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Long-Run Growth and the Evolution of Technological Knowledge

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Abstract: The long-run evolution of per-capita income exhibits a structural break often associated with the Industrial Revolution. We follow Mokyr (2002) and embed the idea that this structural break reflects a regime switch in the evolution of technological knowledge into a dynamic framework, using *Airy* differential equations to describe this evolution. We show that under a non-monotonous income-population equation, the economy evolves from a Malthusian to a Post-Malthusian Regime, with rising per-capita income and a growing population. The switch is brought about by an acceleration in the growth of technological knowledge. The demographic transition marks the switch into the Modern Growth Regime, with higher levels of per-capita income and declining population growth.

Keywords: Industrial Revolution, Technological Change, Malthus, Demographic Transition.

JEL-Classification: J11, O11, O33, O40.

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Long-Run Growth and the Evolution of Technological Knowledge

1 Introduction

The evolution of the per-capita income of today's industrialized economies is characterized by a sharp structural break. Before the break, per-capita income was fairly constant for a very long time. It then took off, and it has been growing at a nearly constant rate ever since. Following Lucas (2002), we use the term *Industrial Revolution* to refer to this structural break – the onset of sustained growth.

Any growth theory that aims to consistently account for this evolution faces the question of how to come to grips with the structural break. The conceptual answer provided by most growth models involves means to overcome corner solutions that characterize the behavior of households and firms. For instance, the models of Becker, Murphy, and Tamura (1990) or Lucas (2002) have two steady states that are meant to capture Malthusian stagnation and modern growth. The switch between these steady states requires an exogenous shock, e. g. a rise in the return on investment in human capital.

Studies that make the transition between a Malthusian and a modern growth regime explicit include those of Galor and Weil (2000), Jones (2001), and Hansen and Prescott (2002). In the latter paper, the corner solution concerns firms that initially employ only a land-intensive technology, leaving a capital-intensive technology idle. Exogenous technical change that is biased towards the productivity of the capital-intensive sector raises its relative productivity. The Industrial Revolution occurs when the efficient allocation has some resources being employed in the capital intensive sector. The corner solution in the use of the aggregate technology is overcome, and biased technical change continues to boost the industrialized sector.

Our paper points to an alternative explanation of the structural break, which does not rely on corner solutions. Rather, we support the view that the evolution of per-capita income reflects two phases of the evolution of technological knowledge that are linked by a smooth transition.

Before the take-off, the path of technological knowledge is approximately without a trend. No trend does not mean gridlock. Indeed, economic historians account for noticeable vicissitudes in the level of accessible technological knowledge. On the one hand, without appropriate storage devices, knowledge got lost over time. For instance, Landes (2006, p. 3) argues that Europe of the tenth century "had lost much of the science it had once possessed." On the other hand, there were inventions, e. g., eyeglasses and the mechanical clock around the end of the thirteenth century, that temporarily boosted the growth of technological knowledge (Landes, 1998, p. 46 f.).

Yet, these early inventions were not sufficient to induce sustained growth of technological knowledge. Recently, Joel Mokyr (2002, 2005) has provided a convincing explanation for this phenomenon. Mokyr considers the evolution of the epistemic base of technological knowledge an essential prerequisite for a sustained process of knowledge accumulation. Moreover, he argues that before 1800 this base was not sufficiently large to allow for a cumulative growth of technological knowledge: "Although new techniques appeared before the Industrial Revolution, they had narrow epistemic bases and thus rarely if ever led to continued and sustained improvements. [...] The widening of the epistemic bases after 1800 signals a phase transition or a regime change in the dynamics of useful knowledge" (Mokyr, 2002, p. 19 f.). Hence, before the take-off, early inventions contributed to the enlargement of the epistemic base as a byproduct. In doing so, they prepared the evolution of knowledge for the take-off. After a smooth take-off, technological knowledge evolves cumulatively.

To capture these properties, we postulate an evolution of technological knowledge following an Airy-type differential equation. This specification is based on Airy functions (named after George Airy, 1801-1892), which commonly appear in physics, especially in quantum mechanics and electromagnetics (cf. Antosiewicz, 1972). We use an Airy-type differential equation to describe the evolution of a summary statistic that captures the effect of technological knowledge on the evolution of aggregate output. Under this differential equation, the character of the evolution of knowledge smoothly changes over time: a long period of approximate stagnation is followed by a take-off, and then by sustained growth. This is the appealing property of the Airy differential equation. Yet, to allow it to serve as a suitable device for our context, we amend the differential equation. The resulting Quasi-Airy specification preserves the central dynamic properties of Airy-type differential equations and eliminates some undesirable features. Under the Quasi-Airy specification, technological knowledge evolves in early periods through oscillations around some average level; in later periods it rises monotonically and approaches an exponential growth path. Since the evolution is continuous, there is a critical period in which a regime change occurs. We refer to this date as the take-off or the Industrial Revolution.

We probe our Quasi-Airy specification in a simple dynamic macroeconomic model. This setting is able to qualitatively replicate the stylized facts concerning the evolution of per-capita income and population that today's industrialized countries have experienced over the last thousands of years. The model is Malthusian in spirit, with an aggregate production technology that uses knowledge, labor, and a fixed amount of land. Moreover, we posit that population growth depends on per-capita income and ask what properties of this function are sufficient to account for the stylized facts. In our framework, during the period of Malthusian stagnation, per-capita income and population oscillate around constant levels. These averages play the same role as in Malthus' (1798) exposition: If changes in the technology and in the availability of land were absent, per-capita income and population would converge towards these levels.

After the Industrial Revolution, the evolution of technological knowledge becomes cumulative. As a consequence, the economy leaves the state of Malthusian stagnation and reaches higher levels of per-capita income. Sustained growth of per-capita income may or may not materialize, depending on how the asymptotic growth rate of technological knowledge relates to the asymptotic population growth rate. Intuitively, if the former exceeds the latter, then there is scope for sustained growth of per-capita income.

For the regime following the Industrial Revolution, we find that Airy growth in conjunction with a Malthusian population equation cannot adequately explain the demographic transition *and* exponential growth of per-capita income. On the one hand, if the economy approaches an exponential growth of per-capita income, then the population growth rate also rises over time. On the other hand, if the economy fails to exhibit exponential growth, then per-capita income and the population growth rate rise following the Industrial Revolution and later decline to approach constant levels.

To improve on this, we follow Kremer (1993) and Hansen and Prescott (2002) and extend the Malthusian population equation to allow for population growth to increase with per-capita income up to a critical level and to decrease at higher levels. In this setting, Airy growth is consistent with stylized empirical facts. The economy starts in the Malthusian Regime and, at the beginning of the Industrial Revolution, switches to the Post-Malthusian Regime. Here, the positive relationship between population and per-capita income is still intact. Moreover, knowledge growth becomes cumulative and per-capita income increases, yet at a slower pace than later on. The demographic transition marks the switch from the Post-Malthusian to the Modern Growth Regime. Per-capita income rises faster than before and higher levels of income slow down population growth.

Observe that the driving force behind the take-off and the subsequent sustained growth of per-capita income is exogenous. Thus, we inquire into the consequences of technological change rather than into its sources. Yet, if we accept the interpretation of our Quasi-Airy specification as a reduced form of Mokyr's proposed feedback between the evolution of new techniques and their epistemic basis, then this paper is indeed about the isolated contribution of the creation of technological knowledge on the evolution of macroeconomic magnitudes.[1] This contribution is shown to be consistent with key empirical facts both before and after the Industrial Revolution.

This paper is organized as follows. Section 2 introduces the Airy-differential equation and derives the properties of the Quasi-Airy specification. Section 3 employs this specification and studies the evolution of a Malthusian economy. In Section 4, we extend the analysis and use a more realistic population-income equation. Here we present the central result of the paper: Airy growth is consistent with a run through three stages of development, the Malthusian, the Post-Malthusian, and the Modern Growth Regime. Section 5 concludes the paper.

2 Airy-Functions and Knowledge Growth

Let A denote technological knowledge that evolves according to some ordinary differential equation. To retain some freedom in modelling, assume an evolution that

[1] Ideally, one would want to develop a micro-foundation for the Quasi-Airy specification of the evolution of knowledge along the lines of the pioneering work of Olsson (2000, 2005). Yet, this lies outside the scope of this paper. We leave it for future research.

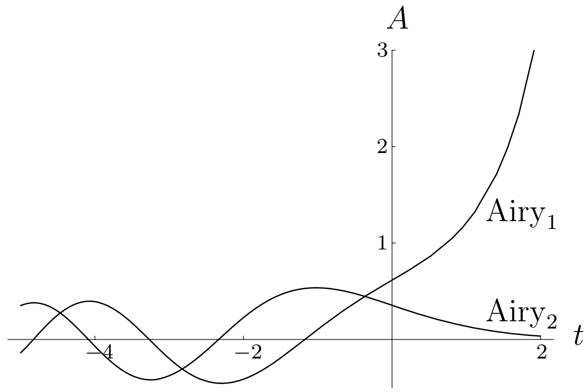

Figure 1: Airy Functions.

is governed by the second-order differential equation

$$\ddot{A} = f(t)\, g(A). \tag{1}$$

This differential equation is non-autonomous and encompasses several specifications of knowledge growth used in growth models.[2]

For simple specifications of $f(t)$ and $g(A)$, equation (1) is compatible with functions that exhibit rich dynamics. Consider $f(t) = t$ and $g(A) = A$. The resulting differential equation

$$\ddot{A} = t\, A \tag{2}$$

is known as the Airy differential equation. It is linear, hence all solutions can be written as a linear combination of two independent solutions, $A_1(t) = \text{Airy}_1(t)$ and $A_2(t) = \text{Airy}_2(t)$. Here, Airy_1 and Airy_2 denote the Airy functions as plotted in Figure 1. Generically, a solution to (2) has the property of oscillating around zero, with increasing cycle length for negative t. For positive t, the solution converges either to ∞ or $-\infty$. In both cases, the growth rate $\dot{A}/A$ converges to $\sqrt{t}$.

[2] Jones (1995) has knowledge growth depending on the existing stock of knowledge and population, $\dot{A} = N^{\lambda}\, A^{\phi}$. Taking derivatives and factoring out gives $\ddot{A} = N^{\lambda}\, A^{\phi}(\lambda\, \dot{N}/N + \phi\, \dot{A}/A)$. Assume $\dot{N} = 0$. Then, upon substitution we obtain $\ddot{A} = N^{2\,\lambda}\, \phi A^{2\,\phi-1}$. This is (1) with $f(t) = \phi\, N^{2\,\lambda}$ and $g(A) = A^{2\phi-1}$. From the special case $\phi = 1$, it is clear that (1) encompasses exponential growth, i. e., $\dot{A} = \gamma\, A$ with $\gamma = N^{\lambda}$. However, $\ddot{A} = \gamma^2 A$ is more general, allowing for exponential growth, decline, and linear combinations of both. For further discussion of the Jones' equation, see Groth, Koch, and Steger (2006) and Hakenes and Irmen (2006).

We define a *take-off* as a phase of transition between a state of approximate stagnation to a state of sustained growth. This raises two questions: does the evolution of A exhibit such a take-off, and, if yes, when does it occur. As to the first question, we note that any linear combination of the two Airy functions in Figure 1 qualifies as a solution to (2). Depending on the sign and the weight of $\text{Airy}_1(t)$, the solution generically converges either to ∞ or to $-\infty$. Thus, a take-off that leads to a sustained increase of A requires the weight of $\text{Airy}_1(t)$ to be positive. As to the second question, we note that the take-off is gradual because of the smoothness of the underlying differential equation. Generically, it occurs in the vicinity of $t = 0$. At this date the time path of A changes its character; the possibility of a monotonous evolution supersedes the evolution through oscillations around zero. Ex ante, this structural break may be hard to detect since for $t > 0$, the path of A may temporarily decline even though $A(0) > 0$. Intuitively, this is the case if the weight of $\text{Airy}_1(t)$ is positive, but not too large relative to the weight of $\text{Airy}_2(t)$. However, if $\text{Airy}_1(t)$ has a positive and sufficiently large weight, a monotonous evolution of A for $t > 0$ is guaranteed.[3]

In the next section we approximate the effect of technological knowledge on aggregate output over time using Airy functions. Then, for early periods this effect materializes through oscillations and, on average, is rather weak. A take-off occurs, and after the take-off, knowledge growth eventually becomes cumulative. This captures, in Mokyr's words, the possibilities of a sufficiently wide epistemic base. As a means to represent the evolution of knowledge, Airy functions have some undesirable properties. For instance, they take on negative values before the take-off and grow faster than exponentially afterwards. To prevent the stock of knowledge from becoming negative and to allow for prospective exponential growth, we introduce the following amendments.

To address the problem of negative knowledge, we stipulate $g(A) = A - A_\emptyset$; here $A_\emptyset > 0$ is the average level of knowledge before the take-off around which knowledge oscillates. To prevent superexponential growth, we adjust the function $f(t)$, assuming that $f(t)$ is monotonically increasing and bounded. Then, the limits

[3] To see this, we show that A can cross the time-line at most once. Assume $A(\bar{t}) = 0$ for $\bar{t} > 0$, and, without loss of generality, $A'(\bar{t}) > 0$. Then $A''(t) > 0$ for $t > \bar{t}$ implies that $A'(t)$ increases for $t > \bar{t}$. Hence, following this intersection, $A(t)$ departs and bends away from the time-line.

$\lim_{t\to-\infty} f(t)$ and $\lim_{t\to\infty} f(t)$ exist. Let $c_{-\infty} := \lim_{t\to-\infty} f(t)$ and $c_\infty := \lim_{t\to\infty} f(t)$ denote these limits, and assume that $c_{-\infty} < 0 < c_\infty$.[4] If these properties hold, we refer to (1) as a Quasi-Airy differential equation. The solution to a Quasi-Airy differential equation exhibits the following properties.

Proposition 1 *Let A evolve according to the Quasi-Airy differential equation. Then, for $t \to -\infty$, A oscillates around $A_\emptyset$ with cycle length $2\,\pi/\sqrt{-c_{-\infty}}$. For $t \to \infty$, A grows exponentially at a growth rate $\sqrt{c_\infty}$.*

Proof: The proof relies on suitable approximations to the Quasi-Airy differential equation.[5] For $t \to -\infty$, we have $f(t) \approx c_\infty$, and the Quasi-Airy differential equation becomes approximately $\ddot{A} = c_{-\infty}\,(A - A_\emptyset)$, with $c_{-\infty} < 0$. The solution is

$$A(t) = A_\emptyset + C_1\,\sin\sqrt{-c_{-\infty}}\,t + C_2\,\cos\sqrt{-c_{-\infty}}\,t, \tag{3}$$

where $C_1, C_2 \in \mathbb{R}$ are integration constants. Since the cycle length of the sine is $2\,\pi$, the cycle length implied by (3) is $2\,\pi/\sqrt{-c_{-\infty}}$.

For $t \to \infty$, we have $f(t) \approx c_{-\infty}$, and the Quasi-Airy differential equation becomes approximately $\ddot{A} = c_\infty\,(A - A_\emptyset)$, with $c_\infty > 0$. The solution is

$$A(t) = A_\emptyset + C_1'\,e^{\sqrt{c_\infty}\,t} + C_2'\,e^{-\sqrt{c_\infty}\,t}, \tag{4}$$

with $C_1', C_2' \in \mathbb{R}$. This is approximately exponential growth at rate $\sqrt{c_\infty}$. ■

Proposition 1 characterizes the asymptotic evolution of A for the Quasi-Airy specification. In contrast to the solutions of the Airy differential equation (2), oscillations now have approximately constant cycle length, and the growth rate after the take-off eventually becomes constant. Both findings can be traced back to the assumed boundedness of $f(t)$.

With respect to a take-off, the implications of the Quasi-Airy specification resemble those of the Airy differential equation. Generically, its solution may converge to

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[4] A closed functional form that satisfies these requirements is the logistic function, $f(t) = c_{-\infty} + (c_\infty - c_{-\infty})\,(1 + \exp\frac{c_\mu - t}{c_\sigma})^{-1}$, with $c_{-\infty} < 0$, c_∞, $c_\sigma > 0$, and $c_\mu \in \mathbb{R}$.

[5] The exact solution to the Quasi-Airy differential equation can be found. It involves complicated algebraic expressions and converges for large $|t|$ to the solution of the differential equations used as approximations. The proof of these assertions is available upon request.

either ∞ or $-\infty$. From (4), it is the sign of C_1' that is decisive: a take-off requires $C_1' > 0$.[6] The take-off must occur in the vicinity of $\tilde{t}$ satisfying $f(\tilde{t}) = 0$. After this date the path of A can exhibit at most one additional wave before it becomes monotonous.[7]

For $t \to -\infty$, the path of A may become negative. From (3), it is the sign and the relative size of $A_\emptyset$, C_1', and C_2' that determine the sign of A. A meaningful interpretation of A as the level of knowledge at t necessitates conditions such that $A > 0$ for all t.[8] Then, the take-off is indeed inevitable:

Corollary 1 *Let A evolve according to the Quasi-Airy differential equation. If $A(t) > 0$ for all $t \in \mathbb{R}$, then a take-off is generically inevitable.*

3 Malthus and the Take-Off

This section studies how per-capita income and population evolve in a simple Malthusian economy if the evolution of technological knowledge satisfies Proposition 1. For small t we find a regime that exhibits the properties of Malthusian stagnation. For $t \geq 0$, there is a take-off and sustained knowledge growth. Yet, the Malthusian interplay between per-capita income and population growth cannot account for both exponential growth of per-capita income and a demographic transition.

Let

$$Y(t) = A(t)\, N(t)^\alpha\, T(t)^{1-\alpha},\ 0 < \alpha < 1, \tag{5}$$

be the aggregate production function, where Y denotes output, A is technological knowledge, N is population, and T is land. We normalize $T(t) \equiv 1$ and retain

[6] Recall that the Airy differential equation exhibits a take-off if and only if the weight associated with $\text{Airy}_1(t)$ is positive. The positive sign of C_1' is the analogous requirement.

[7] The argument is analogous to the one developed in footnote 3, with $A = A_\emptyset$ instead of $A = 0$ and $t = \tilde{t}$ instead of $t = 0$.

[8] For equation (3) the necessary and sufficient condition for $A(t) > 0$ is ${C_1'}^2 + {C_2'}^2 < {A_0}^2$. As a matter of fact, it is also possible to modify the Airy differential equation such that every solution is necessarily positive. To give an example, a solution of the differential equation $\ddot{A} = t \ln A$ oscillates for negative t, but can never become negative.

$Y(t) = A(t)N(t)^{\alpha}$. Accordingly, per-capita income is

$$y(t) = \frac{Y(t)}{N(t)} = A(t)\, N(t)^{\alpha-1}. \tag{6}$$

Let $n(t) = \dot{N}(t)/N(t)$ denote the population growth rate. In a Malthusian manner we assume that $n(t)$ is monotonically rising in $y(t)$. For concreteness, we set

$$n(t) = c_b - \frac{c_d}{y(t)} \quad \text{with} \quad c_b,\, c_d > 0. \tag{7}$$

The parameter c_b can be interpreted as the time-invariant difference between the birth rate and the death rate if per-capita income is infinite. The impact of per-capita income on population growth depends on the size of c_d. For brevity, we henceforth associate c_b with the birth rate and $c_d/y(t)$ with the death rate. Upon combining (6) and (7), we obtain the evolution of population growth as

$$n(t) = c_b - c_d \frac{N(t)^{1-\alpha}}{A(t)} \qquad \Longleftrightarrow \qquad \dot{N}(t) = c_b\, N(t) - \frac{c_d}{A(t)}\, N(t)^{2-\alpha}. \tag{8}$$

To fix ideas, we define a quasi-steady state as a pair $(\bar{N}(t), \bar{y}(t))$ that satisfies $n(t) = 0$ given $A(t)$.

Lemma 1 *There is a unique, globally stable quasi-steady state with*

$$\bar{y}(t) = \bar{y} = \frac{c_d}{c_b} \text{ and } \bar{N}(t) = \left[\frac{A(t)}{\bar{y}}\right]^{\frac{1}{1-\alpha}}. \tag{9}$$

Proof: Equation (9) is immediate from (8) for $n(t) = 0$. For $N(t) \gtrless \bar{N}(t)$, we have $n(t) \lessgtr 0$. Hence global stability follows as (8) implies a negative relationship between $n(t)$ and $N(t)$. ■

Lemma 1 emphasizes the Malthusian character of the economy and the prospective role of technological progress. If A remains constant, then decreasing returns to labor and the positive effect of per-capita income on the growth rate of population imply a self-equilibrating population size. At this size, per-capita income is at the Malthusian subsistence level, $\bar{y}$, and depends only on the exogenous fertility and mortality parameters.

The notion of a quasi-steady state refers to the fact that changes in $A(t)$ must have an effect on $\bar{N}(t)$ while leaving $\bar{y}(t)$ unaffected. Thus, an evolution of technological knowledge according to Proposition 1 moves $\bar{N}(t)$. Proposition 2 states how this impinges on the evolution of $N(t)$, $n(t)$, and $y(t)$. The proof is in the appendix.

Proposition 2 *Let A evolve according to the Quasi-Airy differential equation with $A(t) > 0$ for all $t \in \mathbb{R}$.*

1. *For $t \to -\infty$, $N(t)$, $n(t)$, and $y(t)$ oscillate with cycle length $2\pi/\sqrt{-c_{-\infty}}$ around $(A_\emptyset/\bar{y})^{1/(1-\alpha)}$, 0, and $\bar{y}$, respectively.*

2. *For $t \to \infty$, $N(t)$ and $y(t)$ grow asymptotically at constant rates.*
   If $\sqrt{c_\infty} < (1-\alpha)\,c_b$, then

$$\lim_{t\to\infty} n(t) = \frac{\sqrt{c_\infty}}{1-\alpha} > 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\dot{y}(t)}{y(t)} = 0. \tag{10}$$

   If $\sqrt{c_\infty} > (1-\alpha)\,c_b$, then

$$\lim_{t\to\infty} n(t) = c_b > 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\dot{y}(t)}{y(t)} = \sqrt{c_\infty} - (1-\alpha)\,c_b > 0. \tag{11}$$

Proposition 2 characterizes the asymptotic dynamics of population, its growth rate, and per-capita income. Roughly speaking, two regimes appear. A long period of stagnation is followed by a take-off that leads to higher levels of per-capita income. The take-off occurs when the evolution of technological knowledge starts to become cumulative. Figure 2 provides an illustration of the time paths of $\bar{N}(t)$, $\bar{y}(t)$, and the ensuing evolution of $N(t)$, $n(t)$, and $y(t)$, given initial values for A and N. For our parameter choice, the take-off is in the vicinity of $t = 0$ and leads to a monotonous increase of A for $t > \tilde{t}$.

The period of stagnation corresponds to the time interval to the left of the take-off line. It exhibits several Malthusian features. According to Proposition 1, $A(t)$ oscillates around $A_\emptyset$. Therefore, the quasi-steady state level $\bar{N}(t)$ oscillates around the level $\bar{N}_\emptyset = [A_\emptyset/\bar{y}]^{1/(1-\alpha)}$, marked by a dashed line in Figure 2. The arrows indicate the vector field of population growth. At any t, there are Malthusian forces that push the economy towards the $\bar{N}(t)$-locus. On the dashed curve, these vectors are horizontal, and the economy is in the quasi-steady state ($n(t) = 0$).

The continuous curve in the upper part of Figure 2 marks a population path for given initial conditions. Here, the economy starts with some $N(t) < \bar{N}(t)$, and population initially expands. Had we started with $N(t) > \bar{N}(t)$, population would have contracted initially. This highlights a general property of population dynamics

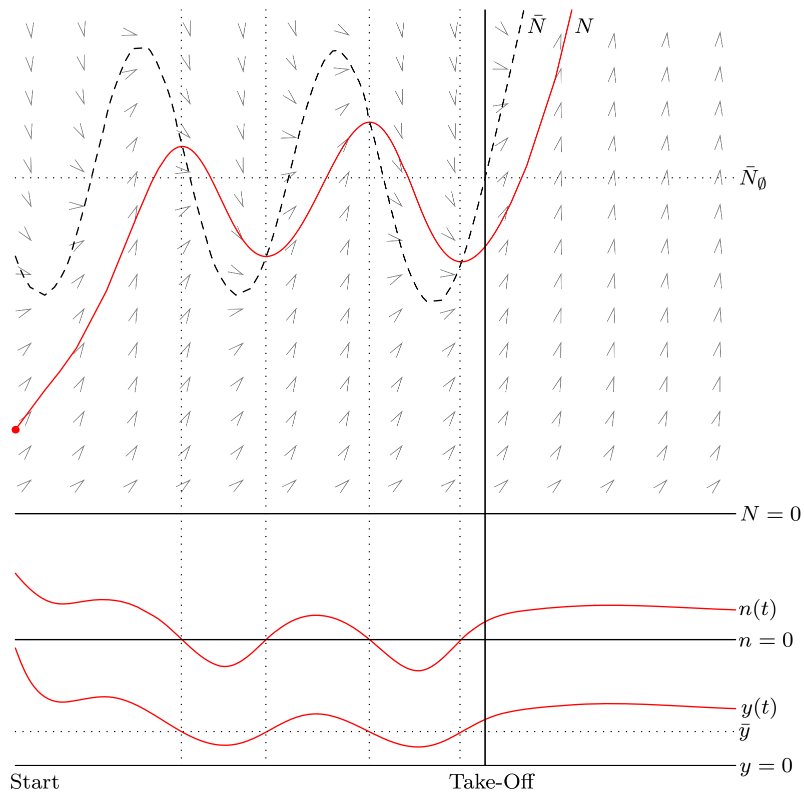

Figure 2: The Evolution of $\bar{N}$, $\bar{y}$, N, n, and y.

in the period of stagnation. Population evolves into the corridor defined by the oscillation of $\bar{N}(t)$, stays inside, and eventually oscillates with the same frequency as $\bar{N}(t)$.

Observe that the evolution of $N(t)$ lags behind that of $\bar{N}(t)$. Whenever $N(t)$ and $\bar{N}(t)$ intersect, the evolution of the former is at a stationary point, thus $n(t) = 0$ and $y(t) = \bar{y}$. If $N(t)$ intersects from below, then for $t' \gtrapprox t$ the local behavior of the economy is determined by a decreasing $\bar{N}(t)$. Accordingly, $N(t') > \bar{N}(t')$, $y(t') < \bar{y}$, and $n(t') < 0$. If $N(t)$ intersects from above, inequalities are reversed. Oscillations of $N(t)$ around $\bar{N}_\emptyset$ imply that, on average, population remains constant until the take-off. Accordingly, $n(t)$ fluctuates around zero, and $y(t)$ oscillates around $\bar{y}$.[9]

[9]Kremer's (1993) collection of population data between 1 million B. C. and 1 A. D. suggests a slightly positive trend in the evolution of the world population. This trend is absent in the evolution described in Proposition 2. We may attribute this to our assumption about the constancy of agricultural land. Indeed, if $T(t)$ was some increasing function of time, then the quasi-steady

With the take-off A starts to grow permanently. From Proposition 1 we know that the growth rate of A approaches the constant $\sqrt{c_\infty}$. Proposition 2 shows that the take-off may or may not result in sustained growth of per-capita income. Intuitively, this depends on how $\sqrt{c_\infty}$ relates to the asymptotic population growth rate. To see this, consider (6) in growth rates

$$\frac{\dot{y}(t)}{y(t)} = \frac{\dot{A}(t)}{A(t)} - (1-\alpha)\, n(t), \tag{12}$$

which requires $n(\infty) = \sqrt{c_\infty}/(1-\alpha)$ if $\lim_{t\to\infty} \dot{y}(t)/y(t) = 0$. According to (10) this outcome obtains if asymptotic knowledge growth is weak, i. e., $\sqrt{c_\infty} < (1-\alpha)\, c_b$. Intuitively, growth of per-capita income peters out since technological progress cannot outweigh the effect of population growth and decreasing returns to labor. Interesting though, (7) implies

$$y(\infty) = \frac{c_d}{c_b - \sqrt{c_\infty}/(1-\alpha)} > \bar{y},$$

i. e., $y(t)$ converges towards a level that strictly exceeds Malthusian subsistence.

If $\lim_{t\to\infty} \dot{y}(t)/y(t) > 0$, then $y(t) \to \infty$, and (7) requires $n(\infty) = c_b$. Moreover, from (12) we must have $\lim_{t\to\infty} \dot{y}(t)/y(t) = \sqrt{c_\infty} - (1-\alpha)\, c_b$. This is positive if the asymptotic knowledge growth is sufficiently pronounced and provides the explanation for equation (11) in Proposition 2.

Thus, for both parameter constellations, the take-off in the evolution of technological knowledge generates a take-off in per-capita income. Yet, an inconsistency with the empirical facts surfaces, too. On the one hand, for the parameter constellation $\sqrt{c_\infty} < (1-\alpha)\, c_b$, the model generates a demographic transition. This is the case depicted in Figure 2. Following the take-off, there is a Post-Malthusian regime with rising $y(t)$ and $n(t)$. Later on, $n(t)$ falls. However, no modern growth regime appears; the economy cannot sustain exponential growth of $y(t)$. On the other hand, if the economy exhibits sustained growth of per-capita income, it also exhibits a large population growth rate. Moreover, the Malthusian population equation (7) implies greater population growth as per-capita income rises and no demographic transition materializes.

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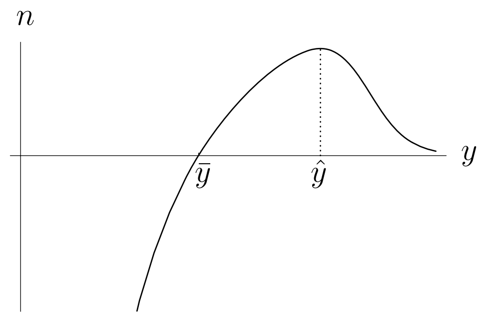
state level of population would be $\bar{N}(t) = T(t)\, [A(t)/\bar{y}]^{1/(1-\alpha)}$ and the oscillations of $N(t)$ and $n(t)$ would exhibit the desired positive trend. Alternatively, we could allow for $A_\emptyset$ to be increasing over time. Neither of these extensions would affect the average level of per-capita income $\bar{y}$.

Figure 3: Population Growth versus Per-Capita Income ($n_\infty = 0$).



The Malthusian population equation is the source of the failure to have increasing levels of $y(t)$ and decreasing levels of $n(t)$ at some stage following the take-off. In the following section we extend this relationship and show that Airy growth is consistent with a Post-Malthusian regime leading to modern growth with sustained per-capita income growth and a declining population growth rate.

4 Malthus to Romer

To cope with the demographic transition that occurred following the take-off, we follow, for example, Kremer (1993) and Hansen and Prescott (2002), and stipulate a non-monotonic relationship between the population growth rate and per-capita income. The key assumption is that $n(y)$ is Malthusian, with $n(\bar{y}) = 0$ and $n'(y) > 0$ up to some level $\hat{y} > \bar{y}$, but $n'(y) < 0$ for $y > \hat{y}$. The level $\hat{y}$ satisfies $n'(\hat{y}) = 0$.[10] Denote $\hat{n} := n(\hat{y})$ the maximum population growth rate. For the sake of simplicity, let $\lim_{y\to\infty} n(y) = 0$. Figure 3 provides an illustration.

Proposition 3 *Let A evolve according to the Quasi-Airy specification with $A(t) > 0$ for all $t \in \mathbb{R}$, and consider $n(y)$ with the above-mentioned properties.*

[10] Several arguments have been developed to explain why $n'(y) < 0$ holds for large y. They emphasize, e. g., a quantity-quality trade-off or an increasing value of women's time (see, e. g., Barro and Becker (1989) or Galor and Weil (1996)). Galor and Weil (1999) review several theories of the demographic transition.

1. *For $t \to -\infty$, population $N(t)$, $n(t)$, and per-capita income $y(t)$ oscillate with cycle length $2\pi/\sqrt{-c_{-\infty}}$ around $(A_\emptyset/\bar{y})^{1/(1-\alpha)}$, 0, and $\bar{y}$, respectively.*

2. *For $t \to \infty$, $N(t)$ and $y(t)$ grow asymptotically at constant rates. If $\sqrt{c_\infty} > (1-\alpha)\,\hat{n}$, then*

$$\lim_{t\to\infty} n(t) = 0 \quad \textit{and} \quad \lim_{t\to\infty} \frac{\dot{y}(t)}{y(t)} = \sqrt{c_\infty} > 0. \tag{13}$$

3. *There is $\hat{t}$ such that $y(\hat{t}) = \hat{y}$. At $\hat{t}$, we have $n(\hat{t}) = \hat{n}$, and in a neighborhood of $\hat{t}$ it holds that*

$$\frac{\dot{y}(t)}{y(t)} < \sqrt{c_\infty} - (1-\alpha)\,\hat{n}. \tag{14}$$

Proof: For $t \to -\infty$ the economy is identical to the one of Proposition 2. The proof of the first claim is included in the proof of Proposition 2. Denote $\tilde{t}$ the date at which the evolution of A changes its character, i. e. $f(\tilde{t}) = 0$. For $t > \tilde{t}$ we invoke Proposition 1 and Corollary 1 to conclude that $A(t) > 0$ for all $t \in \mathbb{R}$ implies $\lim_{t\to\infty} \dot{A}(t)/A(t) = \sqrt{c_\infty}$. If, in addition, $\sqrt{c_\infty} > (1-\alpha)\,\hat{n}$, then for $t \to \infty$ we infer from (12) that $y(t) \to \infty$. From the properties of $n(y)$ we have $n(t) \to 0$ and, as a result, $\dot{y}(t)/y(t) \to \sqrt{c_\infty}$. This proves the second claim. By assumption, $\hat{y}$ is only reached after the take-off. Then, two possibilities arise. If $y(t)$ increases monotonically for $t > \tilde{t}$, there is a unique $\hat{t}$ such that $y(\hat{t}) = \hat{y}$. If the path of $y(t)$ is initially non-monotonic, there may be several dates at which $y(t) = \hat{y}$. In the latter case, consider the last of these dates. Inequality (14) follows from (12) in conjunction with $\dot{A}(\hat{t})/A(\hat{t}) < \sqrt{c_\infty}$. ■

Proposition 3 states the main result of our analysis: Quasi-Airy growth can guide the economy from a phase of Malthusian stagnation into a Post-Malthusian Regime, and then towards a Modern Growth Regime. The phase of Malthusian stagnation has the same characteristics as described in Proposition 2. This is due to the fact that the behavioral specification of $n(y)$ shares the properties of (7) for small levels of per-capita income that satisfy $y < \hat{y}$.

After the take-off, new features can arise. Figure 4 shows a solution with initial conditions such that A and y are monotonous after the take-off. First, a Post-Malthusian Regime appears. It has a growing per-capita income, and the positive relationship between income per capita and population growth is still intact. From

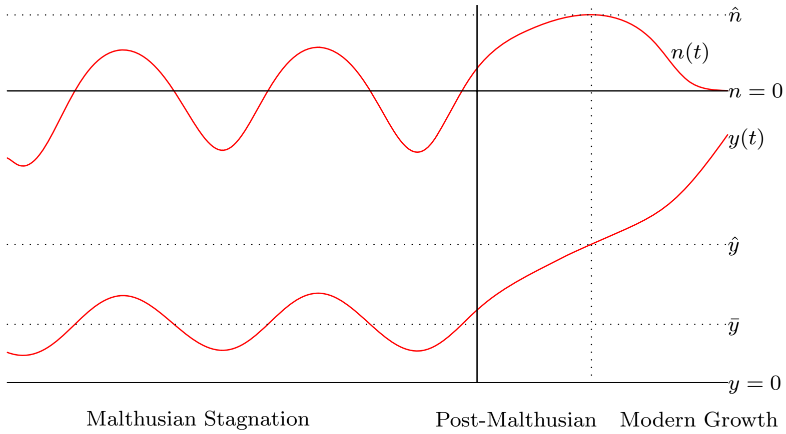


Figure 4: Malthus to Romer.

(13) and inequality (14) we deduce that the growth of per-capita income is slower relative to later periods. Second, there is a Modern Growth Regime. Here, the Malthusian link between per-capita income and population growth breaks down. Population growth slows down, giving rise to a demographic transition. At the same time, growth in per-capita income speeds up. Asymptotically, the economy reaches a balanced growth path with constant population and exponential growth in per-capita income.

Observe that these findings hinge on the condition that the asymptotic growth rate of technological knowledge is high relative to the maximum growth rate of population. If this condition fails to hold, then there is no Modern Growth Regime for the reasons set out in the discussion following Proposition 2. Asymptotically, population growth is positive at $n(\infty) = \sqrt{c_\infty}/(1-\alpha)$, and per-capita income is constant at a level that satisfies $\sqrt{c_\infty}/(1-\alpha) = n(y(\infty)) > \bar{y}$.

5 Concluding Remarks

We associate the Industrial Revolution with the take-off in the evolution of technological knowledge and inquire into its effect on the evolution of per-capita income and population size. The tool that generates the evolution of technological knowledge is based on the Airy differential equation. An evolution that is governed by the proposed differential equation changes its qualitative behavior over time. In particular, it allows for an acceleration in the pace of technological knowledge, following a long period of approximate stagnation. We interpret the beginning of this acceleration as the Industrial Revolution.

From an historical point of view, the Industrial Revolution is often referred to as the event that separates the period of Malthusian stagnation from the modern growth experience. The first step of our analysis shows that a Quasi-Airy growth of knowledge in an otherwise Malthusian economy is consistent with this view. The take-off leads to levels of per-capita output that exceed Malthusian subsistence. However, this setting is not rich enough to exhibit the features of a Post-Malthusian Regime, where the positive link between population growth and per-capita income still works and per-capita income rises. We cope with this deficiency and introduce a behavioral relationship between population growth and per-capita income that allows for a demographic transition. In the ensuing calibrations it is the acceleration in the speed of technological progress that appears as the separating line between the period of Malthusian stagnation and the Post-Malthusian Regime; the demographic transition divides the Post-Malthusian from the Modern Growth Regime.

Recent research suggests that the qualitative change in the evolution of technological knowledge can be attributed to the evolution of its epistemic base (cf. Mokyr, 2002 and Mokyr, 2005). Knowledge growth governed by our Quasi-Airy specification lends itself to an interpretation in this vein.

A Appendix: Proof of Proposition 2

The proof strategy mimics the one used in the proof of Proposition 1. We start with the less involved proof of claim 2.

For large t, (4) implies $A(t) \approx C_1 \, e^{\sqrt{c_\infty}\, t}$ with $C_1 > 0$. Hence, (8) becomes

$$\dot{N}(t) = c_b \, N(t) - \frac{c_d}{C_1 \, e^{\sqrt{c_\infty}\, t}} \, N(t)^{2-\alpha}.$$

This differential equation has the solution

$$N(t) = \Big(- \frac{1-\alpha}{C_1 \, \Phi} \, e^{-\sqrt{c_\infty}\, t} \left(c_d + C_1 \, C_3 \, \Phi \, e^{t\,\Phi} \right) \Big)^{-\frac{1}{1-\alpha}},$$

where $\Phi := \sqrt{c_\infty} - (1-\alpha)\, c_b$. To make sure that $N(t) \geq 0$ for all t, we need either $\Phi < 0$ or $C_3 < 0$.

First, consider the case $\Phi < 0$, i.e. $\sqrt{c_\infty} < /(1-\alpha)c_b$. For large t,

$$N(t) \approx \Big(- \frac{1-\alpha}{C_1\, \Phi}\, e^{-\sqrt{c_\infty}\, t}\, c_d \Big)^{-\frac{1}{1-\alpha}},$$

with growth rate $n(t) = \dot{N}(t)/N(t) = \sqrt{c_\infty}/(1-\alpha) > 0$. Since $y(t) = A(t)\, N(t)^{\alpha-1}$, we have $\dot{y}(t)/y(t) = \sqrt{c_\infty} + (\alpha-1)\sqrt{c_\infty}/(1-\alpha) = 0$. This proves (10).

Second, consider $C_3 < 0$ and $\Phi > 0 \Leftrightarrow \sqrt{c_\infty} > /(1-\alpha)c_b$. For large t,

$$N(t) \approx \Big(- (1-\alpha)\, e^{-\sqrt{c_\infty}\, t}\, C_3\, e^{t\, \Phi} \Big)^{-\frac{1}{1-\alpha}} = \Big(- C_3\, (1-\alpha)\, e^{-(1-\alpha)\, c_b\, t} \Big)^{-\frac{1}{1-\alpha}}.$$

As a consequence, $n(t) = c_b$ and $\dot{y}(t)/y(t) = \sqrt{c_\infty} + (\alpha-1)\, c_b = \Phi > 0$. This is (11).

For small t we use (3) as an approximation. It follows that (8) becomes

$$\dot{N}(t) = c_b\, N(t) - \frac{c_d}{A_\emptyset + C_1\, \sin\sqrt{-c_{-\infty}}\, t + C_2\, \cos\sqrt{-c_{-\infty}}\, t}\, N(t)^{2-\alpha}. \qquad (15)$$

Since all addends in the denominator remain finite, it is inappropriate to neglect any of them. However, observe that the relation between C_1 and C_2 influences only the phasing of cycles. With a focus on the limit behavior for $t \to -\infty$, only the amplitude and the duration of cycles matter. Therefore, it is permissible to set $C_2 = 0$ and $C_1 > 0$. Using this simplification, we next show that, after some time, $N(t)$ falls into the interval,

$$N(t) \in \Big[\Big(\frac{c_b}{c_d}\, (A_\emptyset - C_1) \Big)^{\frac{1}{1-\alpha}};\ \Big(\frac{c_b}{c_d}\, (A_\emptyset + C_1) \Big)^{\frac{1}{1-\alpha}} \Big].$$

To see this, note from (3), with $C_2 = 0$, that $A(t) \leq A_\emptyset + C_1$. In the extreme case where $A(t) = A_\emptyset + C_1$ for all t, (15) becomes

$$\dot{N}(t) = c_b\, N(t) - \frac{c_d}{A_\emptyset + C_1}\, N(t)^{2-\alpha}.$$

If $N(t)$ converges, then $\dot{N}(t) \approx 0$ after some time, and

$$0 \approx c_b\, N(t) - \frac{c_d}{A_\emptyset + C_1}\, N(t)^{2-\alpha}, \quad N(t) \to \Big(\frac{c_b}{c_d}\, (A_\emptyset + C_1) \Big)^{\frac{1}{1-\alpha}}.$$

An analogous argument applies to the extreme case $A(t) = A_\emptyset - C_1$. Hence, once $N(t)$ is inside the indicated interval, it does not escape. In addition, if $N(t)$ is inside the interval, then by the properties of the sine function we have $\dot{N}(t) > 0$ if $t = 2\,\pi\,\sqrt{-c_{-\infty}}\,(i+1/4)$ for some $i \in \mathbb{N}$. Similarly, $\dot{N}(t) < 0$ if $t = 2\,\pi\,\sqrt{-c_{-\infty}}\,(i-1/4)$ for some $i \in \mathbb{N}$. Since $N(t)$ moves back and forth every $2\,\pi\,\sqrt{-c_{-\infty}}$, we know that $N(t)$ oscillates with the according frequency. As a consequence, $n(t)$ oscillates around zero with the same frequency. Since $n(t) = c_b - c_d/y(t)$ implies $y(t) = (c_b - n(t))/c_d$, $y(t)$ oscillates with the same frequency. This completes the proof. ■

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