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Communication network formation with link specificity and value transferability

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Abstract

We propose a model on strategic formation of communication networks with (i) link specificity: the more direct links somebody maintains, the less she can specify her attention per link, the lower her links' value, while this negative externality was previously ignored in the communication context, and (ii) value transferability via indirect links for informational but not for social value from communication, while this positive externality was modeled uniformly before. Assuming only social value to isolate the impact of link specificity, the pairwise stable set includes many non-standard networks under high or particular combinations of fully connected components under low link specificity. Allowing for social and informational value, the joint effect of link specificity and value transferability reduces the stable set to certain fragmented networks under high or the complete network under low link specificity. These extremes are beneficial for efficiency, whereas quite inefficient networks may arise for intermediate link specificity.

JEL classification: A14, C79, D85, M31

Keywords: Bilateral communication links, Link specificity, Value transferability, Social vs. informational value, Strategic network formation

1 Introduction

Structures of who communicates with whom are distinguishing empirical phenomena (e.g., Trier, 2008) and can determine important outcome variables such as the extent to which value is shared throughout a community and how it is distributed (e.g., Granovetter, 2005; Ren et al., 2007). Therefore, in the current paper we study the structure of bilateral communication links among individuals. We model their formation as a game-theoretic network formation process in which agents choose to create and maintain links, only if the participants in the link benefit from doing so, which results in a pairwise stable network (Jackson and Wolinsky, 1996). The model thus describes how agents benefit and lose from being connected and predicts which stable communication networks emerge when agents myopically maximize the resulting payoff value. Herein, we incorporate a combination of two important aspects common to communication networks that has not been investigated before.

First, our model features link specificity in the sense that the more direct connections an individual has to maintain with other individuals, the less she is able to specify her attention per link. Therefore, her value per link for others declines and she also derives less value from each link with others (Currarini, 2007; co-author model Jackson and Wolinsky, 1996). Thus, we go beyond the standard assumption of a fixed cost per link in communication networks (e.g., Bala and Goyal, 2000; connections model Jackson and Wolinsky, 1996).

We assume that two connected agents contribute to their bilateral process of communication value creation according to a standard production function with as inputs the amount of time invested by each agent in the link. Higher link specificity implies higher output elasticities in each bilateral value production process and therefore lower advantage of being connected with several agents. Unit output elasticities are adopted to analyze high link specificity, while constant returns to scale, i.e., both output elasticities equal to $1/2$, reflect low link specificity.

Second, we introduce the important distinction between social and informational value as motivations for bilateral exchange decisions. This typology was suggested by the virtual community literature regarding the question why individuals choose to participate in and contribute to such a community as a whole (e.g., Dholakia et al., 2004). Social value is related to the fact that individuals may enjoy communicating with others, for example because they find it entertaining or because they feel it enhances their self-worth. Informational value refers to the fact that individuals may obtain new valuable knowledge from others when they communicate. Typically, informational value can be transferred relatively easily to third parties through indirect links, whereas social value is more personal and therefore hardly transferable without creating a direct link.

To understand the relative impact of social and informational member orientation, we assume that social value is only experienced from direct neighbors and that informational value flows via any path consisting of bilateral communication links connecting two agents. Hereby, we integrate transferable and nontransferable value in one model, while so far, value transferability was at best incorporated uniformly for all value (e.g., Bala and Goyal, 2000; connections model Jackson and Wolinsky, 1996).

More specifically, we first deal with the case of communication having social value only (Section 2) in order to illustrate the separate impact of link specificity on network structure. When link specificity is high, the set of pairwise stable networks is characterized by two simple conditions and is shown to contain a wide range of non-standard networks, including highly connected and "small world" networks, whereas previous models for social and economic network formation mostly predicted simple networks like stars and wheels. When link specificity is low, particular combinations of fully connected components are pairwise stable, similar to the prediction of Jackson and Wolinsky (1996) for the co-author context.

Next, we deal with the case of communication from which both social and informational value is derived (Section 3) in order to illustrate the impact of value transferability on structure. Under high link specificity, only networks that consist of disjoint star components of two or three agents are shown to be pairwise stable. Apparently, the combination of these two features: high link specificity, which is an example of a negative network externality, and even marginal informational value transferability, which is an example of a positive network externality (Asvanund et al., 2004), has a strong fragmentizing effect on the emerging pairwise stable networks. Under low link specificity, the opposite effect takes place: already with small informational value transferability, only the complete network is pairwise stable.

Section 4 focuses on efficiency properties of the wide variety of stable networks discussed in the previous two sections. In particular, it is found that both the fragmentation under high link specificity and the dense stable networks under low link specificity are most efficient in their own setting. In Section 5, link specificity values other than 1 and 1/2 are investigated by simulations. Especially, it is found that 1 and 1/2 are indeed suitable polar cases and that for intermediate link specificity values the common tension between stability and efficiency (e.g., Jackson and Wolinsky, 1996) is re-established. Subsequently, Section 6 concludes and offers directions for further research.

2 Nontransferable social value

Since the structure of a communication network determines value for participants, we capture its formation in a game-theoretical model. Although we believe that communication networks typically combine social and informational value aspects, we first deal with the simpler case in which only social value is derived from communication. This approach allows us to illustrate the separate impact of *link specificity* on communication structure and to exclude value transferability.

Link specificity (Currarini, 2007; Jackson and Wolinsky, 1996) means that the more direct connections an individual has to maintain with other individuals, the less she is able to specify her attention per link. Therefore, her value per link for others declines and she also derives less value from each link with others. These negative externalities of link formation are crucial in our communication context, since here no benefits arise from individual contributions as such. The reason is that communication is only valuable if it is two-sided, thus effort has to be invested by both sender and receiver.[1]

In short, the objective of this section is to develop a model for communication network formation with only social value from communication. We use the concept of pairwise stability to characterize the collection of stable communication networks.

2.1 Model and stability concept

A communication network is described by (N, g), where $N = \{1, ..., n\}$, $n \geq 3$, is a community of agents. A direct link $g_{i,j}$ between agents i and j in this community $(i, j \in N; i \neq j)$ is interpreted as a communication relationship between i and j which is established if they both wish the link; $g_{i,j}$ indicates with a 1 or a 0 whether i is directly linked to j or not. These relationships are expressed by undirected links: for any two agents i and j, $g_{i,j} = g_{j,i}$. By definition, $g_{i,i} = 0$, as agents do not establish communication links with themselves. In this community agents only derive social value from interaction.

In case of an isolated relationship between two agents, each agent experiences social value $V^{\mathrm{s}} > 0$ as the outcome of their joint communication production process. However, maintenance of the communication relationship costs effort: investment of both agents is needed in order to make the communication specific to their personal circumstances and hence useful. Accordingly, in case of a network where two agents do not form an isolated pair, both agents are assumed to divide their effort equally among all their relationships, as a result of which, in an extreme case, the potential social communication value is divided proportionally by the number of links that agents face. However, since agents may have economies of scale in coping with several links, the extent of link specificity can be smaller.

We assume that the contributions of two agents in their bilateral process of communication value creation are reflected by a Cobb-Douglas production function with the time invested in the link by the agents as inputs. We assume both output elasticities are equal to ρ, where $\rho = 1$ corresponds to the case of high link specificity and $\rho = 1/2$ coincides with constant returns to scale and results in low link specificity. Therefore, the total payoff for agent i in communication network g is given by

$$\Pi_i(g) = \begin{cases} \sum\limits_{j \in N_i(g)} \frac{V^{\mathrm{s}}}{\left(\mu_i(g) \cdot \mu_j(g)\right)^{\rho}} & \text{if } \mu_i(g) > 0 \\ 0 & \text{if } \mu_i(g) = 0, \end{cases} \tag{1}$$

where $N_i(g)$ is the set of agents with whom i has a direct link, agent j is a neighbor of agent i if $j \in N_i(g)$, and $\mu_i(g) = |N_i(g)|$ is the number of neighbors of agent i, which is also referred to as the degree of i; $V^{\mathrm{s}} > 0$ denotes the social value that i would derive from communication with j if neither i nor j were linked to any other agent; and $\rho \leq 1$ indicates the level of link specificity.[2]

[1] In contrast, in the co-author setting, which has been the subject of investigation in earlier research (Jackson and Wolinsky, 1996), each co-author can write independently as well.

[2] For comparison: the payoff function in the co-author model of Jackson and Wolinsky (1996) can be written as
$$\Pi_i(g) = \begin{cases} \sum\limits_{j \in N_i(g)} \left(\frac{V^{\mathrm{s}}}{\mu_i(g)} + \frac{V^{\mathrm{s}}}{\mu_j(g)} + \frac{V^{\mathrm{s}}}{\mu_i(g) \cdot \mu_j(g)} \right) & \text{if } \mu_i(g) > 0 \\ 0 & \text{if } \mu_i(g) = 0. \end{cases}$$

For the model thus described we predict which stable networks emerge by using the concept of pairwise stability (Jackson and Wolinsky, 1996), where a network is stable if no single agent can strictly improve her payoff by deleting one of her direct links and no pair of agents can both weakly improve their payoffs by creating a direct link while at least one of the two members strictly improves her payoff by doing so. This solution concept is weak in the sense that it only assumes stability against deviations of exactly one link (which involves the permission of two agents in the case of link formation), reflecting a form of myopia. Alternatively, the model could be analyzed by applying the Nash solution (Bala and Goyal, 2000), which assumes stability against single-agent deviations of more than one link. Because of the extreme coordination problem of the Nash concept in two-sided link formation and since the weak concept of pairwise stability already clearly and interestingly constrains the number of communication networks that are stable, we choose for the pairwise stability solution. The study of farsighted stability notions in the setting of communication networks, following the approaches of for instance the largest pairwise consistent set (Chwe, 1994) or the pairwise farsightedly stable set (Herings et al., 2009) is left for future research.

In our notation, we have the following definition.

Definition 1 (pairwise stability) The network g is pairwise stable if for all $i, j \in N$ with $g_{i,j} = 1$ it holds that

$$\Pi_i(g) \geq \Pi_i(g') \text{ and } \Pi_j(g) \geq \Pi_j(g'),$$

where g' is such that $g'_{i,j} = 0$ and $g'_{k,\ell} = g_{k,\ell}$ for all $\{k, \ell\} \neq \{i, j\}$, and for all $i, j \in N$ with $g_{i,j} = 0$ it holds that

$$\begin{gathered}\Pi_i(g) > \Pi_i(g') \text{ or} \\ \Pi_j(g) > \Pi_j(g') \text{ or} \\ \left(\Pi_i(g) = \Pi_i(g') \text{ and } \Pi_j(g) = \Pi_j(g')\right),\end{gathered}$$

where g' is such that $g'_{i,j} = 1$ and $g'_{k,\ell} = g_{k,\ell}$ for all $\{k, \ell\} \neq \{i, j\}$.

2.2 Stable networks under high link specificity

First, we evaluate pairwise stability in communication networks under high link specificity, which we obtain by setting $\rho = 1$. We prove that in this case, the collection of pairwise stable networks can be described by two easily verifiable conditions: (i) they are what we call equal neighbor degree networks, meaning that everybody has at least one neighbor and all neighbors of an agent has the same degree, and (ii) there is at most a difference of one between the degrees of agents in the same component.

Definition 2 (equal neighbor degree network) A network g is an equal neighbor degree network when it holds for each $i \in N$ that $\mu_i(g) \geq 1$ and for all $j, j' \in N_i(g)$ that $\mu_j(g) = \mu_{j'}(g)$. Here we adopt the following notation: the own degree of an agent i is denoted by d_i and her neighbors' degree by e_i.

Definition 3 (path) A path in g connecting i and j is a sequence of agents $k_1, \ldots, k_m \in N$ for whom it holds that $g_{i,k_1} = g_{k_1,k_2} = \ldots = g_{k_{m-1},k_m} = g_{k_m,j} = 1$.

Definition 4 (component) A component c in g is a network among a set of agents $C \subseteq N$ for whom it holds that for all $i, j \in C, i \neq j$, there exists a path in c connecting i and j, and for any $i \in C$ and $j \in N$, $g_{i,j} = 1$ implies $c_{i,j} = 1$.

Definition 5 (star) A network g is a star if it has exactly $n - 1$ links and there exists an agent j for whom it holds that $g_{j,i} = 1$ for all $i \neq j$. Similarly, a component c is a star if it has exactly $|C| - 1$ links and it contains an agent j for whom it holds that $g_{j,i} = 1$ for any other $i \in C$. Agent j is called the center agent whereas the other agents are the periphery agents of the star.

Example 1 A network consisting of star components is an equal neighbor degree network.

Example 2 The network given in Figure 1 is an equal neighbor degree network.

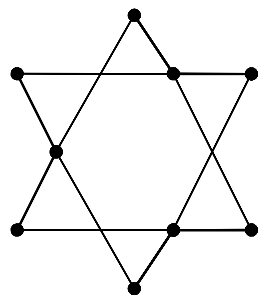


Figure 1: An equal neighbor degree network

Before providing the main result in Proposition 1, we first derive Lemma 1. It observes that under high link specificity, pairwise stable communication networks must be equal neighbor degree networks since it is beneficial for an agent i to delete the link with a neighbor who has to maintain more direct links than i's average neighbor. Furthermore, Lemma 1 expresses a condition to exclude link creation.

Lemma 1 *When $\rho = 1$, a communication network is pairwise stable if and only if it is an equal neighbor degree network where it holds for each not directly linked pair of agents i, j that*

$$e_i \leq d_j \text{ or } e_j \leq d_i \text{ or } (e_i = d_j + 1 \text{ and } e_j = d_i + 1)\,. \tag{2}$$

Proof. ($\Longleftarrow$) Assume that g is an equal neighbor degree network where for each not directly linked pair of agents i, j condition (2) is satisfied. The payoff of an agent i as expressed in equation (1) can be written as

$$\Pi_i(g) = \sum_{j \in N_i(g)} \frac{V^s}{\mu_i(g)\mu_j(g)} = d_i \frac{V^s}{d_i e_i} = \frac{V^s}{e_i},$$

so i does not want to delete a link, for then her payoff would reduce to zero if $d_i = 1$, whereas if $d_i > 1$ it would remain equal:

$$(d_i - 1)\,\tfrac{V^{\mathrm{s}}}{(d_i-1)e_i} = \tfrac{V^{\mathrm{s}}}{e_i}.$$

Moreover, no link between any pair of agents i, j is created if it makes either i or j strictly worse off or both of them equally well off. Therefore, no link is created if

$$\tfrac{V^{\mathrm{s}}}{e_i} > d_i \tfrac{V^{\mathrm{s}}}{(d_i+1)e_i} + \tfrac{V^{\mathrm{s}}}{(d_i+1)(d_j+1)} \quad \text{or} \tag{3}$$

$$\tfrac{V^{\mathrm{s}}}{e_j} > d_j \tfrac{V^{\mathrm{s}}}{(d_j+1)e_j} + \tfrac{V^{\mathrm{s}}}{(d_i+1)(d_j+1)} \quad \text{or} \tag{4}$$

$$\left(\tfrac{V^{\mathrm{s}}}{e_i} = d_i \tfrac{V^{\mathrm{s}}}{(d_i+1)e_i} + \tfrac{V^{\mathrm{s}}}{(d_i+1)(d_j+1)} \text{ and } \tfrac{V^{\mathrm{s}}}{e_j} = d_j \tfrac{V^{\mathrm{s}}}{(d_j+1)e_j} + \tfrac{V^{\mathrm{s}}}{(d_i+1)(d_j+1)}\right). \tag{5}$$

The following shows that $e_i \leq d_j$ implies (3):

$$e_i \leq d_j \implies e_i + d_i(d_j+1) < (d_i+1)(d_j+1) \implies \tfrac{d_i(d_j+1)+e_i}{(d_i+1)(d_j+1)e_i} < \tfrac{1}{e_i}.$$

Analogously, it can be shown that $e_j \leq d_i$ implies (4), and $(e_i = d_j + 1)$ and $(e_j = d_i + 1)$ implies (5). Therefore, g is pairwise stable.

($\implies$) Assume that the network g is pairwise stable. First, suppose that there is an agent i for whom it holds that $\mu_i(g) = 0$. Then her payoff would strictly improve from a link with some other agent k. It is obvious that also k's payoff would strictly increase if $\mu_k(g) = 0$, which contradicts pairwise stability, so consider the case where $\mu_k(g) \geq 1$. The payoff of k without this link equals

$$\sum_{j\in N_k(g)} \tfrac{V^{\mathrm{s}}}{\mu_k(g)\cdot\mu_j(g)} = \tfrac{V^{\mathrm{s}}}{\mu_k(g)}\left(\sum_{j\in N_k(g)} \tfrac{1}{\mu_j(g)}\right),$$

whereas by linking with i it would become

$$\sum_{j\in N_k(g)} \tfrac{V^{\mathrm{s}}}{(\mu_k(g)+1)\cdot\mu_j(g)} + \tfrac{V^{\mathrm{s}}}{(\mu_k(g)+1)\cdot 1} = \tfrac{V^{\mathrm{s}}}{(\mu_k(g)+1)}\left(\sum_{j\in N_k(g)} \tfrac{1}{\mu_j(g)} + 1\right) \geq \tfrac{V^{\mathrm{s}}}{\mu_k(g)}\left(\sum_{j\in N_k(g)} \tfrac{1}{\mu_j(g)}\right).$$

The inequality follows from the observation that the expression before the inequality equals V^{s} times the average of the terms $1/\mu_j(g)$, $j \in N_k(g)$, and 1, the expression after the inequality is equal to V^{s} times the average of the terms $1/\mu_j(g)$, $j \in N_k(g)$, and that $1 \geq 1/\mu_j(g)$ for all $j \in N_k(g)$. This contradicts pairwise stability of g. It follows that $\mu_i(g) \geq 1$ for all $i \in N$.

Secondly, suppose that for some i it does not hold that $\mu_j(g)$ is constant for all $j \in N_i(g)$. Then there is an agent $k \in N_i(g)$ such that

$$\mu_k(g) > \frac{\sum_{j\in N_i(g)} \mu_j(g)}{\mu_i(g)}.$$

The payoff for i is given by

$$\sum_{j \in N_i(g)} \frac{V^{\mathrm{s}}}{\mu_i(g) \cdot \mu_j(g)} = \frac{V^{\mathrm{s}}}{\mu_i(g)} \sum_{j \in N_i(g)} \frac{1}{\mu_j(g)},$$

whereas by deleting the link with k, the payoff for i would become

$$\sum_{j \in N_i(g)} \frac{V^{\mathrm{s}}}{(\mu_i(g)-1) \cdot \mu_j(g)} - \frac{V^{\mathrm{s}}}{(\mu_i(g)-1) \cdot \mu_k(g)} = \frac{V^{\mathrm{s}}}{(\mu_i(g)-1)} \left(\sum_{j \in N_i(g)} \frac{1}{\mu_j(g)} - \frac{1}{\mu_k(g)} \right) > \frac{V^{\mathrm{s}}}{\mu_i(g)} \sum_{j \in N_i(g)} \frac{1}{\mu_j(g)},$$

where the inequality follows immediately from the interpretation of the last two terms as V^{s} times an average of numbers $1/\mu_j(g)$, $j \in N_i(g)$. This contradicts pairwise stability, so $\mu_j(g) = \mu_{j'}(g)$ for all $j, j' \in N_i(g)$. We have shown that a pairwise stable network is an equal neighbor degree network.

Finally, suppose that there exists a not directly linked pair i, j for which condition (2) is not satisfied, implying

$$e_i \geq d_j + 1 \text{ and } e_j \geq d_i + 1 \text{ and } (e_i > d_j + 1 \text{ or } e_j > d_i + 1). \tag{6}$$

Then i and j want to create a link between them, since this would cause the payoff for agent i to become

$$d_i \frac{V^{\mathrm{s}}}{(d_i+1)e_i} + \frac{V^{\mathrm{s}}}{(d_i+1)(d_j+1)} \geq d_i \frac{V^{\mathrm{s}}}{(d_i+1)e_i} + \frac{V^{\mathrm{s}}}{(d_i+1)e_i} = \frac{V^{\mathrm{s}}}{e_i},$$

and for agent j to become

$$d_j \frac{V^{\mathrm{s}}}{(d_j+1)e_j} + \frac{V^{\mathrm{s}}}{(d_j+1)(d_i+1)} \geq d_j \frac{V^{\mathrm{s}}}{(d_j+1)e_j} + \frac{V^{\mathrm{s}}}{(d_j+1)e_j} = \frac{V^{\mathrm{s}}}{e_j},$$

where according to the last condition in (6) at least one of the inequality signs is strict. This contradicts pairwise stability too. Therefore, g is an equal neighbor degree network where condition (2) holds for each not directly linked pair of agents i, j. ■

The condition to exclude link creation in Lemma 1 can be further simplified, leading to the following main result.

Proposition 1 *When $\rho = 1$, a communication network is pairwise stable if and only if it is an equal neighbor degree network where it holds for each pair of agents k, ℓ in the same component that*

$$|d_k - d_\ell| \leq 1. \tag{7}$$

Proof. Considering Lemma 1, it is sufficient to show that in an equal neighbor degree network condition (2) holds for each not directly linked pair i, j if and only if condition (7) is satisfied for each pair k, ℓ in the same component.

($\Longleftarrow$) Assume an equal neighbor degree network where for each pair k, ℓ in the same component condition (7) is satisfied. Let i, j be any not directly linked pair. If $e_i \leq d_j$, condition (2) is satisfied.

If not, then $e_i > d_j$ and we can derive by applying condition (7) twice that

$$e_j \leq d_j + 1 \leq e_i \leq d_i + 1.$$

If $e_j \leq d_i$, condition (2) is satisfied. If not, then $e_j = d_i + 1$ and condition (2) is satisfied if it also holds that $e_i = d_j + 1$. Suppose not, then $e_i \geq d_j + 2$ and we can derive by applying condition (7) that

$$e_i \geq d_j + 2 \geq (e_j - 1) + 2 = d_i + 2,$$

which contradicts condition (7). Therefore, condition (2) is satisfied.

($\Longrightarrow$) Assume an equal neighbor degree network where for each not directly linked pair i, j condition (2) is satisfied. Let k, ℓ be any pair in the same component, so there exists at least one path between k and ℓ. Assume that the total number of agents on any of these paths is odd. Due to the equal neighbor degree network it holds that $d_k = d_\ell$, so condition (7) is satisfied.

Assume that the total number of agents on all of these paths is even. We consider three cases.

(i) $N_k(g) \setminus \{\ell\} = \emptyset$ and $N_l(g) \setminus \{k\} = \emptyset$. It follows that the component consists of k and ℓ only, so condition (7) trivially holds.

(ii) $N_k(g) \setminus \{\ell\} \neq \emptyset$ and $N_l(g) \setminus \{k\} \neq \emptyset$. Consider $m \in N_k(g) \setminus \{\ell\}$. Due to the equal neighbor degree network it holds that

$$d_k = e_m = e_\ell \text{ and } e_k = d_m = d_\ell.$$

Since ℓ and m are not directly linked, by condition (2) we have

$$d_k = e_\ell \leq d_m = d_\ell \text{ or } d_k = e_m \leq d_\ell \text{ or } \left(d_k = e_\ell = d_m + 1 = d_\ell + 1 \text{ and } d_k = e_m = d_\ell + 1\right),$$

so $d_k \leq d_\ell + 1$. By the same argument, using some $n \in N_\ell(g) \setminus \{k\}$, we find $d_\ell \leq d_k + 1$. Consequently, condition (7) is satisfied.

(iii) (Without loss of generality) $N_k(g) \setminus \{\ell\} = \emptyset$ and $N_l(g) \setminus \{k\} \neq \emptyset$. Since k is connected to ℓ, we have $N_k(g) = \{\ell\}$, $d_k = 1$, $k \in N_\ell(g)$, and $d_\ell \geq 2$. As in case (ii), using some $m \in N_\ell(g) \setminus \{k\}$, it follows that $d_\ell \leq d_k + 1 = 2$. Therefore, it holds that $d_\ell = 2$. Due to the equal neighbor degree network we find $d_m = d_k = 1$. We have shown that g is a three-agent star. Clearly, condition (7) holds. ■

The following examples illustrate the wide range of networks thus proven to be pairwise stable in the social value case.

Definition 6 (complete network) A network g is complete if all agents are connected, so for all $i, j \in N$, $i \neq j$, it holds that $g_{i,j} = 1$.

Definition 7 (wheel network) A network g is a wheel if it has exactly n links and there exists a sequence of different agents $k_1, ..., k_n \in N$ for whom it holds that $g_{k_1,k_2} = g_{k_2,k_3} = ... = g_{k_{n-1},k_n} = g_{k_n,k_1} = 1$.

Definition 8 (regular network) A network g is regular if it exists of one component and for each $i \in N$ it holds that $d_i = d$.

Corollary 1 *When $\rho = 1$, the complete, wheel, or any regular communication network is pairwise stable, for it is an equal neighbor degree network where it holds for each pair of agents k, ℓ in the single component that*

$$|d_k - d_\ell| = 0 \leq 1.$$

Example 3 A non-regular communication network that is pairwise stable under $\rho = 1$ is given in Figure 2.

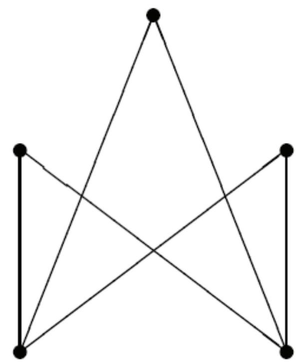


Figure 2: A non-regular pairwise stable communication network for $\rho = 1$

Example 4 A communication network consisting of multiple components that is pairwise stable under $\rho = 1$ is given in Figure 3.

Example 5 A "small world" is a network with local clusters of highly interlinked agents together with agents that link the various clusters. As a consequence, although most agents are not directly connected, every agent is indirectly linked to every other agent by a relatively small number of steps. A regular "small world" communication network that is pairwise stable under $\rho = 1$ is given in Figure 4.

Note that this wide set of stable communication networks includes complex real-life networks (as empirically observed by e.g. Dodds et al., 2003), whereas previous models for social and economic network formation mostly predicted simple networks like stars and wheels (e.g., Bala and Goyal, 2000; Goyal and Vega-Redondo, 2007).

2.3 Stable networks under low link specificity

In this section we study the case of low link specificity, obtained by setting $\rho = 1/2$. We show that particular combinations of fully connected components are pairwise stable communication networks. This is similar to the prediction for the co-author model of Jackson and Wolinsky (1996), where a pairwise stable network can be partitioned into fully connected components, each of which has a

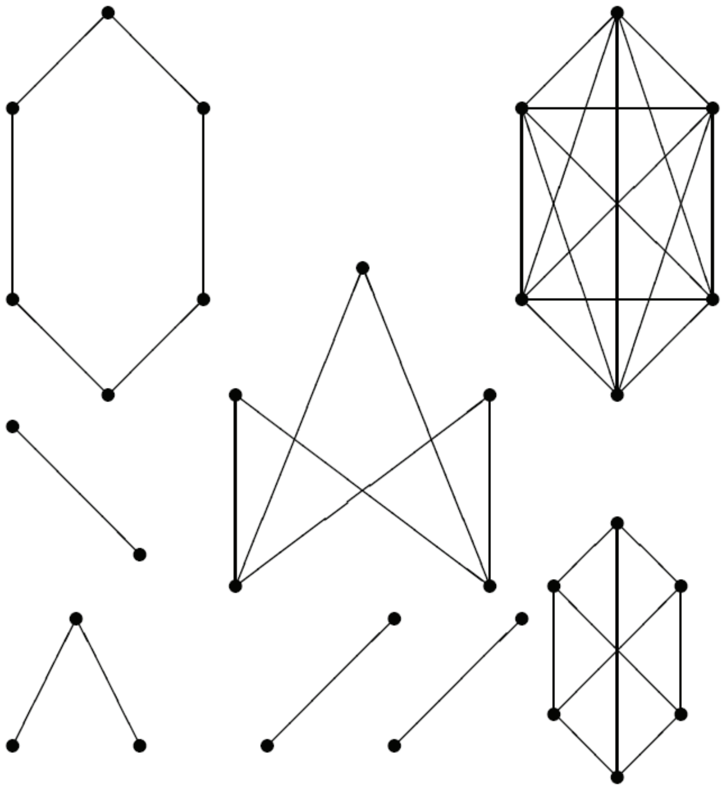


Figure 3: A multiple-component pairwise stable communication network for $\rho = 1$

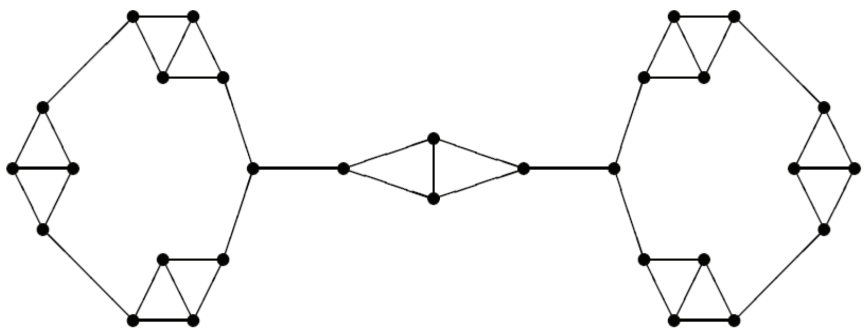


Figure 4: A "small world" pairwise stable communication network for $\rho = 1$

different number of members: if m_{c_1} is the number of members of one such component and m_{c_2} is the next largest size, then $m_{c_1} > (m_{c_2})^2$. We obtain a different condition because of the difference in payoff functions (see Section 2.1).

Proposition 2 *Consider a communication network g consisting of fully connected components $c_1, \ldots, c_k$ with $m_{c_1} \geq m_{c_2} \geq \cdots \geq m_{c_k}$, where m_{c_j} is the number of members of c_j. When $\rho = 1/2$ it holds that g is pairwise stable if and only if $m_{c_k} \geq 2$ and $m_{c_j} \geq 4m_{c_{j+1}} - 2$, $j = 1, \ldots, k-1$.*

Proof. We show first that $m_{c_k} \geq 2$. Since $n \geq 3$ it holds that $m_{c_k} \geq 3$ if $k = 1$. Consider the case $k \geq 2$ and suppose $m_{c_k} = 1$. A member of c_{k-1} has payoff V^{s}, whereas creating a link with the unique member of c_k leads to payoff

$$\left(\frac{m_{c_{k-1}} - 1}{\sqrt{m_{c_{k-1}} - 1 + 1}\sqrt{m_{c_{k-1}} - 1}} + \frac{1}{\sqrt{m_{c_{k-1}} - 1 + 1}} \right) V^{\mathrm{s}} = \frac{\sqrt{m_{c_{k-1}} - 1} + 1}{\sqrt{m_{c_{k-1}}}} V^{\mathrm{s}},$$

a term larger than V^{s}. Since obviously the unique member of c_k wants to link with a member of c_{k-1}, we obtain a contradiction. Consequently, it holds that $m_{c_k} \geq 2$.

Consider a component with m members. No member wants to delete a link, for the current payoff for such an agent is V^{s}, whereas deleting a link would reduce it to 0 when m equals 2 and to

$$\tfrac{m-2}{\sqrt{(m-2)(m-1)}} V^{\mathrm{s}}$$

when $m \geq 3$.

Consider two distinct components, let one component have ℓ members and the other m. A player in the ℓ-sized component looses from establishing a link with a member of the m-sized component if and only if

$$\frac{\ell - 1}{\sqrt{\ell(\ell - 1)}} + \frac{1}{\sqrt{\ell m}} < 1,$$

which is equivalent to $1/\sqrt{m} < \sqrt{\ell} - \sqrt{\ell - 1}$. The latter expression is equivalent to $m > 4\ell - 2 - 1/m$. Using that m is an integer larger than or equal to 2, this last expression is equivalent to $m \geq 4\ell - 2$. ■

Notice that, unless $n = 7$, the collection of pairwise stable networks described in Proposition 2 contains the pairwise stable networks in the co-author model of Jackson and Wolinsky (1996). It is a subset of the collection of pairwise stable networks under high link specificity ($\rho = 1$, Section 2.2).

For $\rho = 1/2$ we did not find any other pairwise stable communication networks. In particular, it is not hard to verify that neither regular communication networks with $d < n - 1$ nor any of the example networks in Figures 2, 3, or 4 are stable when $\rho = 1/2$.

3 Informational as well as social value

This section introduces the case in which both social and informational value is derived from communication in networks. Thus, we can illustrate the impact of *value transferability* on communication structure in combination with the effect of link specificity. Value transferability (Bala and Goyal, 2000) means that value from communication is not only derived by direct neighbors, but can also be transferred via indirect links. More specifically, we make a distinction between social and informational value derived from communication, where only informational value is transferable through the network. For example, social value from communication between two Saab enthusiasts only exists for the two communication partners, but informational value (e.g., from a solution to a technical problem) can exist for others in the network. After proposing a model for network formation in this setting, the pairwise stable networks are characterized again. We show that the set of stable communication networks is much more limited in range than in the purely social value setting (Section 2).

3.1 Model

A communication network is described by (N, g), where $N = \{1, \ldots, n\}$, $n \geq 3$, is a community of agents. A direct link $g_{i,j}$ between agents i and j in this community ($i, j \in N; i \neq j$) can be interpreted as a communication relationship between i and j which is established if they both wish the link. These relationships are expressed by undirected links: for any two agents i and j, $g_{i,j} = g_{j,i}$, and $g_{i,i} = 0$.

In case of an isolated relationship between two agents where interaction only has social value, each agent experiences social value $V^{\mathrm{s}} > 0$ as the outcome of their joint communication production process. In case of an isolated relationship between two agents where interaction only has informational value, each agent experiences informational value $V^{\mathrm{i}} > 0$ as the outcome of their joint communication production process. In general, agents are assumed to give relative attention to informational and social value in the proportions α and $1 - \alpha$ respectively, where α is assumed to be constant satisfying $0 \leq \alpha \leq 1$.

Again we assume that the contributions of two agents in their bilateral process of communication value creation are reflected by a Cobb-Douglas production function with both output elasticities equal to ρ, where $\rho = 1$ corresponds to the case of high link specificity and $\rho = 1/2$ coincides with constant returns to scale and results in low link specificity.

Moreover, informational value is, without any decay except for this effort division, transferred to third parties through indirect links (paths of links), whereas social value is not transferable. This is due to the fact that in the direct communication production process of two agents, any of them can use the informational value that she acquired during the bilateral communication creation with other neighbors. Consequently, agent j_0 experiences not only first-step informational payoff from her direct neighbors:

$$\Pi_{j_0}^{1\mathrm{i}}(g) = \sum_{j_1 \in N_{j_0}(g)} \frac{V^{\mathrm{i}}}{(\mu_{j_0}(g) \cdot \mu_{j_1}(g))^{\rho}},$$

which is similar to the social payoff in equation (1), but also second-step informational payoff:

$$\Pi_{j_0}^{2\mathrm{i}}(g) = \sum_{j_1 \in N_{j_0}(g)} \frac{1}{\left(\mu_{j_0}(g)\cdot\mu_{j_1}(g)\right)^{\rho}} \sum_{j_2 \in N_{j_1}(g)\setminus\{j_0\}} \frac{V^{\mathrm{i}}}{\left(\mu_{j_1}(g)\cdot\mu_{j_2}(g)\right)^{\rho}},$$

third-step informational payoff:

$$\Pi_{j_0}^{3\mathrm{i}}(g) = \sum_{j_1 \in N_{j_0}(g)} \frac{1}{\left(\mu_{j_0}(g)\cdot\mu_{j_1}(g)\right)^{\rho}} \sum_{j_2 \in N_{j_1}(g)\setminus\{j_0\}} \frac{1}{\left(\mu_{j_1}(g)\cdot\mu_{j_2}(g)\right)^{\rho}} \sum_{j_3 \in N_{j_2}(g)\setminus\{j_1,j_0\}} \frac{V^{\mathrm{i}}}{\left(\mu_{j_2}(g)\cdot\mu_{j_3}(g)\right)^{\rho}},$$

and so forth, thus the overall informational payoff for j_0 is equal to

$$\begin{aligned}
\Pi_{j_0}^{\mathrm{i}}(g) &= \sum_{q=1}^{n-1} \Pi_{j_0}^{q\mathrm{i}}(g) = V^{\mathrm{i}} \sum_{q=1}^{n-1} \prod_{r=1}^{q} \sum_{j_r \in N_{j_{r-1}}(g)\setminus\{j_{r-2},j_{r-3},\ldots,j_0\}} \frac{1}{\left(\mu_{j_{r-1}}(g)\cdot\mu_{j_r}(g)\right)^{\rho}} \\
&= \sum_{q=1}^{n-1} \sum_{j_q \in N_{j_{q-1}}(g)\setminus\{j_{q-2},j_{q-3},\ldots,j_0\}} \frac{V^{\mathrm{i}}}{\left(\mu_{j_0}(g)\cdot\prod_{r=1}^{q-1}\left(\mu_{j_r}(g)\right)^2\cdot\mu_{j_q}(g)\right)^{\rho}}.
\end{aligned}$$

Therefore, the total payoff for agent i in communication network g is given by

$$\Pi_i(g) = \begin{cases} \alpha \sum_{j \in \bar{N}_i(g)} \sum_{p \in \mathcal{P}_{i,j}(g)} \frac{V^{\mathrm{i}}}{\left(\mu_i(g)\cdot\prod_{k\in\breve{p}}(\mu_k(g))^2\cdot\mu_j(g)\right)^{\rho}} & \\ +(1-\alpha)\sum_{j\in N_i(g)} \frac{V^{\mathrm{s}}}{\left(\mu_i(g)\cdot\mu_j(g)\right)^{\rho}} & \text{if } \mu_i(g) > 0 \\ 0 & \text{if } \mu_i(g) = 0, \end{cases} \tag{8}$$

where α is the proportion of communication through each link in the community that concerns information and $1-\alpha$ is the proportion of communication through each link in the community that concerns social interaction; $\bar{N}_i(g)$ is the set of agents with whom i has either a direct or an indirect link; $\mathcal{P}_{i,j}(g)$ is the set of paths between i and j, and $\breve{p}$ is the set of agents on path p between i and j; and $V^{\mathrm{i}} > 0$ denotes the informational value that i would derive from communication with j if neither i nor j were linked to any other agent and interaction would only have informational value, and $V^{\mathrm{s}} > 0$ denotes the social value that i would derive from communication with j if neither i nor j were linked to any other agent and interaction would only have social value.

For the model thus described we again use the concept of pairwise stability (Jackson and Wolinsky, 1996) to predict which communication networks are stable.

3.2 Stable networks under high link specificity

For $\rho = 1$ and $0 < \alpha < 1$, it is proven that the pairwise stable communication networks consist of two- and three-agent star components only.[3] First consider the following lemma in which we show

[3] The results in the case where the value derived from communication is only informational ($\alpha = 1$) slightly differ from those in this mixed case ($0 < \alpha < 1$). Specifically, it appears that networks also containing one four-agent star component can be pairwise stable.

that the star communication network becomes unstable when there are more than three agents.

Lemma 2 *When $\rho = 1$ and $0 < \alpha < 1$, the star communication network is pairwise stable if and only if $n = 3$.*

Proof. From the star network, it is not beneficial for any of the periphery agents to delete her link with the center agent as then her payoff will be zero. For the center agent, deleting a link with any of the periphery agents will provide her with the same payoff, since she is not involved in any indirect links to other agents. Periphery agent i does not create a link with another periphery agent *i'* if and only if this would not decrease her payoff:

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{n-1}+\tfrac{n-2}{(n-1)^2}\right)+(1-\alpha)\,V^{\mathrm{s}}\tfrac{1}{n-1}\geq$$

$$\alpha V^{\mathrm{i}}\left(\underbrace{\tfrac{1}{2(n-1)}+\tfrac{1}{8(n-1)}}_{\text{(a)}}+\underbrace{\tfrac{1}{4}+\tfrac{1}{4(n-1)^2}}_{\text{(b)}}+\underbrace{\tfrac{n-3}{2(n-1)^2}+\tfrac{n-3}{8(n-1)^2}}_{\text{(c)}}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{2(n-1)}+\tfrac{1}{4}\right)$$

$$\Longleftrightarrow\ \alpha V^{\mathrm{i}}\,(4-n)+(1-\alpha)\,V^{\mathrm{s}}\,(3-n)\geq 0\ \Longleftrightarrow\ n\leq 3,$$

where the informational payoff elements on the right-hand side of the first inequality are derived from (a) the center agent, (b) agent *i'*, and (c) the other periphery agents consecutively. Since we assumed communities to consist of at least three agents, it holds that $n = 3$. ■

Now the main result can be proven by tracking that when high link specificity is reinforced by value transferability, it is beneficial for agents in communication networks to break cycles and to delete links with tree branches that are longer than one link.

Proposition 3 *When $\rho = 1$ and $0 < \alpha < 1$, a communication network is pairwise stable if and only if it consists of disjoint star components of two or three agents.*

Proof. ($\Longleftarrow$) It is not beneficial for any of the periphery agents in a two- or three-agent star component to delete her single link as then her payoff will be zero. Equivalently, for the center agent in a three-agent component, deleting a link with any of the two periphery agents is not beneficial as it will provide her with the same payoff.

Link creation between the periphery agents of one three-agent star is eliminated by Lemma 2. Therefore, we only have to examine the following cases $(a)-(f)$ related to link formation between two agents in different components:

	pair agent	center agent of 3-agent star	periphery agent of 3-agent star
pair agent	(a)	(b)	(c)
center agent of 3-agent star	x	(d)	(e)
periphery agent of 3-agent star	x	x	(f)

For each of these cases, it can be proven by evaluating the payoffs with and without the link that no link is created: after forming a link in case (a), a pair agent would get payoff

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{2}+\tfrac{1}{4}\right)\leq\alpha V^{\mathrm{i}}+(1-\alpha)\,V^{\mathrm{s}},$$

after forming a link in case (b), the pair agent would get payoff

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{18}+\tfrac{1}{18}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{2}+\tfrac{1}{6}\right)<\alpha V^{\mathrm{i}}+(1-\alpha)\,V^{\mathrm{s}},$$

after forming a link in case (c), the pair agent would get payoff

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{16}+\tfrac{1}{32}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{2}+\tfrac{1}{4}\right)<\alpha V^{\mathrm{i}}+(1-\alpha)\,V^{\mathrm{s}},$$

after forming a link in case (d), a center agent would get payoff

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{9}+\tfrac{1}{27}+\tfrac{1}{27}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{9}\right)\leq\alpha V^{\mathrm{i}}\left(\tfrac{1}{2}+\tfrac{1}{2}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{2}+\tfrac{1}{2}\right),$$

after forming a link in case (e), the center agent would get payoff

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{6}+\tfrac{1}{24}+\tfrac{1}{48}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{3}+\tfrac{1}{3}+\tfrac{1}{6}\right)<\alpha V^{\mathrm{i}}\left(\tfrac{1}{2}+\tfrac{1}{2}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{2}+\tfrac{1}{2}\right),$$

and after forming a link in case (f), a periphery agent would get payoff

$$\alpha V^{\mathrm{i}}\left(\tfrac{1}{4}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16}+\tfrac{1}{32}\right)+(1-\alpha)\,V^{\mathrm{s}}\left(\tfrac{1}{4}+\tfrac{1}{4}\right)\leq\alpha V^{\mathrm{i}}\left(\tfrac{1}{2}+\tfrac{1}{4}\right)+(1-\alpha)\,V^{\mathrm{s}}\tfrac{1}{2}.$$

($\Longrightarrow$) For this part of the proof, we need some extra notation. The payoff function in (8) can be rewritten as

$$\Pi_i\left(g\right)=\tfrac{1}{\mu_i(g)}\sum_{j\in N_i(g)}T_{i,j}\left(g\right),$$

where $T_{i,j}(g)$ is the total payoff that j transmits to i via her direct link with i. Formally,

$$T_{i,j}(g) = \alpha \left(\frac{V^{\mathrm{i}}}{\mu_j(g)} + \sum_{\left(j' \in \bar{N}_j(g) \setminus \{i\}\right)} \sum_{\left(p \in \mathcal{P}_{j,j'}(g) : i \notin \check{p}\right)} \frac{V^{\mathrm{i}}}{\mu_{j'}(g) \cdot \left(\mu_j(g)\right)^2 \cdot \prod_{k \in \check{p}} \left(\mu_k(g)\right)^2} \right) + (1-\alpha) \frac{V^{\mathrm{s}}}{\mu_j(g)}.$$

Assume that g is a pairwise stable network. Let i be an agent in g and $k \in N_i(g)$ be such that

$$T_{i,k}(g) = \min_{j \in N_i(g)} T_{i,j}(g).$$

Suppose that there exists an agent $\ell \in N_i(g)$ for whom it holds that

$$T_{i,\ell}(g) > T_{i,k}(g).$$

Deleting the link between i and k results in network g', where it holds that

$$T_{i,j}(g') \geq T_{i,j}(g), \ \forall j \in N_i(g'),$$

since k, to whom j might be (in)directly linked, has one costly direct link less, so more informational value might flow from j to i via k. The payoff for i then becomes

$$\Pi_i(g') = \frac{1}{\mu_i(g)-1} \sum_{j \in N_i(g')} T_{i,j}(g') > \frac{1}{\mu_i(g)} \sum_{j \in N_i(g)} T_{i,j}(g) = \Pi_i(g),$$

which contradicts pairwise stability of g. It follows that

$$T_{i,j}(g) = T_{i,j'}(g), \ \forall j, j' \in N_i(g). \tag{9}$$

Next, suppose that g contains a cycle, meaning that there exists a sequence of agents $k_1, ..., k_n \in N$ for whom it holds that $g_{k_1,k_2} = g_{k_2,k_3} = ... = g_{k_{n-1},k_n} = g_{k_n,k_1} = 1$. Let i be an agent in this cycle. Deleting the link with one of i's neighbors in the cycle, say k, results in g', where it holds for the other neighbor of i in the cycle, say m, that

$$T_{i,m}(g') > T_{i,m}(g),$$

since k, to whom m is (in)directly linked, has one costly direct link less, so more informational value flows from k to i via m. Moreover,

$$T_{i,j}(g') \geq T_{i,j}(g), \ \forall j \in N_i(g').$$

The payoff for i then becomes

$$\Pi_i(g') = \frac{1}{\mu_i(g)-1} \sum_{j \in N_i(g')} T_{i,j}(g') > \frac{1}{\mu_i(g)-1} \sum_{j \in N_i(g')} T_{i,j}(g) = \frac{1}{\mu_i(g)} \sum_{j \in N_i(g)} T_{i,j}(g) = \Pi_i(g),$$

where the second equality follows from equation (9). This implies that g is not pairwise stable, leading to a contradiction. We have therefore shown that g does not contain any cycle.

Since we have already shown that g contains no cycles, all components of g are trees. In a tree the number of links is one less than the number of agents. Moreover, in a tree there is a unique path between any two agents. A tree that is not a star contains an agent, say i, with a neighbor h that only has i as a neighbor, and, moreover, i is directly linked to an agent j who has another neighbor different from i. According to equation (9) it holds that

$$T_{i,h}(g) = T_{i,j}(g). \tag{10}$$

Since h has only one neighbor, i, it follows that

$$T_{i,h}(g) = \alpha V^{\mathrm{i}} + (1-\alpha)V^{\mathrm{s}}.$$

We now evaluate $T_{i,j}(g)$ and show it is smaller than $T_{i,h}(g)$.

Think of the component to which h belongs as a tree with h as top agent, denoted $\bar{N}_h(g)$. For players $k, k' \in \bar{N}_h(g)$, $k \neq k'$, player k' is a subordinate of k, denoted $k' \in \bar{S}(k)$, if k is on the unique path from h to k'. Player k' is a direct subordinate of k, denoted $k' \in S(k)$, if k' is a subordinate of k and there is a link between k and k'. We write

$$T_{i,j}(g) = \alpha T^{\mathrm{i}}_{i,j}(g) + (1-\alpha)T^{\mathrm{s}}_{i,j}(g),$$

where

$$T^{\mathrm{s}}_{i,j}(g) = \tfrac{V^{\mathrm{s}}}{\mu_j(g)} \leq \tfrac{V^{\mathrm{s}}}{2}, \tag{11}$$

and

$$T^{\mathrm{i}}_{i,j}(g) = \tfrac{V^{\mathrm{i}}}{\mu_j(g)} + \sum_{k \in \bar{S}(j)} \tfrac{V^{\mathrm{i}}}{\mu_k(g)(\mu_j(g))^2 \prod\limits_{k' \in \breve{p}_{j,k}} (\mu_{k'}(g))^2},$$

where $p_{j,k}$ is the unique path between j and k.

Consider $k \in \bar{S}(i)$. We define the informational payoff that k receives from its subordinates by

$$U^{\mathrm{i}}_k(g) = \tfrac{1}{\mu_k(g)} \sum_{k' \in S(k)} T^{\mathrm{i}}_{k,k'}(g),$$

where $T^{\mathrm{i}}_{k,k'}(g)$ is defined analogously to $T^{\mathrm{i}}_{i,j}(g)$. We obtain a recursive relation by observing that

$$T^{\mathrm{i}}_{k,k'}(g) = \tfrac{V^{\mathrm{i}} + U^{\mathrm{i}}_{k'}(g)}{\mu_{k'}(g)}.$$

We show by induction that

$$U^{\mathrm{i}}_k(g) \leq V^{\mathrm{i}}(\mu_k(g) - 1), \tag{12}$$

from which it follows that

$$T^{\mathrm{i}}_{k,k'}(g) \leq \frac{V^{\mathrm{i}}+V^{\mathrm{i}}(\mu_{k'}(g)-1)}{\mu_{k'}(g)} = V^{\mathrm{i}},$$

and, consequently,

$$T^{\mathrm{i}}_{i,j}(g) \leq V^{\mathrm{i}}. \tag{13}$$

Let $K^0 \subset \bar{N}_h(g)$ be the set of agents without subordinates. For $m \geq 1$, let K^m be the set of agents with all subordinates in $K^0 \cup \cdots \cup K^{m-1}$. Let m' be the smallest integer for which $j \in K^{m'}$. First consider an agent k in K^0, the set of agents without subordinates. Then $U^{\mathrm{i}}_k(g) = 0 = V^{\mathrm{i}}(\mu_k(g) - 1)$, so (12) is satisfied.

Suppose that (12) holds for agents in K^m, $m < m'$. Consider an agent $k \in K^{m+1}$.

$$U^{\mathrm{i}}_k(g) = \frac{1}{\mu_k(g)} \sum_{k' \in S(k)} T^{\mathrm{i}}_{k,k'}(g) \leq \frac{1}{\mu_k(g)} \sum_{k' \in S(k)} \left(\frac{V^{\mathrm{i}}+V^{\mathrm{i}}(\mu_{k'}(g)-1)}{\mu_{k'}(g)}\right) = \frac{\mu_k(g)-1}{\mu_k(g)} V^{\mathrm{i}} \leq \tfrac{1}{2} V^{\mathrm{i}}(\mu_k(g) - 1),$$

so (12) holds for all $k \in \bar{S}(i)$.

Combining (11) and (13) implies $T_{i,j}(g) < T_{i,h}(g)$, a contradiction to equation (10), so g consists of star components only.

The proof of Lemma 2 implies that these stars have at most three agents. "Stars" of a single agent cannot be part of g, for it is always strictly beneficial for this single agent to create a link to the center agent of another star, whereas this center agent is indifferent or improves if she is isolated too. ■

Table 1 pictures all communication networks thus proven to be pairwise stable in the case with both social and informational value from communication and $\rho = 1$ for $n \leq 6$. Comparing these results to the purely social value case (Section 2.2), clearly a much smaller range of very fragmented networks turns out to be pairwise stable in the mixed case where transferable informational value also plays a role. Specifically, even with α slightly above zero, regular communication networks are never pairwise stable and also the example networks in Figures 2, 3, and 4 are not stable anymore.

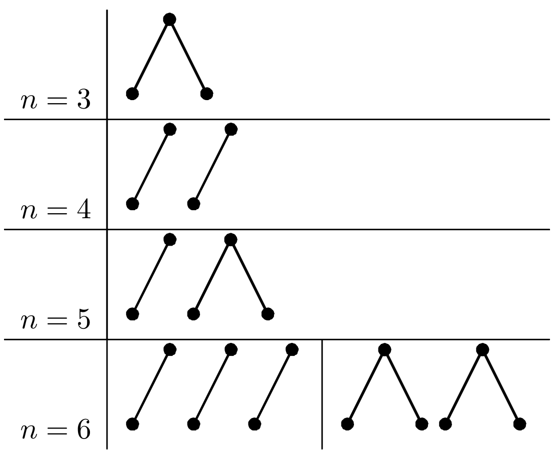


Table 1: Pairwise stable communication networks for $\rho = 1$ and $\alpha > 0$

This may seem counter-intuitive, since apparently transferability of informational value causes networks to become more fragmented and therefore *less* able to transfer information. The intuition behind this finding is that the link specificity property of communication is now strong enough to prevent individuals from maintaining many links since it is strengthened by the transferability of informational value. For example, in a wheel network of three agents, an agent cannot improve (or decrease) her social payoff by deleting one of her links, but she can improve her informational payoff:

$$\frac{V^{\mathrm{i}}}{2} + \frac{V^{\mathrm{i}}}{4} > \frac{2V^{\mathrm{i}}}{4} + \frac{2V^{\mathrm{i}}}{16}.$$

The co-author model of Jackson and Wolinsky (1996) also contains a type of link specificity, but since it is not combined with value transferability, the resulting stable networks are not as fragmented. Similarly, the connections model of Jackson and Wolinsky (1996) contains value transferability, but since it is not combined with link specificity, the resulting networks are not fragmented at all. Likewise, most studies reveal less fragmented stable networks, e.g., Goyal & Vega-Redondo (2007) find large star networks in their setting of structural holes. Therefore, our model can explain real-life phenomena like marriage and the evolvement of threads in online communities into strong reciprocal ties (as empirically observed by Fisher et al., 2006).

3.3 Stable networks under low link specificity

For $\rho = 1/2$, we prove that the complete communication network is pairwise stable by retracing that also with value transferability it is never beneficial for an agent in the complete network to delete one of her links under this low level of link specificity.

Proposition 4 *When $\rho = 1/2$ and $0 \leq \alpha \leq 1$, the complete communication network is pairwise stable.*

Proof. We normalize payoffs by setting $V^{\mathrm{i}} = 1$. For $\alpha = 1$, the payoff for an agent in the complete network is

$$1 + \sum_{q=2}^{n-1} \frac{\prod_{r=2}^{q}(n-r)}{(n-1)^{q-1}},$$

where q indicates the step level, and if she deletes a link her payoff becomes

$$\frac{1+2(n-2)\frac{\sqrt{n-2}}{\sqrt{n-1}}}{n-1} + \sum_{q=3}^{n-1} \frac{\prod_{r=3}^{q}(n-r)}{(n-1)^{q-1}} \left(1 + \frac{n^2-5n+q+4}{\sqrt{n-1}\sqrt{n-2}}\right),$$

where the first term combines payoffs resulting from paths with length 1 and 2. Subtracting the latter from the former gives

$$\sum_{q=3}^{n-1} \left(\frac{\prod_{r=3}^{q}(n-r)}{(n-1)^{q-1}} \left(n - 3 - \frac{n^2-5n+q+4}{\sqrt{n-1}\sqrt{n-2}}\right) + \frac{2(n-2)-2(n-2)\frac{\sqrt{n-2}}{\sqrt{n-1}}}{(n-1)(n-3)} \right). \qquad (14)$$

We have to prove that (14) is nonnegative. Multiplying by $(n-1)$, we find that it is sufficient to show that

$$\sum_{q=3}^{n-1}\left(\left(\prod_{r=3}^{q}\tfrac{n-r}{n-1}\right)(n-3-\tfrac{n^2-5n+q+4}{\sqrt{n-1}\sqrt{n-2}})+\tfrac{2(n-2)}{n-3}-\tfrac{2(n-2)^2}{(n-3)\sqrt{n-1}\sqrt{n-2}}\right)\geq 0. \qquad (15)$$

When we define

$$\begin{aligned} a(q) &= \prod_{r=3}^{q}\tfrac{n-r}{n-1}, \\ b(q) &= n-3-\tfrac{n^2-5n+q+4}{\sqrt{n-1}\sqrt{n-2}}, \end{aligned}$$

then the first term in (15) is given by

$$\sum_{q=3}^{n-1} a(q)b(q).$$

The second minus the third term in (15) is positive. We show by means of contradiction that the first term is nonnegative. Suppose the first term is negative, implying $n\geq 4$.

Notice that $a(q)\geq 0$ and $b(q)$ is decreasing in q, so there is $\bar{q}\geq 3$ such that $3\leq q<\bar{q}$ implies $a(q)b(q)\geq 0$ and $\bar{q}\leq q\leq n-1$ implies $a(q)b(q)<0$. This fact together with the supposition

$$\sum_{q=3}^{n-1} a(q)b(q)<0,$$

implies that

$$\sum_{q=3}^{n-1} a(q)b(q)>\sum_{q=3}^{n-1}\lambda(q)a(q)b(q)$$

for coefficients $\lambda(q)$ larger than or equal to 1 and nondecreasing in q. We define

$$\lambda(q)=\prod_{r=3}^{q-1}\tfrac{n-1}{n-r},$$

with $\lambda(3)=1$ by definition. Then we have obtained the desired contradiction once we show that

$$\sum_{q=3}^{n-1}\left(\tfrac{n-q}{n-1}(n-3-\tfrac{n^2-5n+q+4}{\sqrt{n-1}\sqrt{n-2}})+\tfrac{2(n-2)}{n-3}-\tfrac{2(n-2)^2}{(n-3)\sqrt{n-1}\sqrt{n-2}}\right)\geq 0.$$

It holds that

$$\begin{aligned} \sum_{q=3}^{n-1}\tfrac{n-q}{n-1}(n-3) &= \tfrac{(n-2)(n-3)^2}{2(n-1)}, \\ \sum_{q=3}^{n-1}\tfrac{n-q}{n-1}\tfrac{n^2-5n+4}{\sqrt{n-1}\sqrt{n-2}} &= \tfrac{(n-2)(n-3)(n-4)}{2\sqrt{n-1}\sqrt{n-2}}, \\ \sum_{q=3}^{n-1}\tfrac{n-q}{n-1}\tfrac{q}{\sqrt{n-1}\sqrt{n-2}} &= \tfrac{n(n-3)(n+2)}{2(n-1)\sqrt{n-1}\sqrt{n-2}}-\tfrac{2n^3-3n^2+n-30}{6(n-1)\sqrt{n-1}\sqrt{n-2}}, \\ \sum_{q=3}^{n-1}\tfrac{2(n-2)}{n-3}-\tfrac{2(n-2)^2}{(n-3)\sqrt{n-1}\sqrt{n-2}} &= 2(n-2)-\tfrac{2(n-2)^2}{\sqrt{n-1}\sqrt{n-2}}, \end{aligned}$$

where for the third inequality we use the fact that $1^2 + 2^2 + \cdots + r^2 = \frac{1}{3}r^3 + \frac{1}{2}r^2 + \frac{1}{6}r$. After multiplying by 6 and rewriting we obtain the inequality

$$\tfrac{3n^3-12n^2+27n-30}{n-1} - \tfrac{3n^4-17n^3+45n^2-73n+54}{(n-1)\sqrt{n-1}\sqrt{n-2}} \geq 0.$$

The expression on the left-hand side exceeds

$$\tfrac{3n^3-12n^2+27n-30}{n-1} - \tfrac{3n^4-17n^3+45n^2-73n+54}{(n-1)(n-\frac{8}{5})}.$$

Cross multiplying, we find that the last expression is larger than or equal to zero if and only if

$$3n^4 - 16\tfrac{4}{5}n^3 + 46\tfrac{1}{5}n^2 - 73\tfrac{1}{5}n + 48 \geq 3n^4 - 17n^3 + 45n^2 - 73n + 54.$$

For $n \geq 4$, such is clearly the case. Thus, the complete network is pairwise stable for $\alpha = 1$.

Since it follows from Proposition 2 that the complete network is stable for $\alpha = 0$ and given the linear combination in equation (8), the complete network is pairwise stable for $0 \leq \alpha \leq 1$. ■

The following example illustrates that already at relatively low α, multi-component communication networks (cf. Proposition 2 for $\alpha = 0$) are not pairwise stable anymore.

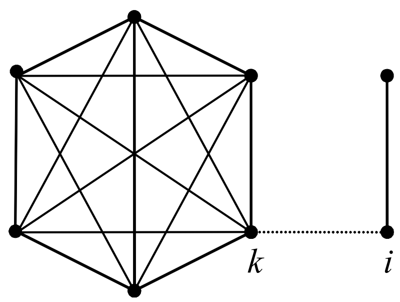


Figure 5: A communication network that is pairwise stable when $\rho = 1/2$ and $\alpha = 0$

Example 6 Assume $\rho = 1/2$ and consider the communication network in Figure 5. When $\alpha = 0$, the current payoff for agent i is V^{s} and if she would create a link with agent k it would become $\left(1/\sqrt{2} + 1/\sqrt{12}\right) V^{\mathrm{s}} \approx 0.99578 V^{\mathrm{s}}$. When $\alpha = 1$, the payoff for i is V^{i} and with a link to k would become $\left(1/\sqrt{2} + 1/\sqrt{12} + 523/250\sqrt{10}\right) V^{\mathrm{i}} \approx 1.65733 V^{\mathrm{i}}$. When $0 < \alpha < 1$, the payoff for i is $\alpha V^{\mathrm{i}} + (1-\alpha) V^{\mathrm{s}}$ and with a link to k would become

$$1.65733 \alpha V^{\mathrm{i}} + 0.99578(1-\alpha) V^{\mathrm{s}},$$

which for $V^{\mathrm{i}} = V^{\mathrm{s}}$ exceeds the current payoff if $\alpha > 0.0064$. Since k is willing to create a link with i for any α, it holds that this network is not pairwise stable when $\alpha > 0.0064$.

4 Efficiency

In this section, the structural results from the previous sections are assessed by their impact on efficiency. We define the efficiency of a network as the sum of payoffs for all agents.[4] It appears that the fragmented stable communication networks under high link specificity as well as the dense stable communication networks under low link specificity can be most efficient in their own setting. Therefore, although not all pairwise stable communication networks found for $\rho = 1$ and $\rho = 1/2$ are most efficient, we conclude in contrast to Jackson and Wolinsky (1996), that stability and efficiency are compatible.

Definition 9 (efficiency) The efficiency provided by network g is given by

$$W(g) = \sum_{i \in N} \Pi_i(g).$$

We conjecture that for $\rho = 1/2$, the densification that characterizes the pairwise stable communication networks has a generally beneficial influence on efficiency, as well as that for $\rho = 1$, the fragmentation that characterizes the pairwise stable communication networks whenever $\alpha > 0$ has a generally beneficial influence on efficiency.[5]

Conjecture 1 *(i) When $\rho = 1$ and n is even, a communication network consisting of disjoint pair components is most efficient, and when $\rho = 1$ and n is odd, a communication network consisting of one three-agent star component and furthermore disjoint pair components is most efficient.*

(ii) When $\rho = 1/2$, the complete communication network is most efficient.

For tractability, we proof the conjecture for low n and for the class of communication networks having only complete components.

Proposition 5 *Conjecture 1 holds when $n = 3, 4, 5,$ or 6.*

Proof. Table 2 lists the normalized efficiency for $\alpha = 1$ and $\alpha = 0$ at $\rho = 1/2$ and $\rho = 1$ for all possible nonisomorphic communication networks when $n = 3, 4, 5,$ and 6, ordered according to their efficiency level at $\alpha = 1$, $\rho = 1/2$. Given the linear combination in equation 8, the conjecture is hereby proven for all $0 \leq \alpha \leq 1$ and $n = 3, 4, 5,$ or 6. ■

Proposition 6 *Consider the class of communication networks consisting of fully connected components $c_1, \ldots, c_k$. When m_{c_j} is the number of members of c_j, the following holds.*

(i) When $\rho = 1/2$, g is most efficient in this class when $k = 1$ and $m_{c_1} = n$, so g is complete.

(ii) When $\rho = 1$ and n is even, g is most efficient when $k = n/2$ and $m_{c_j} = 2$ for all j, so g consists of disjoint pairs.[6]

[4] Alternatively, it may be interesting to study the structural effects on the actual amount of information exchanged, thus disregarding the value derived from social aspects of communication.

[5] For $\alpha = 0$, pairwise stable communication networks tend to be overconnected, cf. Morrill (2010) and Section 5.

[6] Notice that for odd n, the most efficient communication network as predicted by Conjecture 1 includes a component that is not complete.

	$\alpha = 1$		$\alpha = 0$		
	$\rho = \frac{1}{2}$	$\rho = 1$	$\rho = \frac{1}{2}$	$\rho = 1$	
$n = 3$	0.000	0.000	0.000	0.000	
	2.000	2.000	2.000	*2.000*	
	3.828	*2.500*	2.828	*2.000*	3-agent star
	4.500	1.875	*3.000*	1.500	complete
$n = 4$	0.000	0.000	0.000	0.000	
	2.000	2.000	2.000	2.000	
	3.828	2.500	2.828	2.000	
	4.000	*4.000*	*4.000*	*4.000*	2 disjoint pairs
	4.500	1.875	3.000	1.500	
	5.464	2.667	3.464	2.000	
	5.743	3.125	3.828	2.500	
	6.352	2.333	3.788	1.833	
	7.000	2.625	*4.000*	2.000	
	7.121	1.975	3.933	1.556	
	7.556	1.663	*4.000*	1.333	complete
$n = 5$	0.000	0.000	0.000	0.000	
	2.000	2.000	2.000	2.000	
	3.828	2.500	2.828	2.000	
	4.000	4.000	4.000	*4.000*	
	4.500	1.875	3.000	1.500	
	5.464	2.667	3.464	2.000	
	5.743	3.125	3.828	2.500	
	5.828	*4.500*	4.828	*4.000*	3-agent star +
	6.352	2.333	3.788	1.833	disjoint pair
	6.500	3.875	*5.000*	3.500	
	7.000	2.625	4.000	2.000	
$n = 5$ cont.	7.000	2.750	4.000	2.000	
	7.121	1.975	3.933	1.556	
	7.394	3.389	4.540	2.667	
	7.556	1.663	4.000	1.333	
	7.700	3.781	4.828	3.000	
	7.993	2.594	4.414	2.000	
	8.232	2.815	4.609	2.222	
	8.298	3.097	4.864	2.500	
	8.829	3.042	4.788	2.333	
	8.909	2.337	4.625	1.833	
	8.993	2.508	4.828	2.000	
	9.035	2.503	4.788	2.000	
	9.375	3.320	*5.000*	2.500	
	9.407	2.071	4.732	1.667	
	9.654	2.648	4.933	2.056	
	9.676	2.062	4.743	1.625	
	9.826	2.166	4.966	1.778	
	9.865	2.620	4.869	1.722	
	10.245	2.261	4.899	2.000	
	10.257	1.709	4.890	1.514	
	10.570	1.762	4.976	1.556	
	10.766	1.794	4.964	1.375	
	11.094	1.849	*5.000*	1.250	complete
$n = 6$	0.000	0.000	0.000	0.000	
	2.000	2.000	2.000	2.000	
	3.828	2.500	2.828	2.000	
$n = 6$ cont.	4.000	4.000	4.000	4.000	
	4.500	1.875	3.000	1.500	
	5.464	2.667	3.464	2.000	
	5.743	3.125	3.828	2.500	
	5.828	4.500	4.828	4.000	
	6.000	*6.000*	*6.000*	*6.000*	3 disjoint pairs
	6.352	2.333	3.788	1.833	
	6.500	3.875	5.000	3.500	
	7.000	2.625	4.000	2.000	
	7.000	2.750	4.000	2.000	
	7.121	1.975	3.933	1.556	
	7.394	3.389	4.540	2.667	
	7.464	4.667	5.464	4.000	
	7.556	1.663	4.000	1.333	
	7.657	5.000	5.657	4.000	
	7.700	3.781	4.828	3.000	
	7.743	5.125	5.828	4.500	
	7.993	2.594	4.414	2.000	
	8.232	2.815	4.609	2.222	
	8.298	3.097	4.864	2.500	
	8.328	4.375	5.828	3.500	
	8.352	4.333	5.788	3.833	
	8.472	2.800	4.472	2.000	
	8.829	3.042	4.788	2.333	
	8.909	2.337	4.625	1.833	
$n = 6$ cont.	8.932	3.531	5.121	2.750	
	8.993	2.508	4.828	2.000	
	9.000	4.625	*6.000*	4.000	
	9.000	3.750	*6.000*	3.000	
	9.035	2.503	4.788	2.000	
	9.047	3.728	5.285	2.889	
	9.121	3.975	5.933	3.556	
	9.352	4.125	5.616	3.333	
	9.358	4.069	5.540	3.167	
	9.375	3.320	5.000	2.500	
	9.407	2.071	4.732	1.667	
	9.526	2.760	4.948	2.100	
	9.556	3.663	*6.000*	3.333	
	9.654	2.648	4.933	2.056	
	9.676	2.062	4.743	1.625	
	9.678	4.445	5.828	3.500	
	9.826	2.166	4.869	1.722	
	9.865	2.620	4.899	2.000	
	9.882	3.097	5.256	2.417	
	9.937	3.398	5.536	2.750	
	9.955	3.469	5.609	2.722	
	10.099	3.317	5.464	2.667	
	10.204	3.586	5.685	2.889	
	10.245	2.261	4.966	1.778	
	10.257	1.880	4.890	1.514	

	$\alpha=1$		$\alpha=0$	
	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=\frac{1}{2}$	$\rho=1$
$n=6$	10.271	3.788	5.864	3.000
cont.	10.450	3.281	5.414	2.500
	10.529	2.578	5.203	2.000
	10.570	1.943	4.976	1.556
	10.586	2.765	5.424	2.200
	10.645	3.468	5.575	2.667
	10.645	3.494	5.609	2.722
	10.701	2.824	5.454	2.222
	10.709	2.707	5.328	2.125
	10.766	1.688	4.964	1.375
	10.824	2.877	5.500	2.306
	10.824	3.802	5.864	3.000
	10.865	3.224	5.933	2.556
	10.870	3.161	5.746	2.583
	10.894	3.024	5.670	2.417
	10.971	3.034	5.643	2.444
	11.014	3.287	5.864	2.667
	11.066	2.352	5.338	1.867
	11.094	1.516	5.000	1.250
	11.151	3.722	5.788	2.833
	11.271	2.485	5.476	2.014
	11.368	2.907	5.853	2.417
	11.401	2.356	5.360	1.850
	11.410	2.994	5.625	2.333
	11.471	3.189	5.828	2.500
	11.487	3.126	5.754	2.444

	$\alpha=1$		$\alpha=0$	
	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=\frac{1}{2}$	$\rho=1$
$n=6$	11.537	2.464	5.492	1.956
cont.	11.593	3.170	5.788	2.500
	11.597	2.619	5.679	2.100
	11.604	2.943	5.571	2.250
	11.616	2.626	5.638	2.125
	11.625	3.996	*6.000*	3.000
	11.647	2.560	5.593	2.042
	11.716	2.773	5.869	2.222
	11.761	2.717	5.744	2.194
	11.765	3.130	5.754	2.444
	12.004	3.318	5.933	2.556
	12.024	2.203	5.535	1.772
	12.045	2.640	5.678	2.083
	12.104	2.660	5.698	2.111
	12.128	2.786	5.821	2.222
	12.133	2.417	5.814	1.967
	12.143	2.306	5.646	1.875
	12.149	2.126	5.460	1.680
	12.197	2.452	5.785	2.014
	12.217	2.699	5.743	2.125
	12.218	3.319	5.933	2.556
	12.331	2.825	5.933	2.222
	12.348	3.285	5.899	2.500
	12.362	2.269	5.627	1.822
	12.363	2.815	5.869	2.222
	12.373	2.762	5.835	2.167

	$\alpha=1$		$\alpha=0$	
	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=\frac{1}{2}$	$\rho=1$
$n=6$	12.410	2.292	5.669	1.833
cont.	12.434	2.358	5.720	1.903
	12.528	2.605	5.657	2.000
	12.551	2.381	5.780	1.911
	12.567	2.746	5.815	2.139
	12.576	2.347	5.743	1.875
	12.583	2.031	5.631	1.658
	12.591	2.436	5.857	1.958
	12.664	2.125	5.732	1.750
	12.688	2.912	5.966	2.278
	12.792	2.510	5.890	2.014
	12.830	2.026	5.662	1.636
	12.912	2.863	5.933	2.222
	12.938	1.870	5.683	1.550
	12.956	2.456	5.857	1.958
	12.962	2.430	5.837	1.931
	12.974	2.089	5.745	1.692
	13.006	2.481	5.922	1.972
	13.018	2.863	5.933	2.222
	13.035	2.161	5.844	1.756
	13.041	2.153	5.828	1.750
	13.076	2.397	5.805	1.889
	13.207	2.522	5.942	2.000
	13.270	2.142	5.824	1.728
	13.332	2.498	5.922	1.972
	13.361	2.186	5.890	1.764

	$\alpha=1$		$\alpha=0$		
	$\rho=\frac{1}{2}$	$\rho=1$	$\rho=\frac{1}{2}$	$\rho=1$	
$n=6$	13.448	1.893	5.780	1.547	
cont.	13.473	2.222	5.910	1.792	
	13.478	2.206	5.915	1.778	
	13.506	2.541	*6.000*	2.000	
	13.509	1.942	5.865	1.591	
	13.541	1.943	5.852	1.592	
	13.566	2.182	5.878	1.750	
	13.616	2.215	5.952	1.778	
	13.730	2.234	5.964	1.792	
	13.838	1.979	5.920	1.614	
	13.842	1.758	5.848	1.455	
	13.852	2.555	*6.000*	2.000	
	13.861	1.993	5.914	1.625	
	13.912	2.242	5.964	1.792	
	13.967	1.969	5.908	1.600	
	13.976	1.747	5.848	1.440	
	14.058	2.000	5.964	1.625	
	14.069	1.783	5.909	1.472	
	14.122	2.018	5.976	1.639	
	14.305	1.813	5.960	1.492	
	14.411	1.639	5.932	1.365	
	14.484	1.827	*6.000*	1.500	
	14.602	1.660	5.978	1.380	
	14.797	1.524	5.978	1.280	
	15.062	1.417	*6.000*	1.200	complete

Table 2: Normalized efficiency for all nonisomorphic communication networks when $\alpha = 1, 0$, $\rho = 1/2, 1$, and $n = 3, 4, 5, 6$

Proof. Because of symmetry in a complete component it suffices to consider the payoffs for one member and because of the linear combination in equation 8 it suffices to consider the cases $\alpha = 0$ and $\alpha = 1$.

(i) For $\rho = 1/2$ and $\alpha = 0$, the normalized payoff for an agent in component c_j is 0 when $m_{c_j} = 1$ and 1 when $m_{c_j} \geq 2$. For $\rho = 1/2$ and $\alpha = 1$, the normalized payoff for an agent in component c_j is 0 when $m_{c_j} = 1$, 1 when $m_{c_j} = 2$ and

$$1 + \sum_{q=2}^{m_{c_j}-1} \frac{\prod_{r=2}^{q} \left(m_{c_j} - r\right)}{\left(m_{c_j} - 1\right)^{q-1}} = 1 + \sum_{q=2}^{m_{c_j}-1} \prod_{r=2}^{q} \frac{m_{c_j} - r}{m_{c_j} - 1}$$

when $m_{c_j} \geq 3$, which is $3/2$ for $m_{c_j} = 3$ and increasing in m_{c_j}.

(ii) For $\rho = 1$ and $\alpha = 0$, the normalized payoff for an agent in component c_j is 0 when $m_{c_j} = 1$ and $1/\left(m_{c_j} - 1\right)$ when $m_{c_j} \geq 2$, which is 1 for $m_{c_j} = 2$ and decreasing in m_{c_j}. For $\rho = 1$ and $\alpha = 1$, the normalized payoff for an agent in component c_j is 0 when $m_{c_j} = 1$, 1 when $m_{c_j} = 2$ and

$$\frac{1}{m_{c_j}-1} + \sum_{q=2}^{m_{c_j}-1} \frac{\prod_{r=2}^{q} \left(m_{c_j} - r\right)}{\left(m_{c_j} - 1\right)^{2q-1}} = \frac{1 + \sum_{q=2}^{m_{c_j}-1} \prod_{r=2}^{q} \frac{m_{c_j} - r}{\left(m_{c_j} - 1\right)^2}}{m_{c_j} - 1}$$

when $m_{c_j} \geq 3$, which is $5/8$ for $m_{c_j} = 3$ and decreasing in m_{c_j}. ■

5 Other values of link specificity

In this section, other link specificity values than 1 and 1/2 are investigated by simulations of our communication network formation model. They illustrate that 1 and 1/2 are indeed suitable polar cases and that for intermediate values the common tension between stability and efficiency (e.g., Jackson and Wolinsky, 1996) is re-established.

5.1 Method

5.1.1 Pairwise stability

A simulation starts with a random network in the sense that for every $i, j \in N$ with $i \neq j$, the link $g_{i,j}$ is randomly chosen to be equal to 0 or 1.

In every iteration we randomly determine whether there will be an attempt to delete or create a link. If this turns out to be delete, one agent is randomly drawn from the community and subsequently another one. If there exists a link between these two agents, the first agent calculates the payoff she will earn when she deletes this link (equation (8)). If this is strictly higher than the payoff she earns with the current network, the link is deleted. If no link exists between the two agents or the first agent does not gain by its deletion, the current network is maintained until the next iteration.

If the attempt appears to be create, two agents are randomly drawn from the community. If no link exists between these two agents, they both calculate the payoff they will earn when they create such a link. If this is weakly higher for both agents and strictly higher for at least one of them, the link is created. If there already exists a link between these two agents or one of them loses or none of them gains by its creation, the current network is maintained.

In this way, 5,000 iterations are performed consecutively. Afterwards it is verified whether the simulation converged to a pairwise stable network. We perform 500 of these simulations for each of 15 ($\rho = 0.1, 0.2, ..., 1.4, 1.5$) x 3 ($\alpha = 0, 1/2, 1$) parameter value combinations.

5.1.2 Local efficiency

We also perform simulations to compute locally efficient communication networks. This is achieved by repeating the procedure of Section 5.1.1, assuming that agents all hold efficiency as their objective function (Definition 9). Pairwise stability after such a simulation establishes local efficiency.

Definition 10 The network g is locally efficient if for all $i, j \in N$ with $g_{i,j} = 1$ it holds that

$$W(g) \geq W(g'),$$

where g' is such that $g'_{i,j} = 0$ and $g'_{k,l} = g_{k,l}$ for all $\{k, l\} \neq \{i, j\}$, and for all $i, j \in N$ with $g_{i,j} = 0$ it holds that

$$W(g) \geq W(g'),$$

where g' is such that $g'_{i,j} = 1$ and $g'_{k,l} = g_{k,l}$ for all $\{k, l\} \neq \{i, j\}$.

5.1.3 Further specifications

Community size $n = 6$ is chosen for all simulations, since it is large enough to illustrate interesting tendencies as well as small enough to generate reasonable calculation times regarding the exponentially increasing number of paths in the payoff function (equation (8)). Furthermore, we take $V^{\mathrm{i}} = V^{\mathrm{s}} = 6$. We have verified that our results are robust for the case with an odd number of agents $n = 5$.

In order to compare simulation outcomes among levels of ρ and α, we use the density of a network (Wasserman and Faust, 1994, p.164):

$$D(g) = \frac{\sum_{i=1}^{n} \mu_i(g)}{n(n-1)} = \frac{1}{30} \sum_{i=1}^{6} \mu_i(g).$$

For example, the empty network has density 0, a network consisting of three disjoint pair components has density 0.2, a network consisting of two disjoint three-agent star components has density 0.27, a network consisting of a four-agent wheel component and a disjoint pair component or a network consisting of a three-agent complete component and a disjoint three-agent star component has density 0.33, a wheel network or a network consisting of two disjoint complete three-agent components has density 0.4, a network consisting of a complete four-agent component and a disjoint pair component has density 0.47, a regular network with degree 3 has density 0.6, a regular network with degree 4 has density 0.8, and the complete network has density 1.

Our intuitive prediction is that in general, a higher level of link specificity ρ makes communication more costly, and therefore the density of a simulated communication network lower. This intuition is confirmed by the analytically found potential outcomes for the cases $\rho = 1$ and $\rho = 1/2$ (Sections 2 and 3). A higher level of focus on informational value α provides more value spillovers from indirect links, and thus the expected density of a pairwise stable communication network is higher. As noted at the end of Section 3.2, this intuition is not confirmed by the analytically found potential outcomes: whereas in Section 2.2 a large range of possibly dense communication networks is proven to be pairwise stable for $\rho = 1$ and $\alpha = 0$, in Section 3.2 only very fragmented communication networks turn out to be pairwise stable for $\rho = 1$ and $\alpha = 1$.

5.2 Results

It appears that all 500 simulations in every setting converge to pairwise stable networks as described in Section 5.1.1. Therefore, we deal with density and efficiency of these networks.

5.2.1 Density

For each combination of 15 levels of link specificity ρ and three levels of focus on informational versus social value from communication α, the average density of the 500 simulated pairwise stable and locally efficient networks are given in Table 3 and represented in Figure 6.

$\rho \backslash \alpha$	0		$\frac{1}{2}$		1	
	pws	locef	pws	locef	pws	locef
0.1	1.00	1.00	1.00	1.00	1.00	1.00
0.2	1.00	1.00	1.00	1.00	1.00	1.00
0.3	1.00	1.00	1.00	1.00	1.00	1.00
0.4	1.00	1.00	1.00	1.00	1.00	1.00
0.5	1.00	0.58	1.00	1.00	1.00	1.00
0.6	1.00	0.22	1.00	0.43	1.00	0.52
0.7	0.95	0.22	1.00	0.25	1.00	0.40
0.8	0.70	0.23	0.69	0.23	0.54	0.22
0.9	0.38	0.23	0.37	0.24	0.36	0.24
1.0	0.27	0.23	0.23	0.24	0.23	0.24
1.1	0.20	0.20	0.20	0.23	0.20	0.24
1.2	0.20	0.20	0.20	0.20	0.20	0.23
1.3	0.20	0.20	0.20	0.20	0.20	0.20
1.4	0.20	0.20	0.20	0.20	0.20	0.20
1.5	0.20	0.20	0.20	0.20	0.20	0.20

Table 3: Simulated effect of ρ on density of pairwise stable and locally efficient communication networks for $n = 6$ and $\alpha = 0, 1/2, 1$

The basic intuition about the effect of ρ on density in communication networks is thus confirmed by the simulation outcomes as it was by the analytical results of Sections 2 and 3, as the density

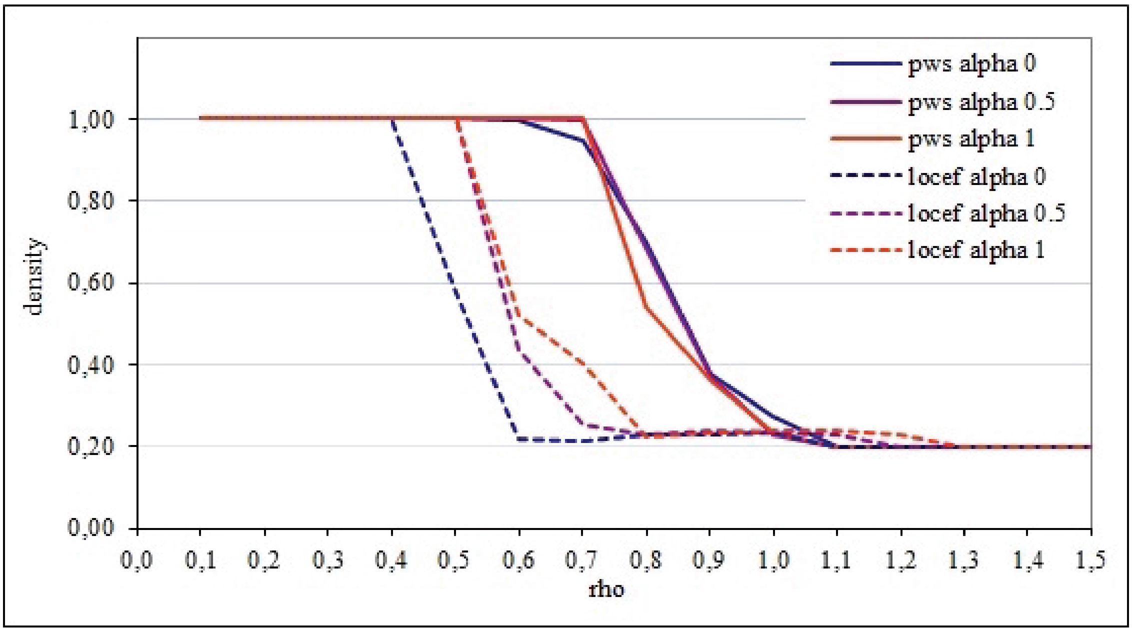


Figure 6: Simulated effect of ρ on density of pairwise stable and locally efficient communication networks for $n = 6$ and $\alpha = 0, 0.5, 1$

of the pairwise stable networks is generally decreasing in ρ. Notice that this also roughly holds for the local efficiency simulation outcomes, although a few small exceptions appear, for example at $\rho = 0.8$.

The intuition about the effect of α on density in communication networks is again contradicted by the simulation outcomes, e.g., for $\rho = 0.8$ we find a higher average density for $\alpha = 1/2$ (0.69) than for $\alpha = 1$ (0.54), and for $\rho = 1.0$ we find a higher average density for $\alpha = 0$ (0.27) than for $\alpha = 1/2$ (0.23).[7] This can again be ascribed to the interaction effect of value transferability and high link specificity (cf. end of Section 3.2). Notice that our intuition about the effect of α on density in communication networks is still confirmed by the local efficiency simulation outcomes, in particular for ρ around 0.6.

The simulations confirm that $\rho = 1$ and $\rho = 1/2$ are suitable polar cases, since outside these boundaries the simulated pairwise stable as well as locally efficient communication networks largely coincide with either the complete network or rather a network consisting of pairs only, whereas in between we find large differences in density between the locally efficient and pairwise stable communication networks, where pairwise stable networks are more connected than locally efficient networks. Thus, for intermediate levels of ρ, the pairwise stable communication networks are highly connected as with lower levels of link specificity, whereas the locally efficient communication networks are very fragmented as with higher levels of link specificity. The following example elaborates such a case.

[7]Thus, for $\alpha = 0$ the simulation process on average selects relatively sparse networks from the wide range of pairwise stable networks, which reduces the conflict with intuition.

Example 7 *Let $\rho = 3/4$, $\alpha = 1/2$. Table 4 provides an overview of the types and numbers of pairwise stable communication networks resulting from 100 simulations and the types and numbers of locally efficient communication networks resulting from another 100 simulations.*

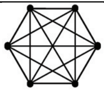
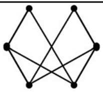
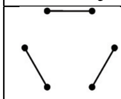
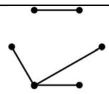
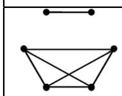
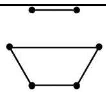
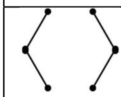
pairwise stable networks		**locally efficient networks**	
 84	 6	 67	 9
 8	 2	 24	

Table 4: Example of simulated communication networks

5.2.2 Efficiency

As a means to investigate to what extent the density differences between the locally efficient and pairwise stable communication networks actually lead to efficiency differences, the average efficiency of the 500 simulated pairwise stable and locally efficient networks are given in Table 5 and represented in Figure **??** for each parameter setting. For comparison, the expected efficiency from a random network (based on 5000 randomly generated networks) and the maximum efficiency have also been calculated for each setting.

$\rho\backslash\alpha$	0				$\frac{1}{2}$				1			
	pws	locef	rand	max	pws	locef	rand	max	pws	locef	rand	max
0.1	130.5	130.5	73.4	130.5	1758.8	1758.8	274.3	1758.8	3387.2	3387.2	471.8	3387.2
0.2	94.6	94.6	59.9	94.6	601.3	601.3	160.3	601.3	1108.1	1108.1	258.8	1108.1
0.3	68.5	68.5	49.1	68.5	243.0	243.0	101.8	243.0	417.4	417.4	154.3	417.4
0.4	49.7	49.7	40.4	49.7	115.8	115.8	70.0	115.8	181.8	181.8	99.2	181.8
0.5	36.0	35.9	33.4	36.0	63.2	63.2	50.4	63.2	90.4	90.4	67.3	90.4
0.6	26.1	34.8	27.6	36.0	38.0	42.6	37.8	43.0	50.0	54.6	48.0	55.4
0.7	19.7	34.5	23.0	36.0	24.5	35.9	29.3	36.0	29.9	43.6	35.7	43.6
0.8	19.3	32.2	19.2	36.0	22.1	33.8	23.2	36.0	30.1	35.7	27.2	36.0
0.9	23.9	31.6	16.1	36.0	26.8	32.0	18.7	36.0	30.0	33.6	21.3	36.0
1.0	29.2	30.2	13.5	36.0	32.3	31.0	15.2	36.0	32.9	32.2	17.0	36.0
1.1	36.0	36.0	11.5	36.0	36.0	31.3	12.5	36.0	36.0	31.1	13.7	36.0
1.2	36.0	36.0	9.7	36.0	36.0	36.0	10.5	36.0	36.0	31.8	11.2	36.0
1.3	36.0	36.0	8.3	36.0	36.0	36.0	8.8	36.0	36.0	36.0	9.4	36.0
1.4	36.0	36.0	7.1	36.0	36.0	36.0	7.5	36.0	36.0	36.0	7.9	36.0
1.5	36.0	36.0	6.1	36.0	36.0	36.0	6.4	36.0	36.0	36.0	6.7	36.0

Table 5: Simulated effect of ρ on efficiency of pairwise stable, locally efficient, random, and optimal communication networks for $n = 6$ and $\alpha = 0, 1/2, 1$

Although we have seen in Section 5.2.1 that a higher α and thus more value transferability does not always lead to more dense communication networks, it apparently does always lead to higher efficiency, as the efficiency of the pairwise stable networks is increasing in α, even strictly so for $\rho \leq 1.0$.[8]

[8] Notice that this claim is dependent on the assumption $V^{\mathrm{i}} = V^{\mathrm{s}}$ chosen for the simulations. For example, it can be shown that for $\rho = 1$ and $V^{\mathrm{s}} = xV^{\mathrm{i}}$, α has an opposite effect on efficiency if $x > 5/4$.

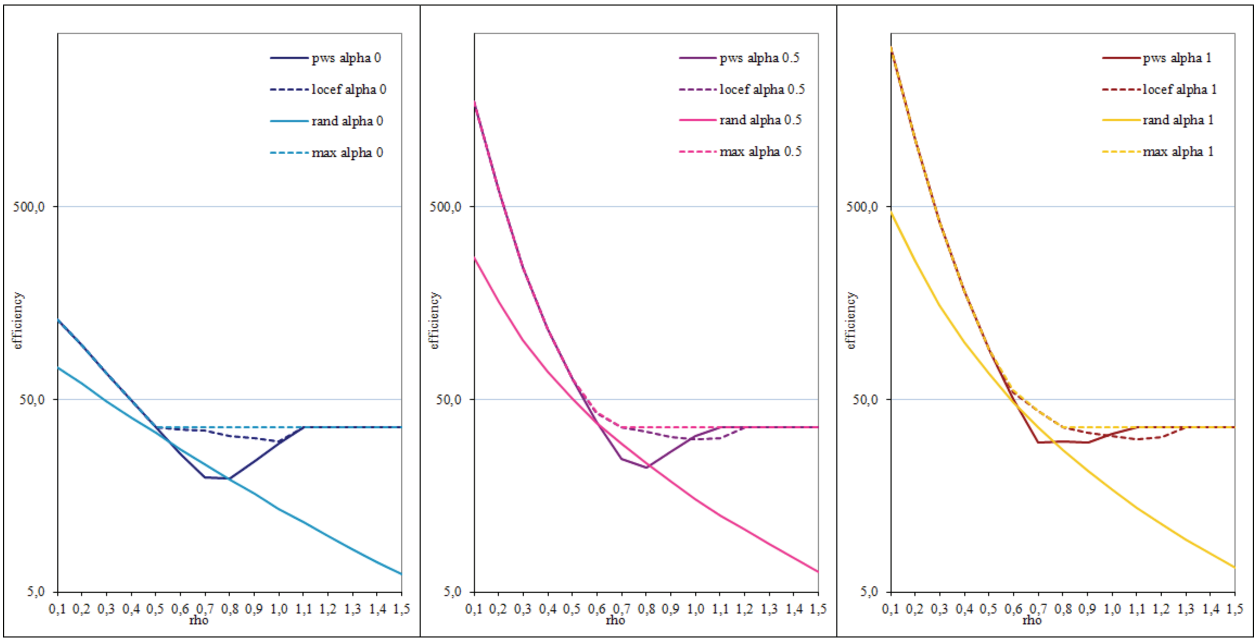


Figure 7: Simulated effect of ρ on efficiency of pairwise stable, locally efficient, random, and optimal communication networks for $n = 6$ and $\alpha = 0, 0.5, 1$

Furthermore, though a lower ρ has been shown to lead to higher density in communication networks, it only leads to subsequent higher efficiency when link specificity is low enough, since for intermediate values of ρ, efficiency is much lower and partly even strictly increases in ρ. Thus, a community should either specialize in the quantity (low ρ) or in the quality (high ρ) of her communication.

Moreover, we find that the density difference between pairwise stability and local efficiency in communication networks found for $\alpha = 0$, $\rho = 0.5$ does not lead to a subsequent efficiency difference, but for intermediate levels of link specificity ($1/2 < \rho < 1$), the tension between local efficiency and pairwise stability is considerable (up to larger than 40% for $\alpha = 0$ and $\rho = 0.7, 0.8$). In the most extreme intervals (i.e., around $\rho = 0.7$), the simulated pairwise stable communication networks are on average even less efficient than a random network.

Interestingly, on the other hand a few parameter combinations exist where social preferences appear to be harmful for efficiency in the sense that the simulated locally efficient communication networks are not globally optimal, whereas individual incentives do lead to maximum efficiency ($\alpha = 1/2$, $\rho = 1.1$ and $\alpha = 1$, $\rho = 1.1, 1.2$).

6 Discussion

6.1 Summary

This paper has shown that the structure of bilateral communication links within communication networks can be appropriately studied using a model based on the game-theoretic literature of social and economic network formation. A combination of important aspects common to communication networks was incorporated that had not been investigated until now: the negative externality of link specificity, which was neglected in the communication network context so far, and the positive externality of informational value transferability, while previous research modeled all value uniformly in this respect.

In the case of communication having nontransferable social value only (Section 2), illustrating the separate impact of link specificity on structure, the set of pairwise stable communication networks was characterized for high link specificity and shown to include a wide range of non-standard networks like highly connected and "small world" networks, whereas previous models for social and economic network formation mostly predicted simple networks like stars and wheels. For low link specificity, particular combinations of fully connected components were proven to be pairwise stable communication networks in line with the co-author model of Jackson and Wolinsky (1996).

In the case of communication from which both social and informational value is derived (Section 3), illustrating the joint impact of link specificity and value transferability on structure, under high link specificity only networks that consist of disjoint star components of two or three agents were shown to be pairwise stable. Herewith, we predict much more fragmentation than usually in the literature about social and economic network formation, where mostly only either of these two features was included. Under low link specificity, the opposite extreme effect takes place: already with small informational value transferability, multi-component communication networks may fail to be pairwise stable.

Both the fragmentation under high link specificity and the dense pairwise stable communication networks under low link specificity are most efficient in their own setting (Section 4), whereas for intermediate link specificity values quite inefficient communication networks may arise (Section 5).

6.2 Implications

Intuition predicts that a higher level of link specificity ρ makes communication more costly, and therefore stable communication networks will be sparser. This is confirmed by the analytical results for the cases $\rho = 1$ and $\rho = 1/2$ (Sections 2 and 3) as well as by the simulation outcomes for other ρ values (Section 5).

However, lower link specificity and thus higher density only leads to subsequent higher efficiency when link specificity is low enough, since for intermediate values of ρ, efficiency is much lower (Section 5). This implies that *enhancing* communication in the community by decreasing link specificity from a high to an intermediate level results in *lower* efficiency from communication. Although at first sight counterintuitive, these results are in line with the laws of specialization: a community should either focus on quantity (low ρ) or on quality (high ρ) in her communication efforts.

A higher level of focus on transferable informational value α provides more value spillovers from indirect links, so intuition predicts higher density for pairwise stable communication networks. This is contradicted by the analytical results for $\rho = 1$ (Sections 2 and 3) and to a smaller extent by the simulation outcomes for relatively high ρ values (Section 5), which can be ascribed to the interaction effect of value transferability and high link specificity.

However, when the potential social and informational value present in a community are equal, higher α does always lead to higher efficiency, even strictly so for $\rho \leq 1.0$ (Section 5). Therefore, stimulating the focus on informational value of communication can be a generally effective tool for boosting efficiency. In line with intuition, this effectivity is larger the smaller ρ.

6.3 Further research

Future studies could introduce valuation heterogeneity in the sense that individuals represent different values for their fellows or have different opinions on the values of their fellows, like Galeotti et al. (2006) for standard communication network formation models. For example, when $\rho = 1$ and $0 < \alpha < 1$, if we assume a valuation pattern deviating from full homogeneity in the sense that there is one agent j who is valued differently than all other agents, it can be proven that all pairwise stable communication networks consist of small star components and one possibly larger component without cycles containing the differing agent j but not at the periphery. In particular, this component may be a star component with agent j at the center.

Another extension of the current model could be to relax the assumption that agents divide their available effort equally among all their relationships, thus entering the subject of link strength and dropping the common one-zero formulation of links. Bloch and Dutta (2009) performed such a task for a different setting with one-sided link formation and found rather similar results as in their discrete variant. However, this similarity is partly ascribed to the assumption of a convex relationship between individual effort and link strength because of a fixed cost of link formation, whereas in our model a linear relationship would be more in line with the assumed costs of link maintenance rather than formation.

Besides, a possible follow-up would be to empirically examine the applicability of the used payoff function in diverse contexts. The model could be tested experimentally, contributing to an emerging literature as surveyed by Kosfeld (2004).

Accordingly, we hope that our current work stimulates future research in the appealing area of communication networks and the roles of affecting link specificity as well as balancing nontransferable social and transferable informational value.

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