\sigma = 2c$ and $\tilde{M}_{0,0}^\sigma = \frac{c}{\frac{1}{2}-G_\sigma(-1)}$. ■

The intuition behind Theorem 1 is straightforward. Suppose $\mathcal{E} = \{0,1\}$. If agent i works ($e_i = 1$) in the prize game and so does his rival, i's expected share of the prize is exactly $P/2$. If he shirks ($e_i = 0$) and his rival still works, his expected payoff does not fall to zero, since with noise he may, with a stroke of luck, win anyway. His expected payoff is $G^\sigma(-1)P$. On net his incentive to work (i.e, the increase in agent's payoff when he switches from shirk to work, ignoring his disutility of effort and assuming that his rival is working) is $P(1/2 - G^\sigma(-1))$. When the wage rate is set equal to $P/2$, his incentive to work in the wage game is $P/2$, no matter what the noise. But if $G^\sigma(-1) > 0$, then $P(1/2 - G^\sigma(-1)) < P/2$. Hence the prize P will need to be more than twice the optimal wage r if $G^\sigma(-1) > 0$, and will never be less.

2.3 The Power of Envy and Pride

Envy makes it easier to motivate the agents to work, via wages or prizes.[7] For wages, this is because shirking entails not only a lesser payment, but also the envy of those who are working and getting paid more.

But the motivating power of envy and pride is even stronger with prizes than with wages. Notice that an agent who shirks not only reduces his (expected) prize, he *increases* the (expected) prize of his rival, generating still more envy.[8] Indeed, Itoh (2004) established in his binary framework (where there are *only* two output levels – success or failure) that, whenever envy and pride are present (no matter to how small a degree), an extreme type of contract is optimal: a prize should be given to the agent who succeeds when his rival fails, and no prize should given in any other circumstance.

The efficacy of a prize can be clearly seen in our model when there is no noise ($\sigma = 0$). Let us assume (as in Itoh (2004) and Fehr and Schmidt (1999)) that the ratio between α and β is constant:

$$\beta = \gamma \cdot \alpha \tag{6}$$

for some fixed $\gamma > 0$. Since the envy parameter α now also determines the pride parameter β, we shall call α the *envy-pride (E-P)* parameter. From (3) and (4) we see at once that, when $\sigma = 0$, the principal needs to pay out total wages $M^0_{\alpha,\gamma\cdot\alpha} = 2c/(1+\alpha)$, but a prize of only $\tilde{M}^0_{\alpha,\gamma\cdot\alpha} = 2c/(1+\alpha+\gamma\alpha)$,in order to motivate both agents to work. Clearly both the required wage bill and the prize become smaller as the E-P parameter α rises. When $\alpha = 0$, $M^0_{0,0} = \tilde{M}^0_{0,0} = 2c$ whereas both $M^0_{\alpha,\gamma\cdot\alpha}$ and $\tilde{M}^0_{\alpha,\gamma\cdot\alpha}$ converge to zero as $\alpha \to \infty$. For high enough α, the principal hardly needs to expend any money at all. But the point is, he expends *less* on prizes than on wages for *any* $\alpha > 0$. When $\gamma = 1$, the maximum savings in absolute terms from using prizes instead of wages occur when $\alpha = 1/\sqrt{2} \approx .70$, yielding a savings of $M^0_{1/\sqrt{2},1/\sqrt{2}} - \tilde{M}^0_{1/\sqrt{2},1/\sqrt{2}} \approx 1.17c - .83c = .34c$. The presence of E-P reduces the wage bill from $2c$ to $1.17c$, or about 41%, and the switch from wages to prizes reduces the total payments from $1.17c$ to $.83c$, or another 29%.

The presence of noise, however, changes the situation in a crucial way, that Itoh's (2004) binary framework cannot account for. As was seen in Theorem 1, if there is no E-P, then a modicum of unbiased noise in output leaves wages intact but harms prizes: if $G^\sigma(-1) > 0$ wages outperform prizes ($M^\sigma_{0,0} < \tilde{M}^\sigma_{0,0}$). By an obvious continuity argument this will also be the case for small but positive E-P ($M^\sigma_{\alpha,\gamma\cdot\alpha} < \tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha}$ for sufficiently small $\alpha > 0$). Thus Itoh's conclusion that prizes are superior to wages for

[7] As envy and pride increase, agents are obviously more easily motivated to work (see Proposition 1). This fact was noted in Grund and Sliwka (2005) in the context of tournaments, and in Nielson and Stowe (2009) for wages.

[8] Notice that this effect relies on the cardinal approach to envy and pride: E&P increase as the gap grows bigger.

any $\alpha > 0$ is not true in the context of a continuum of outputs. Our next theorem shows that, if the noise in output is below a certain bound, a *sufficiently high* E-P is *necessary and sufficient* for prizes to outperform wages.

Theorem 2 (**Prizes outperform Wages iff E-P exceeds Noise-dependent Threshold, provided Noise is not too large**). *Suppose that:*

$$G^{\sigma}(-1) < \min\left(\frac{1}{4}, \frac{\gamma}{2(1+\gamma)}\right). \tag{7}$$

Define the noise-dependent threshold

$$\alpha^* = \frac{2G^{\sigma}(-1)}{\gamma - 2(1+\gamma)G^{\sigma}(-1) + (1-\gamma)\Delta_{\sigma}}. \tag{8}$$

If E-P is greater than the threshold, then prizes outperform wages: $\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} < M^{\sigma}_{\alpha,\gamma\cdot\alpha}$ *if* $\alpha > \alpha^*$. *If E-P is below the threshold, then wages outperform prizes:* $M^{\sigma}_{\alpha,\gamma\cdot\alpha} < \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha}$ *if* $\alpha < \alpha^*$.

Proof. Since $0 \leq \Delta_{\sigma} \leq \frac{1}{2}$, as remarked in the statement of Proposition 1, (7) implies

$$G^{\sigma}(-1) < \frac{\gamma + (1-\gamma)\Delta_{\sigma}}{2(1+\gamma)}, \tag{9}$$

and thus α^* is well-defined and non-negative. The theorem now follows immediately by comparing (3) and (4) in Proposition 1. ■

Example 1. Suppose ε_1^{σ} and ε_2^{σ} are normally distributed with mean zero and standard deviation σ, and $\gamma = 1$. Condition (7) on the noise then amounts to

$$G^{\sigma}(-1) < \frac{1}{4}.$$

Theorem 2 applies whenever $\sigma \leq 1$, since in this case

$$G^{\sigma}(-1) \leq G^{1}(-1) \approx 0.24 < \frac{1}{4}.$$

For instance, if $\sigma = 1/2$ then $G^{\sigma}(-1) \approx 0.08$ and

$$\alpha^* \approx 0.23,$$

i.e. agents need to care one fourth as much about the gap in payments as about their own payment in order for prizes to dominate wages.[9]

[9] Note that, with normally distributed noise, the principal *collects* money from an agent with positive probability in a wage contract (whenever the agent produces negative output, i.e., a "loss"). With prizes, he only *hands out* money. Inspite of this, the principal prefers prizes to wages when the level of E-P is sufficiently high.

Theorem 2 is in line with the results obtained by Itoh (2004) in the *binary* framework: if, as in the binary output case, there is no continuous noise in output (or very little of it, implying $G^{\sigma}(-1) = G^{0}(-1) = 0$), then $\alpha^{*} = 0$, and hence prizes outperform wages with the slightest E-P. More generally, our third Theorem states that given *any* positive E-P α (however small), prizes outperform wages provided the random noise in agents' outputs is sufficiently low (below some upper bound that depends on α) .

Theorem 3 (Given any given positive E-P, Prizes outperform Wages if Noise is small). *Given $\alpha > 0$, there exists $\sigma' > 0$ such that whenever $\sigma \leq \sigma'$, $\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} < M^{\sigma}_{\alpha,\gamma\cdot\alpha}$.*

Proof. Since $\lim_{\sigma\to 0} G^{\sigma}(-1) = 0$ and $\lim_{\sigma\to 0} \Delta^{e}_{\sigma} = 0$ for every $e \in \mathcal{E}\backslash\{1\}$, by Proposition 1

$$\lim_{\sigma\to 0} \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} = \frac{2c}{1+(1+\gamma)\alpha} < \frac{2c}{1+\alpha} = \lim_{\sigma\to 0} M^{\sigma}_{\alpha,\gamma\cdot\alpha}.$$

■

Our next result emphasizes one drawback of prizes, that is not revealed in the framework of Itoh (2004): too much noise destroys their efficacy, no matter how much E-P there may be. The reason is as follows. When wages are based on a noisy measure of output, a worker may be overpaid or underpaid w.r.t. his effort. But as long as the noise is unbiased, and wages are linear, his expected wage is correct. In contrast, when prizes are based on a noisy measure of relative output, the expected payment a worker gets is biased toward $P/2$, diminishing the expected payment to the hard worker and increasing the expected payment to the shirker.

Theorem 4 (If Noise is large, Wages outperform Prizes no matter how high the E-P is). *If*

$$G^{\sigma}(-1) \geq \max\left(\frac{1}{4}, \frac{\gamma}{2(1+\gamma)}\right), \tag{10}$$

then $M^{\sigma}_{\alpha,\gamma\cdot\alpha} < \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha}$ for every $\alpha \geq 0$.

Proof. Since $0 \leq \Delta_{\sigma} \leq \frac{1}{2}$, as remarked in the statement of Proposition 1, (10) implies

$$G^{\sigma}(-1) \geq \frac{\gamma + (1-\gamma)\Delta_{\sigma}}{2(1+\gamma)}. \tag{11}$$

By (11) and Proposition 1 it then follows that

$$\begin{aligned}
\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} &\geq \frac{2\,(1+\gamma)}{1+(\gamma-1)\,\Delta_\sigma}\cdot\frac{c}{1+\alpha\,(\gamma+1)} \\
&= \frac{2}{1+(\gamma-1)\,\Delta_\sigma}\cdot\frac{c}{\frac{1}{1+\gamma}+\alpha} \\
&\geq \frac{2}{1+(\gamma-1)\,\Delta_\sigma}\cdot\frac{c}{\frac{1}{1+(\gamma-1)\Delta_\sigma}+\alpha} \\
&= \frac{2c}{1+\alpha+\alpha\,(\gamma-1)\,\Delta_\sigma} = M^{\sigma}_{\alpha,\gamma\cdot\alpha}.
\end{aligned}$$

■

2.4 Pride vs Envy

Assumption (6), which rigidly ties the envy and pride parameters through the ratio γ, is conducive to the neat statement of Theorem 2, in terms of a *single* threshold value for E-P. But it precludes an inquiry into the separate roles that pride and envy may play in the efficacy of the prize. Now we drop asumption (6).

To gain intuition, consider the simple scenario with no noise ($\sigma = 0$) and binary effort levels $\mathcal{E} = \{0, 1\}$. The incentive to work for wage rate r is

$$r + \alpha r.$$

This the sum of the direct utility of consuming the wage r, and the envy αr suffered when shirking and getting zero while the rival works and gets r. The payoff to an agent who works in the prize game, excluding disutility from work, is $\frac{1}{2}\,(P+\beta P)+\frac{1}{2}\,(-\alpha P)$; if he shirks, he gets $-\alpha P$. Thus the incentive to work with prize P is thus

$$\frac{1}{2}P + \frac{1}{2}\alpha P + \frac{1}{2}\beta P.$$

Setting the prize fund P equal to the total wage bill $2r$, we see that prizes provide an extra incentive of βr. Thus no matter how large or small envy α is, the slightest presence of pride ($\beta > 0$) will cause prizes to outperform wages. And indeed, formally, using Proposition 1 for $\beta > 0$ and $\sigma = 0$ we obtain

$$M^{0}_{\alpha,\beta} = \frac{2c}{1+\alpha} > \frac{c}{\frac{1}{2}}\cdot\frac{1}{1+\alpha+\beta} = \tilde{M}^{\sigma}_{\alpha,\beta}. \tag{12}$$

With noise in output ($\sigma > 0$), a simple continuity argument extends the above claim, showing that prizes outperform wages provided the noise σ is sufficiently small:

Theorem 5 (Given any positive Pride and any Envy, Prizes outperform Wages if Noise is small). *Given any $\beta > 0$ and $\alpha \geq 0$, there exists $\sigma' > 0$ such that $\tilde{M}^{\sigma}_{\alpha,\beta} < M^{\sigma}_{\alpha,\beta}$ whenever $\sigma \leq \sigma'$.*

Proof. Suppose that the assertion is false for some $\beta > 0$ and $\alpha \geq 0$. Then one can find a non-negative sequence $(\sigma_k)_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma_k = 0$, such that

$$M^{\sigma_k}_{\alpha,\beta} \leq \tilde{M}^{\sigma_k}_{\alpha,\beta} \tag{13}$$

for all k. Since $G^{\sigma_k}(-1) \to 0$ and $\Delta_{\sigma_k} \to 0$ as $\sigma_k \to 0$, it follows from (3) and (4) that $\lim_{k\to\infty} M^{\sigma_k}_{\alpha,\beta} = M^0_{\alpha,\beta}$ and $\lim_{k\to\infty} \tilde{M}^{\sigma_k}_{\alpha,\beta} = \tilde{M}^0_{\alpha,\beta}$, and thus taking the limit of both sides of (13) as $k \to \infty$ leads to a contradiction with (12). ■

A more delicate analysis shows that, with *any* fixed (but not too high) level of noise, sufficiently high pride has the effect of making prizes superior to wages (provided the envy parameter does not exceed some fixed multiple of the pride parameter):

Theorem 6 (Prizes outperform Wages with sufficiently high Pride, provided Noise is not too large). *Suppose that $\alpha \leq K\beta$ for some fixed $K \geq 1$, and that*[10]

$$G^{\sigma}(-1) < \frac{1}{2}[1 - \frac{K}{K+1}(1 + \max_{e\in\mathcal{E}\setminus\{1\}} \frac{\Delta^e_{\sigma}}{1-e})]. \tag{14}$$

Then there exists $\beta' > 0$ such that whenever $\beta \geq \beta'$ prizes outperform wages: $\tilde{M}^{\sigma}_{\alpha,\beta} < M^{\sigma}_{\alpha,\beta}$.

Proof. Using the assumption that $\alpha \leq K\beta$ and Proposition 1,

$$\frac{\tilde{M}^{\sigma}_{\alpha,\beta}}{M^{\sigma}_{\alpha,\beta}} = \frac{1 + \alpha + (\beta - \alpha)\,\Delta_{\sigma}}{2(\frac{1}{2} - G_{\sigma}(-1))\,(1 + \alpha + \beta)} = \frac{\frac{1}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}(1 - \Delta_{\sigma}) + \frac{\beta}{\alpha+\beta}\Delta_{\sigma}}{2(\frac{1}{2} - G_{\sigma}(-1))\left(1 + \frac{1}{\alpha+\beta}\right)}$$

$$\leq \frac{\frac{1}{\beta} + \frac{K}{K+1}(1 - \Delta_{\sigma}) + \Delta_{\sigma}}{2(\frac{1}{2} - G_{\sigma}(-1))\left(1 + \frac{1}{(K+1)\beta}\right)} \leq \frac{\frac{1}{\beta} + \frac{K}{K+1}(1 + \max_{e\in\mathcal{E}\setminus\{1\}} \frac{\Delta^e_{\sigma}}{1-e})}{2(\frac{1}{2} - G_{\sigma}(-1))\left(1 + \frac{1}{(K+1)\beta}\right)}.$$

The expression on the right converges to $\frac{\frac{K}{K+1}(1+\max_{e\in\mathcal{E}\setminus\{1\}} \frac{\Delta^e_{\sigma}}{1-e})}{2(\frac{1}{2}-G_{\sigma}(-1))}$ when $\beta \to \infty$, which is below 1 given assumption (14). Thus, there exists $\beta' > 0$ such that whenever $\beta \geq \beta'$, $\frac{\tilde{M}^{\sigma}_{\alpha,\beta}}{M^{\sigma}_{\alpha,\beta}} < 1$, implying that prizes outperform wages. ■

[10] Since $\lim_{\sigma\to 0} G^{\sigma}(-1) = \lim_{\sigma\to 0} \Delta^e_{\sigma} = 0$, the inequality in (14) holds for all sufficiently small σ. For instance, if $K = 1$, (14) holds whenever $G^{\sigma}(-1) < 1/8$.

Remark 1. The situation, when pride is small and envy is large, is murky in the presence of noise. (When there is no noise, recall that no matter what the level of envy α is, the slightest presence of pride ($\beta > 0$) will cause prizes to outperform wages — see (12).) But the following example indicates that large envy by itself may also enable prizes outperform wages. When $\beta \le \alpha$, one can check that

$$\frac{\tilde{M}^{\sigma}_{\alpha,\beta}}{M^{\sigma}_{\alpha,\beta}} = \frac{1+\alpha+(\beta-\alpha)\,\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\,(1+\alpha+\beta)} = \frac{\frac{1}{\alpha+\beta}+\frac{\alpha}{\alpha+\beta}(1-\Delta_\sigma)+\frac{\beta}{\alpha+\beta}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{\alpha+\beta}\right)}$$

$$\le \frac{\frac{1}{\alpha}+(1-\Delta_\sigma)+\frac{1}{2}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{2\alpha}\right)} = \frac{\frac{1}{\alpha}+1-\frac{1}{2}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{2\alpha}\right)}.$$

Since $\lim_{\alpha\to\infty} \frac{\frac{1}{\alpha}+1-\frac{1}{2}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))\left(1+\frac{1}{2\alpha}\right)} = \frac{1-\frac{1}{2}\Delta_\sigma}{2(\frac{1}{2}-G_\sigma(-1))}$, the condition

$$G^{\sigma}(-1) < \Delta_\sigma/4 \tag{15}$$

ensures the existence of $\alpha' > 0$ such that, whenever $\alpha \ge \alpha'$ and $\beta \le \alpha$, prizes outperform wages: $\tilde{M}^{\sigma}_{\alpha,\beta} < M^{\sigma}_{\alpha,\beta}$. While there is no universal condition that would guarantee (15) (unlike the analogous condition in Theorem 6, which was implied by $G^{\sigma}(-1) < \frac{1}{8}$ as pointed out in Footnote 10), (15) holds for normally distributed random noises for some values of the standard deviation σ. For instance, when $\sigma = 0.4$,

$$G^{\sigma}(-1) - \Delta_\sigma/4$$

$$= \frac{1}{2\cdot 0.4\sqrt{\pi}}\int_{-\infty}^{-1} e^{-\frac{t^2}{4\cdot 0.4^2}}\,dt - \frac{1}{4}\left(\frac{1}{2\cdot 0.4\sqrt{\pi}}\int_{0}^{\infty} te^{-\frac{t^2}{4\cdot 0.4^2}}\,dt - \frac{1}{2\cdot 0.4\sqrt{\pi}}\int_{1}^{\infty}(t-1)e^{-\frac{t^2}{4\cdot 0.4^2}}\,dt\right)$$

$$\approx -.0\,156\,8 < 0.$$

Thus, when $\sigma = 0.4$, prizes outperform wages for all sufficiently large α (provided $\beta \le \alpha$).

3 When Agents are not Risk Neutral: the Need for a Bonus

The biggest objection to prizes is that they force agents to face a huge uncertainty about who will get the prize, even if they work hard. It comes to the fore when agents are risk averse. Indeed we will see in Remark 2 that, even when there is no noise in output, risk aversion causes wages to outperform prizes. But Itoh's (2004) intuition regarding prizes still holds in some measure. We find (see Theorem 7) that, with positive envy and pride, wages supplemented by prizes are in many instances an improvement on wages alone. These supplementary prizes are common in practice, in the form of bonuses.

Let the utility function of each of the two agents be given by

$$u(A,B,e) = U(A) - \alpha V(\max\{B-A,0\}) + \beta V(\max\{A-B,0\}) - ce,$$

where, as before, A is the amount paid to the agent, B the amount paid to his rival, α his envy parameter, β his pride parameter, e his choice of effort level, and $c > 0$ the disutility from effort.[11] We assume that U and V are continuously differentiable, and that their derivatives are strictly positive everywhere[12]; furthermore both U and V vanish at zero. We do *not* need to assume that either U or V is concave.

The i.i.d. random noises ε_1^σ and ε_2^σ are now taken to be supported on a compact interval $[-\lambda, \lambda]$, for all σ. As before, G^σ denotes the cumulative distribution function of $\varepsilon_1^\sigma - \varepsilon_2^\sigma$, and we assume that G^σ is convex on $[-1, 0]$.

Up until now we only considered "pure" contracts which could take the form of either a prize P or a piece-rate wage r. Now we allow for *mixed* contracts (P, r) : each agent is paid rq when his output is q, plus a prize (bonus) P if his output is more than his rival's (tossing a coin in case of ties). The contract (P, r) induces[13] a game $\Gamma^\sigma_{\alpha,\beta}(P, r)$ in the obvious manner.

Let $\Pi^\sigma_{\alpha,\beta}$ denote the set of mixed contracts which elicit full effort, i.e.,

$$\Pi^\sigma_{\alpha,\beta} = \left\{(P, r) \in R^2_+ \mid (e_1 = 1, e_2 = 1) \text{ is a Nash equilibrium of } \Gamma^\sigma_{\alpha,\beta}(P, r)\right\}.$$

The principal's payout is $P + 2r$ when $(e_1 = 1, e_2 = 1)$ is played in $\Gamma^\sigma_{\alpha,\beta}(P, r)$. Thus the set of *optimal* contracts is

$$\widetilde{\Pi}^\sigma_{\alpha,\beta} = \arg\min \left\{P + 2r \mid (P, r) \in \Pi^\sigma_{\alpha,\beta}\right\}.$$

With risk neutral agents, there is no need to consider $\Pi^\sigma_{\alpha,\beta}$ because pure contracts are just as good as any mixture: there always exists $(P, r) \in \widetilde{\Pi}^\sigma_{\alpha,\beta}$ such that either $P = 0$ or $r = 0$, at least in the canonical case of $\mathcal{E} = \{0, 1\}$, since then the (unique) incentive constraint defining $\Pi^\sigma_{\alpha,\beta}$ is linear in r and P.

If agents are *not* risk neutral, however, mixed contracts may well beat pure contracts. We leave the exploration of the exact structure of optimal mixed contracts for future research. But we shall delineate two scenarios in which *any optimal mixed contract must necessarily entail a positive bonus*, i.e., $P > 0$ for every $(P, r) \in \widetilde{\Pi}^\sigma_{\alpha,\beta}$.

In the first scenario (Theorem 7 below) envy and pride are fixed at an arbitrary positive level. It turns out that bonuses are needed, provided the noise is sufficiently small. Moreover the optimal contract may often not be a pure prize since, with risk aversion, pure prize tends to be inferior to pure wage (see Remark 2 below). In this case, both the wage and the prize components of the optimal contract (P, r) will be

[11] We could have more generally considered $u(A,B,e) = U(A) - \alpha V_{envy}(\max\{B-A,0\}) + \beta V_{pride}(\max\{A-B,0\}) - ce$ instead of supposing $V = V_{pride} = V_{envy}$. Similar results would obtain but at the cost of more notation.

[12] U is defined on $\mathbb{R}$, while V on $\mathbb{R}_+$.

[13] The underlying components c, U, V of the utility are held fixed, while (P, r), σ, α, and β vary.

positive. In the second scenario (Theorem 8 below), the noise is fixed and not too large. Here for sufficiently high envy and pride, even pure prizes will beat wages, again showing the general need for bonuses.

The intuition for Theorem 7 is roughly as follows. Suppose the two agents are earning only wages. When there is no noise, a hard-working agent knows the wage $w = r \cdot 1$ he will earn for sure. Assuming differentiable utilities, he is nearly risk neutral for small variations in consumption. So consider reducing the piece-rate by ε, and instead awarding a prize of 2ε to the highest performance. Then the expected consumption utility of a hard-working agent stays almost the same. But as we argued before with risk neutrality, the incentive created by envy-pride is greater for the prize than the wage. Thus a small bonus increases incentives without increasing the total expected payout of the principal.[14]

Note that this argument only works for small prizes and small noise. As the prize gets larger, risk aversion kicks in and the prize becomes a less attractive substitute for wages. As noise increases, the luckiest worker, who already has the highest wage and therefore the lowest marginal utility for money, will get the prize, reducing its ex ante consumption utility.

Theorem 7 (Bonus is needed with sufficiently low Noise, for fixed Envy and Pride). *Assume that: (i)*[15] $\alpha \leq \beta$; *(ii) there exist* $B < \infty$ *and* $b > 0$ *such that* $U'(x) \leq B$ *for every* $x \in \mathbb{R}$ *and* $b \leq V'(x) \leq B$ *for every* $x \in \mathbb{R}_+$. *Then there exists* $\sigma' > 0$ *such that* $P > 0$ *for every* $(P, r) \in \widetilde{\Pi}^{\sigma}_{\alpha,\beta}$ *whenever* $\sigma \leq \sigma'$.

Proof. Fix $0 < \alpha \leq \beta$. Suppose to the contrary that there exists a vanishing sequence $\{\sigma_k\}_{k=1}^{\infty}$ of positive numbers and $(P_k^*, r_k^*) \in \widetilde{\Pi}^{\sigma_k}_{\alpha,\beta}$ such that $P_k^* = 0$ (and, w.l.o.g., $r^* \equiv \lim_{k\to\infty} r_k^*$ exists and $0 < r^* < \infty$). Consider $(P^k_{\varepsilon,\delta}, r^k_{\varepsilon,\delta}) = \left(\varepsilon, r_k^* - \frac{1}{2}\varepsilon\,(1+\delta)\right)$. We shall show that there exist small enough $\varepsilon > 0$ and $\delta > 0$ such that $(P^k_{\varepsilon,\delta}, r^k_{\varepsilon,\delta}) \in R^2_+$ elicits full effort from both agents in a Nash equilibrium of $\Gamma^{\sigma_k}_{\alpha,\beta}(P^k_{\varepsilon,\delta}, r^k_{\varepsilon,\delta})$ when k is large (and the noise parameter σ_k is small). Since

$$P_k^* + 2r_k^* = 2r_k^* > 2r_k^* - \varepsilon\delta = P^k_{\varepsilon,\delta} + 2r^k_{\varepsilon,\delta},$$

it will follow that (P_k^*, r_k^*) are not optimal when k is large, a contradiction.

[14] Note that this intuition is somewhat deficient, as a major difference remains between the risk-neutral and the non-risk-neutral cases even when the added prize component is infinitesimal. The *marginal* envy, or pride, may change when a non-risk-neutral agent switches from work to shirk, distorting in a certain way the the incentive to work compared to the exact risk-neutral case. This will be seen in the proof of Theorem 7; it is for this reason that the assumption that $\alpha \leq \beta$ is made in this theorem.

[15] Assumption (i) can be substituted by requiring that V be *convex*.

Now we turn to establishing the existence of the requisite $(P^k_{\varepsilon,\delta}, r^k_{\varepsilon,\delta})$. First notice that in order to implement $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium of $\Gamma^{\sigma_k}_{\alpha,\beta}(P^k_{\varepsilon,\delta}, r^k_{\varepsilon,\delta})$, it is necessary and sufficient for the following incentive conditions to hold:

$$\frac{1}{2}E\left(\left[\begin{array}{c} U(P^k_{\varepsilon,\delta} + r^k_{\varepsilon,\delta}(1+\varepsilon_i^{\sigma_k})) \\ +\beta V(P^k_{\varepsilon,\delta} + r^k_{\varepsilon,\delta}(\varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k})) \end{array}\right] \mid \varepsilon_i^{\sigma_k} > \varepsilon_j^{\sigma_k}\right)$$

$$+\frac{1}{2}E\left(\left[\begin{array}{c} U(r^k_{\varepsilon,\delta}(1+\varepsilon_i^{\sigma_k})) \\ -\alpha V(P^k_{\varepsilon,\delta} + r^k_{\varepsilon,\delta}(\varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k})) \end{array}\right] \mid \varepsilon_i^{\sigma_k} < \varepsilon_j^{\sigma_k}\right) - c$$

$$\geq G^{\sigma_k}(e_i - 1)$$

$$\cdot E\left(\left[\begin{array}{c} U(P^k_{\varepsilon,\delta} + r^k_{\varepsilon,\delta}(e_i+\varepsilon_i^{\sigma_k})) \\ +\beta V(P^k_{\varepsilon,\delta} + r^k_{\varepsilon,\delta}(e_i + \varepsilon_i^{\sigma_k} - 1 - \varepsilon_j^{\sigma_k})) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} > 1 + \varepsilon_j^{\sigma_k}\right)$$

$$+ (1 - G^{\sigma_k}(e_i - 1))$$

$$\cdot E\left(\left[\begin{array}{c} U(r^k_{\varepsilon,\delta}(e_i+\varepsilon_i^{\sigma_k})) \\ -\alpha V(P^k_{\varepsilon,\delta} + r^k_{\varepsilon,\delta}(1 + \varepsilon_j^{\sigma_k} - e_i - \varepsilon_i^{\sigma_k})) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} < 1 + \varepsilon_j^{\sigma_k}\right) - ce_i$$

for every $e_i \in \mathcal{E}\backslash\{1\}$. Denote by $I_k(\varepsilon, \delta, e_i)$ the difference between the left-hand side and the right-hand side of the above inequality. Thus, each of the above incentive conditions is equivalent to

$$I_k(\varepsilon, \delta, e_i) \geq 0. \tag{16}$$

Observe that the derivative of I_k with respect to ε, evaluated at $\varepsilon = 0$, is given by

$$\frac{1}{2}E\left(\left[\begin{array}{c} U'(r_k^*(1+\varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(1+\varepsilon_i^{\sigma_k})\right) \\ +\beta V'(r_k^*(\varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(\varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k})\right) \end{array}\right] \mid \varepsilon_i^{\sigma_k} > \varepsilon_j^{\sigma_k}\right)$$

$$+\frac{1}{2}E\left(\left[\begin{array}{c} U'(r_k^*(1+\varepsilon_i^{\sigma_k}))\left(-\frac{1}{2}(1+\delta)(1+\varepsilon_i^{\sigma_k})\right) \\ -\alpha V'(r_k^*(\varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(\varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k})\right) \end{array}\right] \mid \varepsilon_i^{\sigma_k} < \varepsilon_j^{\sigma_k}\right)$$

$$-G^{\sigma_k}(e_i - 1)$$

$$\cdot E\left(\left[\begin{array}{c} U'(r_k^*(e_i+\varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(e_i+\varepsilon_i^{\sigma_k})\right) \\ +\beta V'(r_k^*(e_i - 1 + \varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(e_i - 1 + \varepsilon_i^{\sigma_k} - \varepsilon_j^{\sigma_k})\right) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} > 1 + \varepsilon_j^{\sigma_k}\right)$$

$$- (1 - G^{\sigma_k}(e_i - 1))$$

$$\cdot E\left(\left[\begin{array}{c} U'(r_k^*(e_i+\varepsilon_i^{\sigma_k}))\left(-\frac{1}{2}(1+\delta)(e_i+\varepsilon_i^{\sigma_k})\right) \\ -\alpha V'(r_k^*(1 - e_i + \varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k}))\left(1 - \frac{1}{2}(1+\delta)(1 - e_i + \varepsilon_j^{\sigma_k} - \varepsilon_i^{\sigma_k})\right) \end{array}\right] \mid e_i + \varepsilon_i^{\sigma_k} < 1 + \varepsilon_j^{\sigma_k}\right),$$

for every $e_i \in \mathcal{E}\backslash\{1\}$. Since the random noises belong to a bounded interval by our assumption in this section, they converge to zero in probability as $\sigma \to 0$. Bearing in mind that $G^{\sigma}(t) \to_{\sigma\to 0} 0$ for $t < 1$ and that U', V' are continuous and bounded, as $k \to \infty$ the above expression converges to:

$$\frac{1}{2}\left(U'(r^*)\left(1 - \frac{1}{2}(1+\delta)\right) + \beta V'(0)\right)$$

$$+\frac{1}{2}\left(U'(r^*)\left(-\frac{1}{2}(1+\delta)\right)-\alpha V'(0)\right)$$

$$-U'(r^*e_i)\left(-\frac{1}{2}(1+\delta)e_i\right)+\alpha V'(r^*(1-e_i))\left(1-\frac{1}{2}(1+\delta)(1-e_i)\right).$$

As $\alpha \leq \beta$, the last expression is bounded from below by

$$-\frac{1}{2}U'(r^*)\delta+\alpha V'(r^*(1-e_i))\left(1-\frac{1-e_i}{2}(1+\delta)\right)$$

$$\geq -\frac{1}{2}B\delta+\alpha b\left(\frac{1}{2}(1-\delta)\right).$$

This is positive for $\delta^* \equiv \frac{\alpha b}{2(\alpha b+B)}$, and so $\frac{\partial}{\partial \varepsilon} I_k(\varepsilon, \delta^*, e_i)\mid_{\varepsilon=0}> 0$ for all large enough k. Thus, since the incentive constraint (16) for any given $e_i \in \mathcal{E}\setminus\{1\}$ holds for $(P_k^*, r_k^*) = (P^k_{0,\delta^*}, r^k_{0,\delta^*})$, it also holds for $(P^k_{\varepsilon,\delta^*}, r^k_{\varepsilon,\delta^*})$ for all large enough k and some $\varepsilon = \varepsilon(k) > 0$. Since $\mathcal{E}$ is finite, there are only finitely many incentive constraints, and thus all of them hold simultaneously for $(P^k_{\varepsilon,\delta^*}, r^k_{\varepsilon,\delta^*})$ for all large enough k and some $\varepsilon = \varepsilon^*(k) > 0$. Therefore $(P^k_{\varepsilon^*(k),\delta^*}, r^k_{\varepsilon^*(k),\delta^*})$ elicits full effort from both agents in a Nash equilibrium of $\Gamma^{\sigma_k}_{\alpha,\beta}(P^k_{\varepsilon^*(k),\delta^*}, r^k_{\varepsilon^*(k),\delta^*})$. As was said, this contradicts the optimality of (P_k^*, r_k^*) when k is large. ■

Remark 2 (Optimal contracts tend to be strictly mixed.) Assume that there are just two effort levels, i.e. $\mathcal{E} = \{0, 1\}$, and that the agents are strictly risk averse, i.e U is strictly concave. If there is no noise ($\sigma = 0$) and no envy-pride ($\alpha = \beta = 0$), it is obvious that the minimal prize that implements $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium in the prize game $\tilde{\Gamma}^0_{0,0}$ is $\tilde{M}^0_{0,0} = U^{-1}(2c)$ and the minimal wage payout by the principal that implements $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium in the wage game $\Gamma^0_{0,0}$ is $M^0_{0,0} = 2U^{-1}(c) < U^{-1}(2c)$. Consequently, for all sufficiently low envy and pride and noise, $M^\sigma_{\alpha,\beta} < \tilde{M}^\sigma_{\alpha,\beta}$. This shows that the set of optimal contracts $\tilde{\Pi}^\sigma_{\alpha,\beta}$ does *not* contain a pure prize contract. Since it does not contain a pure wage contract either (by Theorem 7), we conclude that any optimal contract is strictly mixed: $P > 0$ and $r > 0$ for every $(P, r) \in \tilde{\Pi}^\sigma_{\alpha,\beta}$.

Theorem 9 below supplements Theorem 7 and shows that for any fixed noise (below some reasonably large upper bound) a bonus is again needed if there is enough envy and pride. The intuition for this result is that as envy and pride get very large, the optimal piece-rate (assuming no prize) goes to zero. Since the noise is bounded, the final consumption, being the product of the piece-rate and output, also goes to zero. Thus consumption is practically certain, and the agents become nearly risk neutral. Hence, as in the previous sections, even the pure prize outperforms wages:

Theorem 8 (Prizes outperform Wages even without Risk Neutrality, provided there is sufficient E-P and Noise is not too large). *Suppose that $\beta = \gamma \cdot \alpha$ for a fixed $\gamma > 0$ and that condition (7) of Theorem 2 holds. Then there exists $\alpha' > 0$ such that $\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} < M^{\sigma}_{\alpha,\gamma\cdot\alpha}$ if $\alpha > \alpha'$.*

The following is an obvious corollary of Theorem 8:

Theorem 9 (Bonus is needed with sufficiently high Envy-Pride, given any fixed and not too large Noise). *Under the assumptions of Theorem 8, there exists $\alpha' > 0$ such that, if $\alpha > \alpha'$, then $P > 0$ for every $(P, r) \in \widetilde{\Pi}^{\sigma}_{\alpha,\gamma\cdot\alpha}$.*

Proof of Theorem 8. Let $e \in \mathcal{E}\backslash\{1\}$. Denote by r^e_α the minimal piece-rate at which, in the wage game $\Gamma^{\sigma}_{\alpha,\gamma\cdot\alpha}$, effort level 1 is not less attractive to an agent than effort level e, given that his rival chooses effort level 1. Thus r^e_α is the *smallest* among all non-negative numbers r that satisfy the inequality

$$\begin{aligned}
& EU(r(1+\varepsilon^\sigma_i)) + \gamma\alpha EV(\max\{r\,(1+\varepsilon^\sigma_i) - r(1+\varepsilon^\sigma_j), 0\}) && \text{(17a)}\\
& -\alpha EV(\max\{r\left(1+\varepsilon^\sigma_j\right) - r(1+\varepsilon^\sigma_i), 0\}) - c && \text{(17b)}\\
\geq\; & EU(r(e+\varepsilon^\sigma_i)) + \gamma\alpha EV(\max\{r(e+\varepsilon^\sigma_i) - r(1+\varepsilon^\sigma_j), 0\}) && \text{(17c)}\\
& -\alpha EV(\max\{r(1+\varepsilon^\sigma_j) - r(e+\varepsilon^\sigma_i), 0\}) - ce, && \text{(17d)}
\end{aligned}$$

or

$$\begin{aligned}
& E\left[U(r(1+\varepsilon^\sigma_i)) - U(r(e+\varepsilon^\sigma_i))\right] && \text{(18a)}\\
& +\gamma\alpha E\left[\begin{array}{c} V(\max\{r\,(1+\varepsilon^\sigma_i) - r(1+\varepsilon^\sigma_j), 0\}) \\ -V(\max\{r(e+\varepsilon^\sigma_i) - r(1+\varepsilon^\sigma_j), 0\}) \end{array}\right] && \text{(18b)}\\
& +\alpha E\left[\begin{array}{c} V(\max\{r(1+\varepsilon^\sigma_j) - r(e+\varepsilon^\sigma_i), 0\}) \\ -V(\max\{r(1+\varepsilon^\sigma_j) - r\,(1+\varepsilon^\sigma_i)\,, 0\}) \end{array}\right] && \text{(18c)}\\
\geq\; & c\,(1-e)\,. && \text{(18d)}
\end{aligned}$$

Let

$$K \equiv \min_{0 \leq x \leq 1+2\lambda} V'(x) > 0.$$

Then, for all $r \leq 1$

$$\begin{aligned}
& E\left[U(r(1+\varepsilon^\sigma_i)) - U(r(e+\varepsilon^\sigma_i))\right] && \text{(19a)}\\
& +\gamma\alpha E\left[\begin{array}{c} V(\max\{r\,(1+\varepsilon^\sigma_i) - r(1+\varepsilon^\sigma_j), 0\}) \\ -V(\max\{r(e+\varepsilon^\sigma_i) - r(1+\varepsilon^\sigma_j), 0\}) \end{array}\right] && \text{(19b)}\\
& +\alpha E\left[\begin{array}{c} V(\max\{r(1+\varepsilon^\sigma_j) - r(e+\varepsilon^\sigma_i), 0\}) \\ -V(\max\{r(1+\varepsilon^\sigma_j) - r\,(1+\varepsilon^\sigma_i)\,, 0\}) \end{array}\right] && \text{(19c)}\\
\geq\; & \alpha E\left[\begin{array}{c} V(\max\{r(1+\varepsilon^\sigma_j) - r(e+\varepsilon^\sigma_i), 0\}) \\ -V(\max\{r(1+\varepsilon^\sigma_j) - r\,(1+\varepsilon^\sigma_i)\,, 0\}) \end{array}\right] && \text{(19d)}\\
\geq\; & \alpha\frac{1}{2}Kr\,(1-e) && \text{(19e)}
\end{aligned}$$

Consequently, for all large enough α, substituting $r = \frac{c}{\frac{1}{2}\alpha K(1-e)} \leq 1$ into (17) turns it into a valid inequality by (19), and hence $r^e_\alpha \leq \frac{c}{\frac{1}{2}\alpha K(1-e)}$. (In particular, $\lim_{\alpha\to\infty} r^e_\alpha = 0$.) Substituting $r = r^e_\alpha$ in (18), we can therefore use the first-order (linear) approximation $U'(0)\cdot x$ for $U(x)$, and $V'(0)\cdot x$ for $V(x)$, around 0, to derive an existence of $\tau^e_\alpha \geq 0$ such that

$$\begin{aligned}
& U'(0)\cdot r^e_\alpha (1-e) \\
& +\gamma\alpha V'(0)\cdot r^e_\alpha \left(E\left[\max\{1+\varepsilon^\sigma_i - 1 - \varepsilon^\sigma_j, 0\}\right] - E\left[\max\{e+\varepsilon^\sigma_i - 1 - \varepsilon^\sigma_j, 0\}\right]\right) \\
& +\alpha V'(0)\cdot r^e_\alpha (E\left[\max\{1+\varepsilon^\sigma_j - e - \varepsilon^\sigma_i, 0\}\right] - E\left[\max\{1+\varepsilon^\sigma_j - 1 - \varepsilon^\sigma_i, 0\}\right]) \\
\geq\ & c(1-e) - \tau^e_\alpha
\end{aligned}$$

holds for every α, and $\lim_{\alpha\to\infty}\tau^e_\alpha = 0$. Using the definition of Δ^e_σ in (1), this can be rewritten as

$$(U'(0) + \alpha V'(0) + (\gamma-1)\,\alpha V'(0)\frac{\Delta^e_\sigma}{1-e})\cdot r^e_\alpha \geq c - \frac{\tau^e_\alpha}{1-e},$$

or

$$r^e_\alpha \geq \frac{c - \frac{\tau^e_\alpha}{1-e}}{U'(0) + \alpha V'(0) + (\gamma-1)\,\alpha V'(0)\frac{\Delta^e_\sigma}{1-e}}.$$

The minimal piece rate that implements $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium in the wage game $\Gamma^\sigma_{\alpha,\gamma\cdot\alpha}$ should therefore be at least

$$\frac{c - \tau_\alpha}{U'(0) + \alpha V'(0) + (\gamma-1)\,\alpha V'(0)\Delta_\sigma},$$

for Δ_σ defined in (2) and $\tau_\alpha \equiv \max_{e\in\mathcal{E}\setminus\{1\}} \frac{\tau^e_\alpha}{1-e}$. Consequently,

$$M^\sigma_{\alpha,\gamma\cdot\alpha} \geq \frac{2(c-\tau_\alpha)}{U'(0) + \alpha V'(0) + (\gamma-1)\,\alpha V'(0)\Delta_\sigma} \tag{20}$$

for all sufficiently large α.

Arguing as in the end of the proof of Proposition 1, one can show that the minimal prize $\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha}$ that implements $(e_1 = 1, e_2 = 1)$ as a Nash equilibrium in the prize game $\tilde{\Gamma}^\sigma_{\alpha,\gamma\alpha}$ satisfies

$$\begin{aligned}
& \frac{1}{2}(U(\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha}) + \gamma\alpha V(\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha})) - \frac{1}{2}\alpha V(\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha}) - c & (21)\\
=\ & G^\sigma(-1)(U(\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha}) + \gamma\alpha V(\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha})) - [1-G^\sigma(-1)]\,\alpha V(\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha}). & (22)
\end{aligned}$$

It follows that $\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha} = F^{-1}_\alpha\left(\frac{c}{\frac{1}{2}-G^\sigma(-1)}\right)$, where[16] $F_\alpha(x) \equiv U(x) + (1+\gamma)\,\alpha V(x)$. Since $(1+\gamma)\,\alpha V \leq F_\alpha$ on $\mathbb{R}_+$,

$$\tilde{M}^\sigma_{\alpha,\gamma\cdot\alpha} \leq V^{-1}\left(\frac{1}{(1+\gamma)\,\alpha}\cdot\frac{c}{\frac{1}{2}-G^\sigma(-1)}\right) \leq \frac{\widetilde{K}}{\alpha}$$

[16]Since U and V are strictly increasing, F_α is invertible.

for some $\widetilde{K} > 0$ and for all large enough α (and in particular $\lim_{\alpha\to\infty} \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} = 0$). We can therefore use (21) and the linear approximation $U'(0) \cdot x$ for $U(x)$, and $V'(0) \cdot x$ for $V(x)$, around 0, to derive the existence of a $\delta_\alpha \geq 0$ such that

$$\begin{aligned} & \frac{1}{2}(U'(0) + \gamma\alpha V'(0)) \cdot \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} - \frac{1}{2}(\alpha V'(0)) \cdot \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} - c - \delta_\alpha \\ \leq \;& G^{\sigma}(-1)(U'(0) + \gamma\alpha V'(0)) \cdot \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} - [1 - G^{\sigma}(-1)]\, \alpha V'(0) \cdot \tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} \end{aligned}$$

holds for every α, and $\lim_{\alpha\to\infty} \delta_\alpha = 0$. Thus

$$\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} \leq \frac{c + \delta_\alpha}{(U'(0) + (1+\gamma)\,\alpha V'(0))} \cdot \frac{1}{\frac{1}{2} - G^{\sigma}(-1)} \tag{23}$$

for all sufficiently large α.

It follows from (20) and (23) that

$$\limsup_{\alpha\to\infty} \frac{\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha}}{M^{\sigma}_{\alpha,\gamma\cdot\alpha}} \leq \frac{1 + (\gamma - 1)\,\Delta_\sigma}{2\,(1+\gamma)} \cdot \frac{1}{\frac{1}{2} - G^{\sigma}(-1)} < 1,$$

where the last inequality holds by (7)[17], and thus indeed $\tilde{M}^{\sigma}_{\alpha,\gamma\cdot\alpha} < M^{\sigma}_{\alpha,\gamma\cdot\alpha}$ for all sufficiently large α. ■

4 Non-linear Wages

If we were to allow non-piece-rate contracts, based on more general wage functions, the performance of wage contracts would obviously improve. Thus wages would outperform prizes for sufficiently small levels of envy-pride or sufficiently big noise in output, as implied by Theorems 2 and 4. However we will show here that prizes may outperform even non-linear wages when noise is *small*.

A non-linear wage is given by a function w, defined for all possible outputs. These functions are assumed to be nondecreasing, bounded from above by some constant $W > 0$, and to have the property that expected wages are nonnegative even with zero effort, i.e. $Ew(\varepsilon^{\sigma}_i) \geq 0$ for $i = 1, 2$. This guarantees that agents do not get expected negative wages under any level of effort. Denote by $\bar{M}^{\sigma}_{\alpha,\beta}(w)$ the expected payment by the principal under wage function w, when both agents make effort 1. Also let $\bar{M}^{\sigma}_{\alpha,\beta}$ be the infimum of $\bar{M}^{\sigma}_{\alpha,\beta}(w)$ over all (non-linear) w which implement maximal effort by both agents in Nash equilibrium.

First suppose that there is no random noise at all ($\sigma = 0$): agent i's output precisely equals his effort e_i. It is easy to see that there is an optimal w achieving $\bar{M}^{0}_{\alpha,\beta}$. This w pays zero for all output levels below 1 (i.e. $w(x) = 0$ for $x < 1$), and

[17] Or, more precisely, by (9), as was explained in the proof of Theorem 2.

$w\,(1)$ is the minimal payoff under which no agent i prefers $e_i = 0$ to $e_i = 1$ given that his opponent j is choosing $e_j = 1$. As in the computation of $M^{\sigma}_{\alpha,\beta}$ in the proof of Proposition 1, $w\,(1) = c/(1+\alpha)$, and so

$$\bar{M}^{0}_{\alpha,\beta} = \frac{2c}{1+\alpha} = M^{0}_{\alpha,\beta}.$$

It now follows from (12) that

$$\tilde{M}^{0}_{\alpha,\beta} < \bar{M}^{0}_{\alpha,\beta} \tag{24}$$

for all $\beta > 0$. Thus, when there is no noise, prizes outperform *all* wage contracts for any given positive level of envy and pride. When the noise is sufficiently low, this continues to hold, at least when $\alpha \geq \beta$:

Theorem 10 (Given any positive Pride and Envy, Prizes outperform all Wages if Noise is small). *Given $\alpha \geq \beta > 0$, there exists $\sigma' > 0$ such that $\tilde{M}^{\sigma}_{\alpha,\beta} < \bar{M}^{\sigma}_{\alpha,\beta}$ whenever $\sigma \leq \sigma'$.*

Proof. Suppose that the assertion is false for some $\alpha \geq \beta > 0$. Then one can find two non-negative sequences, $(\sigma_k)_{k=1}^{\infty}$ with $\lim_{k\to\infty} \sigma_k = 0$, and a sequence $(w_k)_{k=1}^{\infty}$ of wage contracts, such that

$$\bar{M}^{\sigma_k}_{\alpha,\beta}(w_k) \leq \tilde{M}^{\sigma_k}_{\alpha,\beta} \tag{25}$$

for all k, and w_k implements maximal effort by both agents in Nash equilibrium when agents' outputs are affected by noises $\varepsilon_1^{\sigma_k}$, $\varepsilon_2^{\sigma_k}$. From (3), (4) and the fact that $G^{\sigma_k}\,(-1) \to 0$ as $\sigma_k \to 0$, we obtain

$$\tilde{M}^{\sigma_k}_{\alpha,\beta} = \frac{c}{\frac{1}{2} - G^{\sigma_k}\,(-1)} \cdot \frac{1}{1+\alpha+\beta} \longrightarrow_{k\to\infty} \frac{2c}{1+\alpha+\beta} = \tilde{M}^{0}_{\alpha,\beta}. \tag{26}$$

On the other hand, we claim that

$$\liminf_{k\to\infty} \bar{M}^{\sigma_k}_{\alpha,\beta}(w_k) \geq \bar{M}^{0}_{\alpha,\beta}. \tag{27}$$

Indeed, there is a subsequence of $(w_k)_{k=1}^{\infty}$ (which w.l.o.g. is taken to be the sequence itself) such that the limit

$$\overline{r} \equiv \lim_{k\to\infty} Ew_k\,(1+\varepsilon_1^{\sigma_k}) = \frac{1}{2} \liminf_{k\to\infty} \bar{M}^{\sigma_k}_{\alpha,\beta}(w_k) \tag{28}$$

exists[18]. Since agent i prefers $e_i = 1$ to $e_i = 0$ given $e_j = 1$ when there is noise $\varepsilon_i^{\sigma_k}$ that affects his (and independently his opponent's) output under wage function w_k,

$$\begin{aligned}
& E(w_k(1+\varepsilon_i^{\sigma_k})) + \beta E(\max\{w_k\,(1+\varepsilon_i^{\sigma_k}) - w_k(1+\varepsilon_j^{\sigma_k}), 0\}) \\
& -\alpha E(\max\{w_k(1+\varepsilon_j^{\sigma_k}) - w_k\,(1+\varepsilon_i^{\sigma_k})\,, 0\}) - c \\
\geq\ & E(w_k\,(\varepsilon_i^{\sigma_k})) + \beta E(\max\{w_k\,(\varepsilon_i^{\sigma_k}) - w_k(1+\varepsilon_j^{\sigma_k}), 0\}) \\
& -\alpha E(\max\{w_k(1+\varepsilon_j^{\sigma_k}) - w_k\,(\varepsilon_i^{\sigma_k})\,, 0\}).
\end{aligned}$$

[18] Note that $\overline{r} < \infty$ since the wage functions are uniformly bounded.

But $E(w_k(\varepsilon_i^{\sigma_k})) \geq 0$ and $\beta \leq \alpha$, and hence it follows that

$$\begin{aligned} & E(w_k(1+\varepsilon_i^{\sigma_k})) - c & (29) \\ \geq \quad & -\alpha E(w_k(1+\varepsilon_j^{\sigma_k})) + (\beta - \alpha) E(\max\{w_k(\varepsilon_i^{\sigma_k}) - w_k(1+\varepsilon_j^{\sigma_k}), 0\}). & (30) \end{aligned}$$

Note that as the functions $(w_k)_{k=1}^{\infty}$ are uniformly bounded and $\lim_{k\to\infty} G^{\sigma_k}(-1) = 0$,

$$\lim_{k\to\infty} E(\max\{w_k(\varepsilon_i^{\sigma_k}) - w_k(1+\varepsilon_j^{\sigma_k}), 0\}) = 0.$$

Thus, taking the limit as $k \to \infty$ of both sides of (29)–(30) yields

$$\overline{r} - c \geq -\alpha \overline{r} \tag{31}$$

for $\overline{r}$ defined in (28). Accordingly, agent i would prefer $e_i = 1$ to $e_i = 0$ (or any other effort level) under piece rate $\overline{r}$ when there is no noise. This shows that

$$\lim_{k\to\infty} \bar{M}_{\alpha,\beta}^{\sigma_k}(w_k) = 2\overline{r} \geq \bar{M}_{\alpha,\beta}^{0},$$

and establishes (27).

Now the combination of (25), (26), and (27) contradicts (24), which proves the theorem. ■

5 Multiple Agents

When there are many agents, the scope for envy and pride increases. Coming first (or last) among one hundred contestants gives more pleasure (or pain) than beating a single opponent. The principal can take advantage of this situation to pay less, whether he uses wages or prizes. We suppress this effect, and assume that agents care only about the *average* of others' receipts.

But multiple agents bring another benefit to prizes alone. With two contestants, an agent who shirks might get lucky and beat the other agent who works. However, with ninety-nine other agents working, the shirker is almost sure to come behind one of them. Thus sufficiently many agents tend to ameliorate the drawback of noise, helping prizes to become more efficacious than wages as long as there is some envy and pride.

Suppose that there are n identical agents. We assume that if agents $1, ..., n$ get $A_1, ..., A_n$ and agent i is exerting effort e_i, then i's utility is

$$u_i(A_1, ..., A_n, e_i) = A_i + \alpha(A_i - \frac{\sum_{j\neq i} A_j}{n-1}) - ce_i$$

(for simplicity, we take $\alpha = \beta$ and refer to their common value as the envy-pride (E-P) paramater). We also assume that the random noise variables $(\varepsilon_k^{\sigma})_{k=1}^{\infty}$ have bounded support $[-\sigma, \sigma]$ and possess a continuously differentiable and strictly positive density function f^σ on it. It is shown in Lemmas 1 and 2 in the Appendix that, under these assumptions, there exists $N > 0$ such that the cumulative distribution function G_n^σ of the random variable $\left(\max_{1\le j\le n, j\ne i} \varepsilon_j^\sigma\right) - \varepsilon_i^\sigma$ is convex on $[-1, 0]$ provided $n \ge N$, and

$$\lim_{n\to\infty} n^2 G_n^\sigma(-1) = 0. \tag{32}$$

As in Proposition 1, one can now see that

$$M_{\alpha,\alpha}^\sigma = \frac{nc}{1+\alpha}$$

and

$$\tilde{M}_{\alpha,\alpha}^\sigma = \frac{c}{1+\frac{n}{n-1}\alpha} \cdot \frac{1}{\frac{1}{n} - G_n^\sigma(-1)}.$$

Furthermore, it is easy to verify the following analogue of Theorem 2:

Theorem 11 (Prizes outperform Wages iff E-P exceeds Noise-dependent Threshold). *Suppose*

$$n^2 G_n^\sigma(-1) < 1, \tag{33}$$

and define

$$\alpha^*(n) = \frac{n(n-1)G_n^\sigma(-1)}{1-n^2 G_n^\sigma(-1)}. \tag{34}$$

If $\alpha > \alpha^*(n)$ *then* $\tilde{M}_{\alpha,\alpha}^\sigma < M_{\alpha,\alpha}^\sigma$, *and if* $\alpha < \alpha^*(n)$ *then* $\tilde{M}_{\alpha,\alpha}^\sigma > M_{\alpha,\alpha}^\sigma$.

Due to (32), condition (33) holds for all large enough n. This means that the scenario described in Theorem 4 in the two-agent case is precluded when there are sufficiently many agents: no matter how large the noise in output is, with sufficiently many agents the E-P will kick in above some threshold, making prizes optimal. Moreover, the minimal level of E-P required for prizes to outperform wages becomes vanishingly small as the number of competitors increases:

Theorem 12 (Given any positive E-P, Prizes outperform Wages if there are enough Competitors). *The threshold* $\alpha^*(n)$ *becomes vanishingly small as* n *increases:* $\lim_{n\to\infty} \alpha^*(n) = 0$. *In particular, given* $\alpha > 0$, *there exists* $n' > 0$ *such that whenever* $n \ge n'$, $\tilde{M}_{\alpha,\alpha}^\sigma < M_{\alpha,\alpha}^\sigma$.

Proof. Follows immediately from (34), given (32). ■

6 Appendix

Assume, as in Section 5, that random variables $(\varepsilon_k^\sigma)_{k=1}^\infty$ have bounded support $[-\sigma,\sigma]$ and possess a continuously differentiable and strictly positive density function f^σ on it.

Lemma 1. There exists $N>0$ such that the cumulative distribution function G_n^σ of the random variable $\left(\max_{1\le j\le n, j\ne i}\varepsilon_j^\sigma\right)-\varepsilon_i^\sigma$ is convex on $(-\infty,0]$ provided $n\ge N$.

Proof. For every $-2\sigma\le t\le 0$, G_n^σ is given by

$$G_n^\sigma(t)=\int_{-\sigma-t}^{\sigma}\Pr(\max_{1\le j\le n, j\ne i}\varepsilon_j^\sigma-\varepsilon_i^\sigma\le t\mid\varepsilon_i^\sigma=y)f^\sigma(y)\,dy=\int_{-\sigma-t}^{\sigma}F^\sigma(y+t)^{n-1}f^\sigma(y)\,dy, \tag{35}$$

where F^σ denotes the cumulative distribution function of each ε_j^σ. Using (35),

$$\begin{aligned}
&\frac{\partial}{\partial t}G_n^\sigma(t)\\
&=\int_{-\sigma-t}^{\sigma}(n-1)F^\sigma(y+t)^{n-2}\frac{\partial}{\partial t}F^\sigma(y+t)\,f^\sigma(y)\,dy\\
&\quad+F^\sigma((-\sigma-t)+t)^{n-1}f^\sigma(-\sigma-t)\\
&=\int_{-\sigma-t}^{\sigma}(n-1)F^\sigma(y+t)^{n-2}f^\sigma(y+t)\,f^\sigma(y)\,dy,
\end{aligned}$$

and (for $n\ge 3$)

$$\begin{aligned}
\frac{\partial}{\partial^2 t}G_n^\sigma(t)&=\int_{-\sigma-t}^{\sigma}(n-1)(n-2)F^\sigma(y+t)^{n-3}f^\sigma(y+t)^2f^\sigma(y)\,dy\\
&\quad+\int_{-\sigma-t}^{\sigma}(n-1)F^\sigma(y+t)^{n-2}\frac{\partial}{\partial t}f^\sigma(y+t)\,f^\sigma(y)\,dy\\
&\quad+(n-1)F^\sigma((-\sigma-t)+t)^{n-2}f^\sigma((-\sigma-t)+t)\,f^\sigma(-\sigma-t)\\
&=(n-1)\int_{-\sigma-t}^{\sigma}F^\sigma(y+t)^{n-3}\left[(n-2)f^\sigma(y+t)^2+F^\sigma(y+t)\frac{\partial}{\partial t}f^\sigma(y+t)\right]f^\sigma(y)\,dy.
\end{aligned}$$

Since $\min_{y\in[-\sigma,\sigma]}f^\sigma(y)>0$, it is clear that

$$(n-2)f^\sigma(y+t)^2+F^\sigma(y+t)\frac{\partial}{\partial t}f^\sigma(y+t)>0$$

for every $y \in [-\sigma - t, \sigma]$ and for all sufficiently large n. We conclude that $\frac{\partial}{\partial^2 t} G_n^\sigma(t) > 0$ and thus the function G_n^σ is convex on $[-2\sigma, 0]$ for all sufficiently large n. Since $G_n^\sigma \equiv 0$ on $(-\infty, -2\sigma]$, G_n^σ is in fact convex on the entire $[-\infty, 0]$. ■

Lemma 2. $\lim_{n\to\infty} n^2 G_n^\sigma(-1) = 0$.

Proof. Using (35) in the proof of Lemma 1,

$$G_n^\sigma(-1) \le F^\sigma(\sigma - 1)^{n-1}$$

if $2\sigma \ge 1$, and $G_n^\sigma(-1) = 0$ otherwise. Since $F^\sigma(\sigma - 1) < 1$, obviously

$$\lim_{n\to\infty} n^2 G_n^\sigma(-1) = 0.$$

■

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