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Revisiting the optimal population size problem under endogenous growth: minimal utility level and finite lives

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Abstract

In this paper, we devise a social criterion in the spirit of the critical utility level of Blackorby-Donaldson (1984) to study an optimal population size problem in an endogenously growing economy populated by workers living a fixed amount of time and without capital accumulation. Population growth is endogenous. The problem is analytically solved, yielding closed-form solutions to optimal demographic and economic dynamics. It is shown that provided the economy is not driven to optimal finite time extinction, the optimal solution is egalitarian for appropriate choices of the critical utility levels: all individuals of any cohort are given the same consumption. The results obtained do not require any priori restriction of the values of the elasticity of intertemporal substitution unlike in several related papers.

Key words: Optimal population size, finite life span, critical utility value, optimal extinction, balanced growth paths

JEL numbers: D63, D64, C61, 040

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1 Introduction

The problem of optimal population size is an old problem which has been tackled by many economists. An early modern reference is the work of Edgeworth (1985) who was the first to claim that total utilitarianism leads to a bigger population size and lower standard of living, leading to the *repugnant conclusions* literature in population ethics as formulated mainly by Parfit (1984). A great deal of papers has been devoted to assess the robustness of such claims to different departures from the benchmark theory. For example, Razin and Sadka (1985) introduced parental altruism in the analysis and found that Edgeworth's claim is corroborated. Dynamic extensions including capital accumulation and growth have been considered. In a remarkable contribution, Palivos and Yip (1993) showed that Edgeworth's claim cannot hold for the realistic parameterizations of an AK model with endogenous population growth. Boucekkine, Fabbri and Gozzi (2011) did also mitigate the scope of Edgeworth's claim in an alternative endogenous growth model without capital where the economy is populated by successive cohorts of workers with given finite life span, say T. In the former paper, total utilitarianism (through the traditional Benthamite social welfare function) leads to higher economic growth rates and lower demographic growth rates compared to average utilitarianism (through the traditional Millian social welfare function) for realistic values of the elasticity of intertemporal substitution. In the latter, it is shown that average utilitarianism always leads to optimal finite time extinction while total utilitarianism may lead to balanced growth paths if individuals' life span is large enough.

A common characteristic of the models of this literature stream is the general form of the social welfare function: $\int_0^\infty u(c,n)\, N^\gamma\, dt$, where $u(c,n)$ is the instantanenous utility function, the population growth rate n features intratemporal altruism, and parameter γ, $0 \leq \gamma \leq 1$, measures the strength of intertemporal altruism as captured by the term N^γ. When $\gamma = 1$, one gets the Benthamite social welfare function while the Millian criterion is obtained when $\gamma = 0$. For example, both forms of altruism are present in Palivos and Yip (1993) while Boucekkine et al. (2011) only rely on intertemporal altruism. This paper takes another avenue: the social welfare function considered is of the Blackorby-Donaldson (1984) type, a criterion putting forward a critical level of utility for individuals. There is a large literature on the relative merits of this criterion (for example, Ng, 1986). In its original formulation, Blackorby-Donaldson criterion does not rely on intertemporal optimization: roughly speaking, it states that a social rule to evaluate population change should make sure that the total welfare of surviving individuals is the highest possible above a certain non-negative critical value, say $\bar{u}$. A first contribution of our paper is to devise a Blackorby-Donaldson intertemporal (or intergenerational) social welfare function. To this end, we rely on the endogenous growth model developed by Boucekkine et al. (2011): in particular we keep the assumptions of no capital accumulation and finite life spans. The latter assumption particularly enriches the model since it opens the door to transition dynamics (including optimal finite time extinction) in contrast to the earlier related AK models due to Palivos and Yip (1993) and Razin and Yuen (1996). A major finding in Boucekkine et al. (2011) is that the Benthamite social welfare function (that is the case $\gamma = 1$ with the notations given above) is the only one yielding an egalitarian solution for all generations, that is assigning the same consumption per capita to all individuals of any cohort. A second contribution of this paper is to study to which extent this property is robust to the introduction of a critical utility level in the spirit of Blackorby and Donaldson (1986), where the critical level can vary in appropriate intervals. The same possibility is opened to assess the robustness of the optimal extinction results obtained in the benchmark model of Boucekkine et al. (2011).

A third contribution of the paper is to extend notably the parametric scope of the results established in Boucekkine et al. (2011). In the latter, if the elasticity of intertemporal substitution of the involved isoelastic instantaneous utility functions is lower than unity, the economy is optimally driven to extinction at finite time, balanced growth is only possible if this elasticity is larger than unity. As raised by Palivos and Yip (1993), the latter case is not corroborated by the data. In this paper, an appropriate introduction of critical utility values allows to get rid of this strong asymetry: in particular, low values of the elasticity of intertemporal substition, more consistent with the data, will not necessarily give rise to finite time extinction. In this sense, this paper significantly generalizes the optimal extinction and optimal growth results previously established by Boucekkine et al. (2011) using the more standard Benthamite social welfare function.

The paper is organized as follows. Section 2 describes the model, with a special emphasis on the specification of the social welfare function in the spirit of Blackorby-Donaldson. Section 3 gives the main results Section 4 concludes. The Appendices A and B are devoted to collect most of the proofs.

2 The model

We shall keep the same demographic and production structures as in Boucekkine, Fabbri and Gozzi (2011). Briefly, we consider a population in which every cohort has a fixed finite life span equal to $T < \infty$ during which individuals remain perfectly active (i.e. they have the same productivity and the same procreation ability). If $N(t)$ denotes the size of population at t, the size $n(t)$ of the cohort born at time t is bounded by $M \cdot N(t)$, where $M > 0$ measures the maximal (time-independent) biological reproduction capacity of an individual, and the demographic dynamics are given by the following delay differential equation (in integral form):

$$N(t) = \int_{t-T}^{t} n(s)\,\mathrm{d}s, \tag{1}$$

and

$$n(t) \in [0, MN(t)], \qquad t \geq 0. \tag{2}$$

The past history of $n(r) = n_0(r) \geq 0$ for $r \in [-T, 0)$ is known at time 0: it is in fact the initial datum of the problem. Note that the constraint (2) together with the positivity of n_0 ensure the positivity of $N(t)$ for all $t \geq 0$.

We consider a closed economy, with a unique consumption good, characterized by a labor-intensive aggregate production function exhibiting constant returns to scale, that is

$$Y(t) = aN(t). \tag{3}$$

Note that by equation (1) individuals born at any date t start working immediately after birth till death. Note also that there is no capital accumulation in our model. Output is partly consumed, and partly devoted to raising the newly born cohort, say rearing costs. In this benchmark we assume that the latter costs are linear in the size of the cohort, which leads to the following resource constraint:

$$Y(t) = N(t)c(t) + bn(t) \tag{4}$$

where $b > 0$, $c(t)$ being consumption per capita. As shown in Boucekkine et al. (2011), considering strictly convex rearing costs would rather reinforce some the conclusions of this paper (notably the optimal extinction results).

We now come to main departure with respect to Boucekkine et al. (2011). In the spirit of Blackorby and Donaldson (1984), we consider the following social welfare function

$$\int_{0}^{+\infty} e^{-\rho t}\,(u(c(t)) - \bar{u})\,N(t)\,\mathrm{d}t, \tag{5}$$

where $\rho > 0$ is the time discount factor, $u\colon (0, +\infty) \to \mathbb{R}$ is a continuous, strictly increasing and concave function, and $\bar{u} \in \mathbb{R}$. Instead, Boucekkine et al. (2011) set $\bar{u}$ to zero and the demographic weight to $N^{\gamma}(t)$, where $\gamma \in [0\ 1]$. In this paper, $\gamma = 1$. Boucekkine et al. (2011) are following indeed the framework opened by Nerlove, Razin and Sadka (1985), and initially extended by Palivos and Yip (1993) to integrate endogenous growth. In such a framework, that is when $\bar{u} = 0$, γ is usually interpreted as an intertemporal altruism parameter; when it is zero, we recover the Millian social welfare function, and when it is equal to one (as in this paper), we get the typical Benthamite social welfare function representing total utilitarianism. In this paper, and in the spirit of Blackorby and Donaldson (1984), we do not vary γ but $\bar{u}$ which we interpret as the required minimal utility level. This formulation of the social welfare function is usually referred to as critical-level utilitarianism. Within the original non-dynamic framework of Blackorby and Donaldson (1984), it has been shown that allowing the maximizer to fix the value of $\bar{u}$ between 0, which corresponds to the typical Benthamite social welfare function, and the maximum average utility, which is less trivially shown to yield the average utilitarianism criterion, the theory is rich enough to study the optimal population size problem. Of course, as always, the theory has its drawbacks (see for example, Ng, 1986) but it is now widely recognized as a major normative theory under population change.

In this paper, we borrow the idea of minimal utility level from Blackorby and Donaldson (1984) and combine it with the intertemporal social welfare functions traditionally adopted in the study of the optimal population size problem. There are two main apparent differences with respect to the initial Blackorby-Donaldson theory: first, the minimal utility level in the latter is on **lifetime** utilities of (surviving) individuals. In our framework, $\bar{u}$ is a critical value for instantaneous utility. It is obvious that a critical value for instantaneous utility implies a critical value for lifetime utility given that all individuals have the same life span, T. Henceforth, this first difference is only apparent. Second, we do allow here for non-positive minimal utility levels while as $\bar{u} \geq 0$ in the original Blackorby and Donaldson's theory. As one will see along the way, this is not a crucial difference: we let the minimal utility level to be possibly negative only to accomodate the cases where utility from consumption is negative, which is the case when for example $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$, with $\sigma > 1$, which is the realistic range of values for σ as the elasticity of intertemporal substitution is admittedly quite low on real data. In such a case, requiring $\bar{u} \geq 0$ is certainly not credible.

3 Minimal utility, finite lives and optimal population size

Before studying the implications of minimal utility levels under finite lives for optimal population size, some technicalities are needed. The role of the sign of the instantaneous utility function, $u(c)$, is in particular emphasized.

3.1 Boundedness of value function

This section is adapted from Section 2.2 in Boucekkine et al. (2011). To assess under which conditions the value function of our problem makes sense, we consider the admissible control defined as $c_{MAX} \equiv 0$ giving the maximal population size allowed associated with $n_{MAX}(t) = \frac{a}{b}N(t)$. Call the trajectory related to such a control $N_{MAX}(\cdot)$. By definition $N_{MAX}(\cdot)$ is a solution to the following delay differential equation (written in integral form):

$$N_{MAX}(t) = \int_{(t-T)\wedge 0}^{0} n_0(s)\,\mathrm{d}s + \frac{a}{b}\int_{(t-T)\vee 0}^{t} N_{MAX}(s)\,\mathrm{d}s. \tag{6}$$

The characteristic equation of such a delay differential equation is

$$z = \frac{a}{b}\left(1 - e^{-zT}\right). \tag{7}$$

It can be readily shown (see e.g. Fabbri and Gozzi, 2008, Proposition 2.1.8) that if $\frac{a}{b}T > 1$, the characteristic equation has a unique strictly positive root ξ. This root belongs to $(0, \frac{a}{b})$ and it is also the root with maximal real part. If $\frac{a}{b}T \leq 1$, then all the roots of the characteristic equation have non-positive real part and the root with maximal real part is 0. In that case, we define $\xi = 0$.

With the maximal root of population dynamics precisely defined, we can prove the following sufficient condition for the value function of the problem to be bounded:

Proposition 3.1 *The following hypothesis*

$$\rho > \xi \tag{8}$$

is sufficient to ensure that the value function

$$V(n_0) := \sup_{\substack{\hat{c}(\cdot) \\ \hat{c}(t)\in[0,a],\ \forall t\geq 0}} \int_0^{+\infty} e^{-\rho t}\left(u(\hat{c}(t)) - \bar{u}\right)\hat{N}(t)\,\mathrm{d}t$$

is finite (here we denoted with $\hat{N}(\cdot)$ the trajectory related to the control $\hat{c}(\cdot)$).

This proposition and its proof are adapted from Proposition 2.1 in Boucekkine et al. (2011). Beside the innocuous fact that the so-called interetemporal altruism parameter, γ, is set to one in the current paper, we have an explicit minimal utility level, $\bar{u}$. Nonetheless, it is trivial that the sufficient condition (8) is also sufficient to "stabilize" the additional term due to minimal utility level.

3.2 A preliminary remark on the sign of the instantaneous utility function and optimal extinction

In this section, we shall make a simple but quite useful methodological remark. Let us fix $\bar{u} = 0$. In such a case, negative instantaneous utility functions, $u(c)$, imply that extinction is optimal at finite time.

Proposition 3.2 *Let $\bar{u} = 0$. If $u(a) \leq 0$ then the optimal strategy is $n^*(\cdot) \equiv 0$ so the system is driven to extinction at finite time T.*

The proof is elementary. Consider the admissible strategy $n^*(\cdot) \equiv 0$. Then the associated welfare value is

$$\int_0^T e^{-\rho t} u(a)\, N^*(t) dt \leq 0$$

Take any other admissible strategy $\hat{n}(\cdot)$. Since $\hat{c}(t) \leq a$ and u is increasing we have $u(c(t)) \leq u(a) \leq 0$ for every $t \in [0, T]$ and $u(c(t)) < u(a) \leq 0$ when $c(t) \neq a$. Moreover it must be, by (1),

$$\hat{N}(t) \geq N^*(t).$$

Then the claim follows.

The economic intuition behind this result is trivial. If instantaneous utility is negative, this means that the option to be alive brings less well-being than not being born, which implicitly gives zero, therefore living is worthless, and procreation is nonsense. A direct corollary of this property is that one of the most common utility functions used in intertemporal macroeconomics turns out disqualified.

Corollary 3.1 *Set $\bar{u} = 0$. If $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$ with $\sigma > 1$ then the system is driven to extinction at finite time T.*

So positivity of the utility function is needed at least for certain consumption values. If one wants to pick reasonable parameter values (for example, $\sigma > 1$), then the use of a common utility function like the one of the Corollary 3.1 is improper (in the sense that it leads to obvious "negative" results). The introduction of a nonzero minimal utility level allows to get rid of the previous limitation. Indeed, call $v(c) = u(c) - \bar{u}$. Then the following condition is enough to make the problem nontrivial:

$$v(a) > 0 \tag{9}$$

From now on, we choose $u(c) = \frac{c^{1-\sigma}}{1-\sigma}$. Suppose $\sigma > 1$, then $u(c) < 0$ for any (finite positive) value of consumption. Accordingly, introduce a negative minimal utility level, $\bar{u} < 0$. Condition (9) then writes as:

$$\frac{a^{1-\sigma}}{1-\sigma} - \bar{u} > 0,$$

or

$$a^{1-\sigma} < R,$$

with $R = (1-\sigma)\bar{u} > 0$. For fixed a, one can see that the condition is checked for R large enough, that is when the negative number $\bar{u}$ is low enough, which makes perfect sense. Otherwise, the system is driven optimally to extinction at finite time. For fixed R (or $\bar{u}$), we get the same outcome if the economy is not enough productive, that when a is too low, which is again sensible. Notice that we obtain things in the same vein when $\sigma < 1$: in this case; $u(c) > 0$, $\bar{u}$ is non-negative and again $R = (1-\sigma)\bar{u} > 0$, but the condition goes obviously in the other direction:

$$a^{1-\sigma} > R.$$

Herafter, we don't restrict σ to be lower or larger than one. But consistently with the development just above, we require $\bar{u}$ to be negative when $\sigma > 1$, and positive in the other case. A synthetic condition (9) is then:

$$\frac{a^{1-\sigma} - R}{1-\sigma} > 0. \tag{10}$$

covering all possible (positive) values for σ, which extends substantially the set of utility functions studied in Boucekkine et al. (2011). The latter considers the special case $\bar{u} = 0$, which in turn restricts σ to be lower than 1 to avoid the trivial finite time extinction outcome.

Finally, notice that thanks to our utility function choice and the notations above, one can write the functional to maximize as:

$$\int_0^{+\infty} e^{-\rho t} \, \frac{c^{1-\sigma} - R}{1-\sigma} \, N(t) \, \mathrm{d}t.$$

3.3 The optimal population size problem

First of all, we have to outline the fact that the mathematics needed to characterize the optimal control problem are rather complex, relying on advanced dynamic programming techniques in infinite-dimensioned hilbertian functional spaces. Technical details are given in the Appendix . More details on the methodology can be found in Fabbri and Gozzi (2008).

Some preliminary manipulations are needed. In particular we need to rewrite the optimal control problem using $n(\cdot)$ as a control instead of $c(\cdot)$: using (3) and (4) we obtain

$$c(t) = \frac{aN(t) - bn(t)}{N(t)}. \tag{11}$$

Since we want per-capita consumption to remain positive, we need $n(t) \leq \frac{a}{b}N(t)$, so that:

$$0 \leq n(t) \leq \frac{a}{b}N(t) \qquad t \geq 0. \tag{12}$$

The previous constraint can be rewritten by requiring $n(t)$ to be in the set

$$\mathcal{V}_{n_0} := \left\{ n(\cdot) \in L^1_{\text{loc}}(0, +\infty; \mathbb{R}_+) \ : \ \text{conditions (12) hold for all } t \geq 0 \right\}. \tag{13}$$

We now come to the function to maximize. With $n(t)$ as the control variable, the functional (5) can be rewritten as

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(\frac{aN(t)-bn(t)}{N(t)}\right)^{1-\sigma} - R}{1-\sigma} N(t)\, \mathrm{d}t \\ = \int_0^{+\infty} e^{-\rho t} \left(\frac{(aN(t) - bn(t))^{1-\sigma}\, N^{\sigma}(t)}{1-\sigma} - \frac{RN(t)}{1-\sigma} \right) \mathrm{d}t. \tag{14}$$

Remember that the functional is bounded under condition (8), $\rho > \xi$, as stated in Proposition 3.1. Next, we define, as in Boucekkine et al. (2011), the adjusted productivity parameter β as:

$$\beta := \frac{a}{b}(1 - e^{-\rho T}). \tag{15}$$

As one can immediately see from the expression above, β is indeed a net productivity parameter when the cost of rearing children and the finite characteristic of human life are accounted for. The longer an individual lives, the more she can contribute to society (since she works all her life), and the more her economic value increases. It can be readily shown that the sufficient condition (8) implies

$$\rho > \beta \quad \Longleftrightarrow \quad \frac{\rho}{1-e^{-\rho T}} > \frac{a}{b}. \tag{16}$$

It is now possible to state the main theorem of this paper characterizing the optimal solution to our problem.

Theorem 3.1 *Suppose that (8) (and then (16)) holds, $R > 0$ and $\sigma > 0$ (with $\sigma \neq 1$). Assume (10) to be satisfied and call α_1^R the unique positive solution (recall that $\rho - \beta > 0$ from (16)) of*

$$0 = g(\alpha) := \alpha(\rho-\beta) + \frac{R}{1-\sigma} - \frac{\sigma}{1-\sigma}\left(\frac{\beta}{a}\right)^{1-1/\sigma} \alpha^{1-1/\sigma}.$$

If

$$\frac{\rho}{\beta} a^{1-\sigma} < \frac{a^{1-\sigma}-R}{1-\sigma} \tag{17}$$

then there exist a unique optimal control/trajectory and, if we define

$$\theta^R := \frac{a}{b}\left(1 - \left(\alpha_1^R \beta\right)^{-\frac{1}{\sigma}}\right) \in \left(0, \frac{a}{b}\right), \tag{18}$$

the optimal control $n^*(\cdot)$ and the related trajectory $N^*(\cdot)$ satisfy the following equation:

$$n^*(t) = \theta^R N^*(t), \tag{19}$$

while optimal trajectory the per-capita consumption is constant and its value is

$$c^*(t) = \frac{aN^*(t) - bn^*(t)}{N^*(t)} = a - b\theta^R \in (0, a). \tag{20}$$

If

$$\frac{\rho}{\beta} a^{1-\sigma} \geq \frac{a^{1-\sigma}-R}{1-\sigma} \tag{21}$$

then there exist a unique optimal control/trajectory: the optimal control $n^*(\cdot)$ is identically zero and we have finite time extinction up to time T.

Theorem 3.1 generalizes in several directions the central Theorem 4.1 in Boucekkine et al. (2011). Among others, the results are now obtained for any (positive and not equal to 1) σ value while they are only established for $\sigma \in (0\ 1)$ in the latter. It is also trivial to check that when setting $R = 0$ (or equivalently $\bar{u} = 0$, one gets exactly the results in Theorem 4.1 in Boucekkine et al. (2011). The interpretation of the findings is similar. Briefly, one has to outline the following main properties.

1. Importantly enough, the theorem identifies the conditions under which finite time extinction is optimal or not. The key condition is condition (17), which can be rewritten as:

$$\beta > \frac{\rho(1-\sigma)}{1 - Ra^{\sigma-1}}.$$

Theorem 3.1 implies that the economy should optimally escape finite time extinction if net productivity, β, is large enough. By definition of β, this is possible when: (i) the productivity parameter a is large enough, (ii) when the cost parameter b is low enough, or (iii) when life span is longer. Concerning parameter a, one can readily observe that the right-hand side of the inequality above is decreasing in a, therefore a rising productivity pattern will definitely ease the escape from finite time extinction. The result that demographic stagnation should be avoided if life span is long enough is fully compatible with unified growth theories in the line of Galor and Moav (2007) and Boucekkine et al. (2002). Finally, one has to observe that the impact of a rising parameter R depends on the position of σ with respect to 1: the right-hand side of the inequality increases (Resp. decreases) when $\sigma < 1$ (Resp. $\sigma > 1$). In other words, a rising R makes finite time extinction more likely when $\sigma < 1$, and less likely when $\sigma > 1$. This is not surprising at all: since $R = (1-\sigma)\bar{u}$, a larger R under $\sigma < 1$ means a larger (positive) critical utility level, and a larger R under $\sigma > 1$ means a very low (negative) critical utility level.

2. As in Boucekkine et al. (2011), the theorem also delivers a fine characterization of the optimal solution when the economy is not driven to finite time extinction. In particular, one can see that in such a case both per-capita consumption and the fertility rate are constant over time. Indeed, by equation (19), one can see that the optimal ratio $\frac{n(t)}{N(t)}$, which can be interpreted as a fertility or a reproduction rate, is constant, equal to θ^R. As long as the economy is driven optimally out of finite time extinction, this egalitarian solution will prevail, whatever the parameter R, that is whatever the minimal utility level $\bar{u}$. This introduces an interesting asymmetry with respect to the study of Boucekkine et al. (2011) who establish that when life spans are finite and $\bar{u} = 0$, the egalitarian solution only occurs when the intermporal altruism parameter, γ, is equal to 1. Of course, our finding does not contradict the latter since in both cases γ is set to 1. This said, it is a remarkable property that for all critical values ruling out finite time

extinction, the egalitarian solution will always be optimal. Of course, the value of $\bar{u}$ (or R) will have a quantitative impact on the optimal egalitarian solution, we shall examine this aspect later.

While optimal consumption per capita and the reproduction rate are constant over time (in case the economy does not go into finite time extinction), the demographic variables, $N(t)$ and $n(t)$ do deliver transition dynamics despite the linear production function. This is due to the finite lifetime assumption as already shown in Boucekkine et al. (2011).[1] In our case, using equation (19) and the state equation (1), one finds the following law of motion for optimal $n(t)$:

$$\begin{cases} \dot{n}(t) = \theta^R \left(n(t) - n(t-T)\right), & \text{for } t \geq 0 \\ n(0) = \theta^R N_0 \\ n(s) = n_0(s), & \text{for all } s \in [-T, 0). \end{cases} \tag{22}$$

This linear delay differential equation is similar to the one analyzed by Boucekkine et al. (2005) and Fabbri and Gozzi (2008). The dynamics depends on the initial function, $n_0(t)$ and on the parameters θ and T in a way that will be described below. Indeed, the next proposition gives the asymptotic behavior of the differential equation above when finite time extinction is ruled out, that is under condition (17). Two configurations are identified: asymptotic extinction and balanced growth, depending on the value of the life span, T.

Proposition 3.3 *Suppose that (8) (and then (16)) holds, $R > 0$ and $\sigma > 0$ (with $\sigma \neq 1$). Assume that (10) and (17) are satisfied, so θ^R, defined in (18), is in $(0\,, \frac{a}{b})$. Then*

- *If $\theta^R T < 1$ then $n^*(t)$ (and then $N^*(\cdot)$) goes to 0 exponentially.*
- *If $\theta^R T > 1$ then the characteristic equation of (22)*

$$z = \theta^R \left(1 - e^{-zT}\right), \tag{23}$$

has a unique strictly positive solution h belonging to $(0, \theta^R)$ while all the other roots have negative real part. Moreover

$$\lim_{t\to\infty} \frac{n^*(t)}{e^{ht}} = \frac{\theta^R}{1 - T(\theta^R - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s$$

and

$$\lim_{t\to\infty} \frac{N^*(t)}{e^{ht}} = \frac{1 - e^{-hT}}{h} \frac{\theta^R}{1 - T(\theta^R - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s > 0$$

[1] A similar property can be established when capital goods have a finite lifetime, see Boucekkine et al. (2005), and Fabbri and Gozzi (2008).

This proposition generalizes Proposition 4.2 in Boucekkine et al. (2011). Indeed, it can be shown (see below) that θ^R is an increasing function of T, there the product $T\theta^R$ is also increasing in T. It follows that Proposition 3.3 identifies a threshold value for life span (by the cut-off condition $T\theta^R = 1$) below which extinction results optimal asymptotically, and above which a balanced growth path set in. It is not difficult to prove that condition (17) in Theorem 3.1 identifies another threshold, lower than the one above identified, under which finite time extinction turns out to be optimal. All these properties can be seen as a generalization of the results of Boucekkine et al. (2011), notably Proposition 4.2 and Corollary 4.2 of the latter paper. The same can be said about the asymptotic characterization of the balanced growth paths generated, notably about the impact of the initial conditions $n_0(t)$ on the long-run levels of demographic variables. To close the analysis, we rather concentrate on the quantitative impact of the minimal utility level (through R) on the shape of these balanced growth paths (BGPs). The next final proposition summarizes the results.

Proposition 3.4 *Under the hypotheses of Proposition 3.3 we have the following facts:*

1. θ^R *is increasing in* T*. If* $\sigma \in (0, 1)$ *then* θ^R *is decreasing in* R*, if* $\sigma > 1$ *then* θ^R *is increasing in* R*.*

2. h *is increasing in* T*. If* $\sigma \in (0, 1)$ *then* h *is decreasing in* R*, if* $\sigma > 1$ *then* h *is increasing in* R*.*

Let us start with the comparative statics with respect to life span, T. Not surprisingly, longer lives imply larger growth rates in the BGPs: in our model, the longer individuals live, the more they can contribute to the economy since they are assumed to work all their lives. Correlatively, longer lives induce larger reproduction rates. This goes against the mechanics of the demographic transition but again this is not a surprising outcome: in our model, longer lives do not come with an additional cost, exogenous increments in life expectancy necessarily increase the profitability of procreation. Now, what is the impact of a larger R? As before, it depends on the position of σ with respect to 1. Suppose $\sigma < 1$, then increasing R means increasing the positive critical utility level $\bar{u}$: in such a case, the economy cannot afford to increase its demographic growth rate h because the critical utility level (and therefore the consumption level) is higher. As a result, the reproduction rate should go down, and so does the BGP growth rate h. When $\sigma > 1$, increasing R means decreasing the negative critical value $\bar{u}$, and we get the reverse mechanism.

4 Conclusion

In this paper, we have solved an optimal population size problem incorporating a social criterion in the spirit of Blackorby-Donaldson (1984) in an endogenous growth model without capital accumulation, where workers have a finite life span. By doing so, we substantially generalize the optimal extinction and optimal growth results previously established by Boucekkine et al. (2011) on a similar economic model using the more standard Benthamite social welfare function.

Of course, the results are established on a stylized model but still the obtention of clear-cut analytical results for optimal dynamics in this case is far from trivial (as one can see in the Appendix). This said, enriching the model to account for capital accumulation, to incorporate natural resource depletion or to endogenize the length of life spans is highly desirable. It is on the top of our research program.

References

A. Bensoussan, G. Da Prato, M.C. Delfour and S.K. Mitter. *Representation and control of infinite dimensional system. Second edition.* Birkhäuser, Boston, 2007.

C. Blackorby and D. Donaldson. Social criteria for evaluating population change. Journal of Public Economics 25, 13-33 (1984)

R. Boucekkine, G. Fabbri and F. Gozzi. Life span and the problem of optimal population size. DP 2011-18, IRES, Universié catholique de Louvain (2011).

R. Boucekkine, O. Licandro, L. Puch and F. del Río. Vintage capital and the dynamics of the AK model. Journal of Economic Theory 120, 39-76 (2005).

R. Boucekkine, D. de la Croix and O. Licandro. Vintage human capital, demographic trends and growth. Journal of Economic Theory 104, 340-375 (2002).

O. Diekmann, S. A. Van Gils, S. M. Verduyn Lunel and H. O. Walther. *Delay equations.* Springer, Berlin, 1995.

G. Fabbri and F. Gozzi. Solving optimal growth models with vintage capital: The dynamic programming approach. Journal of Economic Theory 143, 331-373 (2008).

O. Galor and O. Moav. The Neolithic revolution and contemporary variations in life expectancy . DP 2007-14, Brown University (2007).

J. K. Hale and S. M. Verduyn Lunel. *Introduction to functional differential equations*. Springer-Verlag, Berlin, 1993.

M. Nerlove, A. Razin and E. Sadka. Population size: individual choice and social optima. The Quarterly Journal of Economics 100(2), 321-334 (1985).

Y.K. Ng. Social criteria for evaluating population change: an alternative to the Blackroby-Donaldson criterion. Journal of Public Economics 29, 375-381 (1986).

T. Palivos and C. K. Yip. Optimal population size and endogenous growth. Economics letters 41(1), 107-110 (1993).

D. Parfit. *Reasons and persons*. Oxford University Press, Oxford, 1984.

A. Pazy. *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag, New York, 1983.

A. Razin and C. Yuen. Utilitarian tradeoff between population growth and income growth. Journal of Population Economics 8, 81-87 (1995).

R. B. Vinter and R. H. Kwong. The infinite time quadratic control problem for linear system with state control delays: An evolution equation approch. SIAM Journal of Control and Optimization 19, 139-153 (1981).

A The M^2 setting and the proofs of Theorem 3.1

We denote by $L^2(-T,0)$ the space of all functions f from $[-T,0]$ to $\mathbb{R}$ that are Lebesgue measurable and such that $\int_{-T}^0 |f(x)|^2\,\mathrm{d}x < +\infty$. It is an Hilbert space when endowed with the scalar product $\langle f,g\rangle_{L^2} = \int_{-T}^0 f(x)g(x)\,\mathrm{d}x$. We consider the Hilbert space $M^2 := \mathbb{R}\times L^2(-T,0)$ (with the scalar product $\langle (x_0,x_1),(z_0,z_1)\rangle_{M^2} := x_0z_0 + \langle x_1,z_1\rangle_{L^2}$). Following Bensoussan et al. (2007) Chapter II-4 and in particular Theorem 5.1[2], given an admissible control $n(\cdot)$ and the related trajectory $N(\cdot)$, if we define $x(t) = (x_0(t), x_1(t)) \in M^2$ for all $t\ge 0$ as

$$\begin{cases} x_0(t) := N(t) \\ x_1(t)(r) := -n(t-T-r), \qquad \text{for all } r\in[-T,0), \end{cases} \tag{24}$$

we have that $x(t)$ satisfy the following evolution equation in M^2:

$$\dot{x}(t) = A^* x(t) + B^* n(t). \tag{25}$$

where A^* is the adjoint of the generator of a C_0-semigroup[3] A defined as[4]

$$\begin{cases} D(A) := \{(\psi_0,\psi_1)\in M^2 \ : \ \psi_1\in W^{1,2}(-T,0),\ \psi_0 = \psi_1(0)\} \\ A\colon D(A)\to M^2, \qquad A(\psi_0,\psi_1) := (0, \frac{\mathrm{d}}{\mathrm{d}s}\psi_1) \end{cases} \tag{26}$$

and B^* is the adjoint of $B\colon D(A)\to\mathbb{R}$ defined as $B(\psi_0,\psi_1) := (\psi_1(0) - \psi_1(-T))$. Moreover, using the new variable $x\in M^2$ defined in (24) we can rewrite the welfare functional (14) as

$$\int_0^{+\infty} e^{-\rho t} \frac{\left(\frac{ax_0(t)-bn(t)}{x_0(t)}\right)^{1-\sigma} - R}{1-\sigma} x_0(t)\,\mathrm{d}t$$

$$= \int_0^{+\infty} e^{-\rho t}\left(\frac{(ax_0(t)-bn(t))^{1-\sigma} x_0^\sigma(t)}{1-\sigma} - \frac{Rx_0(t)}{1-\sigma}\right)\mathrm{d}t. \tag{27}$$

Then our optimal control problem of maximizing the welfare functional (14) over the set $\mathcal{V}_{n_0}$ in (13) with the state equation (1) can be equivalently rewritten as the problem of maximizing the functional (27) over the same set $\mathcal{V}_{n_0}$ in (13) and with the state equation (25). The value function V depends now on the new variable x

[2] The result is originally due to Vinter and Kwong (1981)

[3] See e.g. Pazy (1983) for a standard reference to the argument.

[4] $W^{1,2}(-T,0)$ is the set $\{f\in L^2(-T,0) \ : \ \partial_\omega f\in L^2(-T,0)\}$ where $\partial_\omega f$ is the distributional derivative of f.

that can be expressed in term of the datum n_0 using (24) for $t = 0$. The associated Hamilton-Jacobi-Bellman equation for the unknown v is[5]:

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \sup_{n \in [0, \frac{a}{b} x_0]} \left(nBDv(x) + \frac{(ax_0 - bn)^{1-\sigma}}{1-\sigma} x_0^{\sigma} \right) - \frac{R}{1-\sigma} x_0. \quad (28)$$

As far as

$$BDv > a^{-\sigma} b \quad (29)$$

the supremum appearing in (28) is a maximum and the unique maximum point is strictly positive (since $x_0 > 0$) and is

$$n_{max} := \frac{a}{b} \left(1 - \left(\frac{BDv(x)}{a^{-\sigma} b} \right)^{-1/\sigma} \right) x_0 \quad (30)$$

so (28) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{a}{b} x_0 BDv(x) + \frac{\sigma}{1-\sigma} x_0 \left(\frac{1}{b} BDv(x) \right)^{1-\frac{1}{\sigma}} - \frac{R}{1-\sigma} x_0. \quad (31)$$

When

$$BDv \leq a^{-\sigma} b \quad (32)$$

then the supremum appearing in (28) is a maximum and the unique maximum point is $n_{max} := 0$. In this case (28) can be rewritten as

$$\rho v(x) = \langle x, ADv \rangle_{M^2} + \frac{(a^{1-\sigma} - R) x_0}{1-\sigma} \quad (33)$$

We expect that the value function of the problem is a (the) solution of the HJB equation. Since it is not hard to see that the value function is 1-homogeneous, we look for a linear solution of the HJB equation. We have the following proposition.

Proposition A.1 *Suppose that (8) (and then (16)) holds, $R > 0$ and $\sigma > 0$ (with $\sigma \neq 1$). Assume (10) to be satisfied and call α_1^R the unique positive solution (recall that $\rho - \beta > 0$ from (16)) of*

$$0 = g(\alpha) := \alpha(\rho - \beta) + \frac{R}{1-\sigma} - \frac{\sigma}{1-\sigma} \left(\frac{\beta}{a} \right)^{1-1/\sigma} \alpha^{1-1/\sigma}.$$

If

$$\frac{\rho}{\beta} a^{1-\sigma} < \left(\frac{a^{1-\sigma} - R}{1-\sigma} \right) \quad (34)$$

then the function

$$v(x) := \alpha_1^R \left(x_0 + \int_{-T}^{0} x_1(r) e^{\rho r} \, \mathrm{d}r \right) \quad (35)$$

[5] Dv is the Gateaux derivative.

is a solution of (31) in all the points s.t. $x_0 > 0$.

On the other side, if

$$\frac{\rho}{\beta}a^{1-\sigma} \geq \left(\frac{a^{1-\sigma}-R}{1-\sigma}\right) \tag{36}$$

then the function

$$v(x) := \alpha_2^R \left(x_0 + \int_{-T}^0 x_1(r)e^{\rho r}\,\mathrm{d}r\right) \tag{37}$$

where

$$\alpha_2^R = \frac{a^{1-\sigma}-R}{\rho(1-\sigma)}$$

is a solution of (33) in all the points s.t. $x_0 > 0$.

Proof. We try directly the claim. Setting $\phi(r) = e^{\rho r}$, $r \in [-T, 0]$ we see that $Dv(x) = \alpha_i^R(1,\phi)$ for all $x \in M^2$, $ADv(x) = (0, \alpha_i^R\rho\phi)$ and $BDv(x) = \alpha_i^R(1 - e^{-\rho T})$. Let us first look at the case $i = 1$. We observe that (34) implies that

$$\frac{a^{1-\sigma}}{1-\sigma}\left[\frac{\rho}{\beta}(1-\sigma) - \left(1 - \frac{R}{a^{1-\sigma}}\right)\right] = g\left(\frac{a^{1-\sigma}}{\beta}\right) < 0,$$

now, since g is strictly increasing and $g(\alpha_1^R) = 0$, we have $\alpha_1^R > \frac{a^{1-\sigma}}{\beta}$ that is equivalent to (29). Analogously (36) ensures that (32) is satisfied. So the HJB can be written in the form (31) [resp. (33)]. To verify the statement we have only to check directly: the left hand side of (31) [resp. (33)] is equal to $\rho\alpha_i^R\left(x_0 + \langle x_1, \phi\rangle_{L^2}\right)$ while the right hand side is, for $i = 1$

$$\left\langle x_1, \alpha_1^R\rho\phi\right\rangle_{L^2} + \frac{a}{b}x_0\alpha_1^R(1-e^{-\rho T}) + \frac{\sigma}{1-\sigma}x_0\left(\frac{1}{b}\alpha_1^R(1-e^{-\rho T})\right)^{1-\frac{1}{\sigma}} - \frac{R}{1-\sigma}x_0 \\ = \rho\alpha_1^R\left(x_0 + \langle x_1, \phi\rangle_{L^2}\right) \tag{38}$$

thanks to the fact that $g(\alpha_1^R) = 0$. So we have the claim for $i = 1$.

For $i = 2$ the right hand side is (using the expression of α_2^R above)

$$\left\langle x_1, \alpha_2^R\rho\phi\right\rangle_{L^2} + \frac{a^{1-\sigma}}{1-\sigma}x_0 - \frac{R}{1-\sigma}x_0 = \left\langle x_1, \alpha_2^R\rho\phi\right\rangle_{L^2} + \alpha_2^R\rho x_0$$

and this proves the claim for $i = 2$. □

Once we have a solution of the Hamilton-Jacobi-Bellman equation we can prove that it is the value function and use it to find a solution of our optimal control problem in feedback form.

Theorem A.1 *Suppose that (8) (and then (16)) holds,* $R > 0$ *and* $\sigma > 0$ *(with* $\sigma \neq 1$*). Assume (10) to be satisfied. If (34) holds then the function* v *defined in (35) is the value function* V *and there exist a unique optimal control/trajectory.*

The optimal control $n^*(\cdot)$ and the related trajectory $x^*(\cdot)$ satisfy the following equation:

$$n^*(t) = \frac{a}{b}\left(1 - \left(\alpha_1^R \beta\right)^{-\frac{1}{\sigma}}\right) x_0^*(t) = \theta^R x_0^*(t). \tag{39}$$

If (36) is satisfied then the function v defined in (37) is the value function V and there exist a unique optimal control/trajectory. The optimal control $n^*(\cdot)$ is identically zero.

Proof. We do not write the details, it can be proved arguing in a not-too-different way than in Fabbri and Gozzi (2008) Proposition 2.3.2. □

Proof of Theorem 3.1. Follows from Theorem A.1 once we write again $N^*(\cdot)$ instead of $x_0^*(\cdot)$. □

B Other proofs

Proof of Proposition 3.3. Since $n^*(\cdot)$ solves (22) it can be written (see Diekmann et al., 1995, page 34) as a series

$$n^*(t) = \sum_{j=1}^{\infty} p_j(t) e^{\lambda_j t}$$

where $\{\lambda_j\}_{j=1}^{+\infty}$ are the roots of of the characteristic equation (23) (studied in Fabbri and Gozzi, 2008, Proposition 2.1.8) and $\{p_j\}_{j=1}^{N}$ are $\mathbb{C}$-valued polynomial. If $\theta^R T > 1$, there exists a unique strictly positive root $\lambda_1 = h$. Moreover $h \in (0, \theta^R)$ and it is also the root with biggest real part (and it is simple). The polynomial p_1 associated to h is a constant (since h is simple) and can be computed explicitly (see for example Hale and Lunel (1993) Chapter 1, in particular equations (5.10) and Theorem 6.1) obtaining

$$p_1(t) \equiv \frac{\theta^R}{1 - T(\theta^R - h)} \int_{-T}^{0} \left(1 - e^{(-s-T)h}\right) n_0(s)\, \mathrm{d}s$$

this gives the limit for $n(t)^*/e^{ht}$. Observe that $\left(1 - e^{(-s-T)h}\right)$ is always positive for $s \in [-T, 0]$ and the constant $\frac{1}{1-T(\theta^R-h)}$ can be easily proved to be positive too so the expression above is positive. The limit for $N(t)^*/e^{ht}$ follows from the relation $N^*(t) = \int_{t-T}^{t} n^*(s)\, \mathrm{d}s$.

If $\theta^R T < 1$ each λ_j, for $j \geq 2$, has negative real while $\lambda_1 = 0$ is the only real root. But again if we compute explicitly the polynomial p_1 (again a constant value) related to the root 0 we have

$$p_1(t) \equiv \frac{\theta^R N_0 + (-\theta^R) \int_{-T}^{0} n_0(r)\, \mathrm{d}r}{1 + \theta T} = \frac{\theta^R N_0 - \theta^R N_0}{1 + \theta^R T} = 0.$$

so only the contributions of the roots with negative real parts remain. This concludes the proof. □

Proof of Proposition 3.4. We use the implicit function theorem. From the definition of α_1^R we have

$$0 = g(\alpha_1^R) := \alpha_1^R(\rho - \beta) + \frac{R}{1-\sigma} - \frac{\sigma}{1-\sigma}\left(\frac{\beta}{a}\right)^{1-1/\sigma}(\alpha_1^R)^{1-1/\sigma}. \tag{40}$$

If we call $K := \alpha_1^R\beta$ we can rewrite (40) as

$$0 = \tilde{g}(\beta, K) = K\rho/\beta - K + \frac{R}{1-\sigma} - \frac{\sigma}{1-\sigma}\left(\frac{K}{a}\right)^{1-1/\sigma}.$$

We compute now

$$\frac{\mathrm{d}K}{\mathrm{d}\beta} = -\frac{\partial\tilde{g}}{\partial\beta}\left(\frac{\partial\tilde{g}}{\partial K}\right)^{-1} = +\frac{\rho}{\beta^2}K\left(\frac{\rho}{\beta} - 1 + \frac{K^{-1/\sigma}}{a^{1-1/\sigma}}\right)^{-1} > 0$$

(the last inequality follows from the fact that $\rho > \beta$). Finally, thanks to the form of θ^R given in (18), $\frac{\mathrm{d}\theta^R}{\mathrm{d}\beta} > 0$ and then, since β is strictly increasing in T, $\frac{\mathrm{d}\theta^R}{\mathrm{d}T} > 0$. The behavior of K and then θ^R on R follows using the same arguments. This concludes the first part of the claim.

To study the behavior of h on R observe that $\frac{\partial h}{\partial\theta^R} > 0$ and then the claim follows from the study of the behavior of θ^R on R.

To prove that h is increasing in T one can first observe (see below) that $\frac{\mathrm{d}h}{\mathrm{d}T} > 0$. Then using this fact together with $\frac{\mathrm{d}\theta^R}{\mathrm{d}T} > 0$ (as we have already proved) we have the claim.

To prove that $\frac{\mathrm{d}h}{\mathrm{d}T} > 0$ we use again the implicit function theorem: define

$$F(\lambda, T) = \theta^R(T)(1 - e^{-T\lambda}) - \lambda.$$

Given T such that $\theta^R(T)T > 1$ one has that $F(\lambda, T)$ is concave in λ, $F(0, T) = 0$ and $F(h, T) = 0$ for some $h \in (0, \theta(T))$. So it must be

$$\frac{\partial}{\partial\lambda}F(\lambda, T)\bigg|_{\lambda=h} = \theta^R(T)Te^{-Th} - 1 < 0.$$

Moreover one can easily see that $\frac{\mathrm{d}\theta^R}{\mathrm{d}T} > 0$, so we have:

$$\frac{\partial F(h, T)}{\partial T} = \frac{\mathrm{d}\theta^R}{\mathrm{d}T}(1 - e^{-Th}) + \theta^R(T)he^{-Th} > 0$$

So, by the implicit function theorem we have

$$\frac{\mathrm{d}h}{\mathrm{d}T} = -\frac{\partial F}{\partial T}\left(\frac{\partial F}{\partial\lambda}\bigg|_{\lambda=h}\right)^{-1} > 0$$

and this concludes the proof. □