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RESEARCH MEMORANDUM

BRIDGING THE GAP BETWEEN A STATIONARY POINT PROCESS AND ITS PALM DISTRIBUTION

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Summary. Let P be the distribution of a stationary point process on the real line and let P^0 be its Palm distribution. In this paper we consider probability measures which are equivalent to P^0, having simple relations with P. Relations between P and P^0 are derived with these intermediate measures as bridges. With the resulting Radon-Nikodym derivatives several well-known results can be proved easily. New results are derived. As a corollary of cross ergodic theorems a conditional version of the well-known inversion formula is proved. Several approximations of P^0 are considered, for instance the local characterization of P^0 as a limit of conditional probability measures $P_{1,n}, n \in \mathbb{N}$. The total variation distance between P^0 and $P_{1,n}$ can be expressed in terms of the P-distribution function of the forward recurrence time.

AMS 1980 *subject classifications.* Primary 60G55; secondary 60G10. *Key words and phrases.* Palm distribution, local characterization, inversion formula, ergodicity.

1 Introduction

In queueing theory it is often wanted to express expectations of time-stationary processes in terms of expectations of customer-stationary sequences. It turns out that the underlying theory for many problems of this type concerns the relationship between two probability measures, the distribution P of a stationary (marked) point process and the Palm distribution P^0 (intuitively arising from P by conditioning on the occurrence of a point (with some mark) in the origin). See e.g. Franken et al. (1982) and Baccelli & Brémaud (1987). As an example we mention Little's law (cf. page 41 of the second reference), linking quantities as the mean number of customers in a queueing system and the mean waiting time. The first mean is considered under P and the second under P^0. For this reason it is important to obtain a good understanding of the relationship between P and P^0.

In this paper we will try to bridge the gap between P and P^0. We will confine ourselves to unmarked point processes, although in the final section a generalization to marked point processes is briefly indicated.
The approach in this paper could be called the *Radon-Nikodym approach.* Several probability measures are considered which are equivalent to P^0 (in the sense of mutual domination), having simple relations with P. The resulting Radon-Nikodym derivatives are used to express P^0-expectations in terms of P-expectations (and vice versa).

Some of the results in this paper are also obtained elsewhere by more conventional methods. Usually, however, our approach is faster and more natural, adding some special elements.

The formal definition of the Palm distribution (see (1.3) below) is one possibility to go from P to P^0. With the classical inversion formula (see (1.5) below) we can go the other way. We will, however, consider probability measures which are intermediate between P and P^0, having simple relations with both.
Examples of such intermediate probability measures are considered in Section 2. They are equivalent to P^0 or to P with simple Radon-Nikodym derivatives. The advantages of using these measures as a bridge is illustrated. As a result of this approach some cross

ergodic theorems are proved in Section 3. No ergodicity conditions are assumed here. As a corollary a conditional version of the well-known inversion formula (1.5) is derived. Starting from P, some strong or pointwise approximations of P^0 are considered in Section 4. For these approximations to hold necessary and sufficient conditions are formulated. For this purpose a notion weaker than ergodicity of the point process is introduced. Some other intermediate probability measures, all equivalent to P^0, are considered. The well-known (and intuitively clear) uniform approximation of P^0 by conditional probability measures $P_{1,n}$, usually referred to as local characterization of the Palm distribution (cf., e.g., Franken et al. (1982; Th. 1.3.7)), is also considered. We derive a very simple expression for the total variation distance of P^0 and $P_{1,n}$. Conditions are given such that the rate of the resulting uniform convergence is of order $1/n$. In Section 5 a generalization to marked point processes is briefly indicated.

At the end of this section we formalize some of the notions mentioned above and give some other definitions and notations.

A *point process* on $\mathbf{R}$ is a random element Φ in the class M of all integer-valued measures φ on the σ-field Bor $\mathbf{R}$ of Borel sets on $\mathbf{R}$ for which

$$\varphi(B) < \infty \text{ for all bounded } B \in \text{ Bor } \mathbf{R}.$$

Let $\mathcal{M}$ be the σ-field generated by the sets $[\varphi(B) = k] := \{\varphi \in M : \varphi(B) = k\}, k \in \mathbf{N}_0$ and $B \in$ Bor $\mathbf{R}$. See Matthes, Kerstan & Mecke (1978), Kallenberg (1983/86) or Daley & Vere-Jones (1988) for more information. Set

$$\begin{aligned} M^\infty &:= \{\varphi \in M : \varphi(-\infty, 0) = \varphi(0, \infty) = \infty;\ \ \varphi\{x\} \leq 1 \text{ for all } x \in \mathbf{R}\}, \\ \mathcal{M}^\infty &:= M^\infty \cap \mathcal{M}. \end{aligned}$$

We will always assume that Φ (or rather its distribution P) is *stationary* (i.e., $\Phi(t+\cdot) =_d \Phi$ for all $t \in \mathbf{R}$). We also assume that $\Phi \neq 0$ wp1, that Φ is simple and that the intensity λ is finite; or, equivalently,

$$P(M^\infty) = 1 \text{ and } \lambda := \mathbf{E}\Phi(0,1] < \infty. \tag{1.1}$$

The atoms of $\varphi \in M^\infty$ are denoted by $X_i(\varphi)$, $i \in \mathbf{Z}$, with the convention that

$$\ldots X_{-1}(\varphi) < X_0(\varphi) \leq 0 < X_1(\varphi) < X_2(\varphi) < \ldots .$$

We interpret $X_i(\varphi)$ as the time of the ith *arrival* (or *point*) and $\alpha_i(\varphi) := X_{i+1}(\varphi) - X_i(\varphi)$ as the ith *interarrival* time (or *interval* length). We have $\Phi(B) := \#\{i \in \mathbf{Z} : X_i \in B\}$ and $[\alpha_i \in B] := [\alpha_i(\varphi) \in B] := \{\varphi \in M^\infty : \alpha_i(\varphi) \in B\}$, $B \in$ Bor $\mathbf{R}$.

For $t \in \mathbf{R}$ the *time shift* $T_t : M \to M$ is defined by $T_t\varphi := \varphi(t + \cdot)$, $\varphi \in M$. By stationarity it is obvious that these mappings are measure preserving under P. The atoms of $T_t\varphi$ are $X_i(\varphi) - t$, $i \in \mathbf{Z}$. For $n \in \mathbf{Z}$ the *point shift* $\theta_n : M^\infty \to M^\infty$ is defined by $\theta_n\varphi := \varphi(X_n(\varphi) + \cdot)$, $\varphi \in M^\infty$. Note that $\theta_n(\theta_1\varphi) = \theta_{n+1}\varphi$.
A random sequence $(\xi_i) := (\xi_i)_{i\in\mathbf{Z}}$ with $\xi_i : M^\infty \to \mathbf{R}$ is *generated by* the point shift θ_1 if $\xi_n(\theta_1\varphi) = \xi_{n+1}\varphi$ for all $\varphi \in M^\infty$ and $n \in \mathbf{Z}$. See also Nieuwenhuis (1989; p. 600). Examples of such sequences are (α_i) and $(1_A \circ \theta_i)$, $A \in \mathcal{M}^\infty$. The general form is $(f \circ \theta_i)$, $f : M^\infty \to M^\infty$ measurable.

The distribution P_n of $\theta_n\Phi$ plays an important role in this paper. It arises from P by shifting the origin to the nth arrival.

$$P_n := P\theta_n^{-1}, \; n \in \mathbf{Z}. \tag{1.2}$$

We now consider the Palm distribution P^0 of Φ. An intuitive definition of P^0 was stated before. The formal definition of the *Palm distribution* P^0 is

$$P^0(A) := \frac{1}{\lambda}\mathbf{E}\left[\sum_{i=1}^{\Phi(0,1]} 1_A(\theta_i\Phi)\right], \; A \in \mathcal{M}^\infty. \tag{1.3}$$

Set $M^0 := \{\varphi \in M^\infty : \varphi\{0\} = 1\}$ and $\mathcal{M}^0 := M^0 \cap \mathcal{M}$. It is obvious that P^0 is a probability measure on $(M^\infty, \mathcal{M}^\infty)$ with $P^0(M^0) = 1$. Note also that $P^0[\alpha_0 = 0] = 0$ by

(1.3), since Φ has no multiple points wp1. According to Franken et al. (1982; Th. 1.2.7) P^0 has the following important property:

$$P^0 = P^0\theta_n^{-1} \quad \text{for all } \; n \in \mathbf{Z}. \tag{1.4}$$

Consequently, any sequence (ξ_i) generated by θ_1 is P^0-*stationary*, i.e., $(\xi_1, \ldots, \xi_n)$ and $(\xi_{k+1}, \ldots, \xi_{k+n})$ have the same distribution under P^0, all $n \in \mathbf{N}$ and $k \in \mathbf{Z}$. Particularly, (α_i) is P^0-stationary.

Definition (1.3) allows us to express P^0 in terms of P. The following inversion formula expresses P in terms of P^0 (cf. Franken et al. (1982; p. 27)).

$$P(A) = \lambda \int_0^\infty P^0[X_1(\varphi) > u; \;\; \varphi(u + \cdot) \in A]du, \; A \in \mathcal{M}. \tag{1.5}$$

Substituting $A = M$ yields

$$E^0\alpha_0 = \frac{1}{\lambda}. \tag{1.6}$$

For $\varphi \in M$ we define

$$N(t, \varphi) := N_t(\varphi) := \begin{cases} \varphi(0, t] & \text{if } \; t \geq 0 \\ -\varphi(t, 0] & \text{if } \; t < 0. \end{cases} \tag{1.7}$$

We will sometimes write $N(t)$ instead of N_t.

The *total variation distance* d between two probability measures Q_1 and Q_2 on a common probability space, both dominated by a σ-finite measure μ and having densities h_1 and h_2 respectively, is defined by

$$d(Q_1, Q_2) := \int |h_1 - h_2| d\mu. \tag{1.8}$$

It is well-known that

$$d(Q_1, Q_2) = 2 \sup_A |Q_1(A) - Q_2(A)| = 2(Q_1[h_1 \geq h_2] - Q_2[h_1 \geq h_2]). \tag{1.9}$$

Expectations with respect to the probability measures P, P_n and P^0, all considered on $(M^\infty, \mathcal{M}^\infty)$, are denoted by E, E_n and E^0, respectively. In particular the distinction between P_0 and P^0 and between E_0 and E^0 should be noted. Expectation with respect to an universal probability space $(\Omega, \mathcal{F}, \mathbf{P})$ is denoted by $\mathbf{E}$.

Let Q_1 and Q_2 be probability measures on a common probability space. We say that Q_1 is *dominated by* Q_2 (notation $Q_1 \ll Q_2$) if the Q_2-null-sets are also Q_1-null-sets. A Radon-Nikodym derivative of Q_1 with respect to Q_2 will be denoted by $\frac{dQ_1}{dQ_2}$. The supplement Q_2-almost surely will usually be suppressed. Q_1 and Q_2 are *equivalent* (notation $Q_1 \sim Q_2$) if they dominate each other; they are *singular* (notation $Q_1 \perp Q_2$) if an event A exists such that $Q_1(A) = 0$ and $Q_2(A) = 1$.

Independence is denoted by $\amalg$ and Lebesgue measure on $\mathbf{R}$ by Leb. Random variable is abbreviated to rv and almost surely to as.

2 Intermediate probability measures

Although P and P^0 are mutually singular, shifts of P are equivalent to P^0 and have simple Radon-Nikodym derivatives. We collect formulas and conclusions that follow from this observation.

In (1.2) the probability measures P_n, $n \in \mathbf{Z}$, were introduced. They can be considered as intermediate between P and P^0 since they have simple relations with both. P_n is related to P in a simple way because of its definition. The relationship to P^0 follows from the following theorem (see Nieuwenhuis (1989; Th. 2.1)).

Theorem 2.1. *Let* $n \in \mathbf{Z}$. *Then*

(i) $P_n \sim P^0$,

(ii) $\rho_n(\varphi) := \lambda \alpha_{-n}(\varphi)$, $\varphi \in M^0$, *defines a Radon-Nikodym derivative of* P_n *with respect to* P^0.

Since P_n and P^0 apparently have the same null-sets, it is clear that convergence wp1 (just as convergence in probability) holds equivalently under both probability measures. This observation leads immediately to some cross ergodic theorems. See Section 3. In Nieuwenhuis (1989) Theorem 2.1 was applied to prove (under some mixing condition) the equivalence of a special type of functional central limit theorems under P and P^0. The relation in (ii) can serve as a tool for transforming formulas involving P into formulas involving P^0 and vice versa. We will give here some examples.

Suppose that $f : M^0 \to \mathbf{R}$ is P^0-integrable. Since

$$E^0 f = \frac{1}{\lambda} E_n \left(\frac{1}{\alpha_{-n}} f \right) \quad \text{and} \quad \alpha_{-n} \circ \theta_n = \alpha_0,$$

we have

$$E^0 f = \frac{1}{\lambda} E \left(\frac{1}{\alpha_0} f \circ \theta_n \right), \quad n \in \mathbf{Z}. \tag{2.1}$$

This relation expresses P^0-expectations in terms of P-expectations and may be an alternative to (1.3). For a P-integrable function $g : M^\infty \to \mathbf{R}$ with $Eg = Eg \circ \theta_0$ it follows immediately from Theorem 2.1(ii) that

$$Eg = \lambda E^0(\alpha_0 g). \tag{2.2}$$

This relation is the counterpart of (2.1). If $Eg = Eg \circ \theta_0$ it is just a reformulation of (1.5), since by (1.5) and Fubini's theorem

$$\begin{aligned} Eg \circ \theta_0 &= \lambda \int_0^\infty E^0 \left(1_{[\alpha_0 > u]} g \circ \theta_0 \circ T_u \right) du \\ &= \lambda E^0 \left(\int_0^{\alpha_0} g \circ \theta_0 \circ T_u du \right) = \lambda E^0(\alpha_0 g). \end{aligned} \tag{2.3}$$

(In the last equality it was used that $\theta_0(T_u \varphi) = \varphi$ for all $\varphi \in M^0$ and $u \in (0, \alpha_0(\varphi))$.)

The formulation in (2.2) is of special interest when g is a function of some sequence generated by θ_1.

To illustrate the simplicity of this Radon-Nikodym approach we will derive some short results here. The well-known relation (1.6) can be obtained by (2.1) by choosing $n = 0$ and $f = \alpha_0$. Other formulas on (α_i) can be obtained by making simple choices for f, g and n, or (probably even faster) by applying Theorem 2.1 directly.

$$\begin{aligned} E\frac{1}{\alpha_0} &= \lambda, \\ E\alpha_k &= \lambda E^0(\alpha_0\alpha_k) = E^0\alpha_0 + \operatorname{cov}_{P^0}(\alpha_0, \alpha_k)/E^0\alpha_0, \quad k \in \mathbf{Z}, \end{aligned} \tag{2.4}$$

(cf. Cox & Lewis (1966; (4.28)) and McFadden (1962; (3.12)),

$$E\frac{\alpha_k}{\alpha_0} = 1, \quad E\alpha_k = E\alpha_{-k}, \quad E\alpha_k\alpha_n = E\alpha_{-k}\alpha_{n-k}, \quad k, n \in \mathbf{Z}.$$

Let $n \in \mathbf{N}_0$. If the P^0-distribution of $(\alpha_0, \ldots, \alpha_n)$ is dominated by Lebesgue measure with density f_n, then the P-distribution of $(\alpha_0, \ldots, \alpha_n)$ is also dominated by Lebesgue measure, with density g_n defined by

$$g_n(x_0, \ldots, x_n) = \lambda x_0 f_n(x_0, \ldots, x_n), \quad x_0, \ldots, x_n \in (0, \infty). \tag{2.5}$$

This relation holds since for $y_0, \ldots, y_n \in (0, \infty)$, $A := [\alpha_0 \leq y_0, \ldots, \alpha_n \leq y_n]$, and $B := X_{i=0}^{n}(0, y_i]$ we have

$$P(A) = P_0(A) = \lambda E^0(\alpha_0 1_A) = \lambda \int_B x_0 f_n(x_0, \ldots, x_n) dx_n \ldots dx_0.$$

In Cox & Lewis (1966; p. 61) Relation (2.5) is proved by heuristic arguments.
Since $X_1(T_u\varphi) = \alpha_0(\varphi) - u$ for all $\varphi \in M^0$ and $u \in (0, \alpha_0(\varphi))$, we have by (1.5) and (2.1) that

$$E\left[1_{[\alpha_0 \in B]} E(g(X_1)|\alpha_0)\right] = E\left(1_{[\alpha_0 \in B]}(g(X_1)\right)$$

$$\begin{aligned}
&= \lambda E^0 \left[\int_0^{\alpha_0} g(X_1 \circ T_u) 1_{[\alpha_0 \in B]} \circ T_u du\right] \\
&= \lambda E^0 \left[\int_0^{\alpha_0} g(s) ds 1_{[\alpha_0 \in B]}\right] \\
&= E\left[1_{[\alpha_0 \in B]} \frac{1}{\alpha_0} \int_0^{\alpha_0} g(s) ds\right]
\end{aligned}$$

for all $B \in$ Bor $\mathbf{R}^+$ and $g : \mathbf{R}^+ \to \mathbf{R}$ such that $E|g(X_1)| < \infty$. Consequently,

$$\text{the conditional } P\text{-distribution of } X_1 \text{ given } \alpha_0 \text{ is } U(0, \alpha_0). \tag{2.6}$$

This well-know result will be applied next. By (2.6), Fubini's theorem and Theorem 2.1 we obtain

$$\begin{aligned}
P[X_1 \le x] &= EP[X_1 \le x|\alpha_0] = E\left[\frac{1}{\alpha_0}\int_0^x 1_{(0,\alpha_0)}(s) ds\right] \\
&= \int_0^x E\left(\frac{1}{\alpha_0} 1_{[\alpha_0 > s]}\right) ds = \lambda \int_0^x P^0[\alpha_0 > s] ds.
\end{aligned}$$

It follows immediately that

$$P_{X_1} \ll \text{Leb and } \frac{dP_{X_1}}{d \text{ Leb}}(s) = \lambda P^0[\alpha_0 > s] \quad \text{Leb ae.} \tag{2.7}$$

Relation (2.7) can also be derived from (1.2.21) in Franken et al. (1982).

The following result will be applied in Section 4. By (2.7) and Fubini's theorem we have

$$P[X_1 > t] = \lambda \int_t^\infty P^0[\alpha_0 > s] ds = \lambda E^0\left[(\alpha_0 - t) 1_{[\alpha_0 > t]}\right], \quad t \in [0, \infty).$$

By Theorem 2.1 we obtain for $t \in [0, \infty)$:

$$P[X_1 > t] = P[\alpha_0 > t] - \lambda t P^0[\alpha_0 > t], \tag{2.8}$$

$$P[X_1 \le t] = P[\alpha_0 \le t] + \lambda t P^0[\alpha_0 > t]. \tag{2.9}$$

We will prove another corollary of Theorem 2.1 which will be useful in Section 3. Let $\mathcal{I}$ be the invariant σ-field under the point shift θ_1, i.e.,

$$\mathcal{I} := \{A \in \mathcal{M}^\infty : \theta_1^{-1} A = A\}. \tag{2.10}$$

Note that $P(A) = P_1(A)$ for all $A \in \mathcal{I}$. Hence,

$$P\big|_{\mathcal{I}} \sim P^0\big|_{\mathcal{I}}. \tag{2.11}$$

In Baccelli & Brémaud (1987; p. 28) Relation (2.11) is proved directly from the definition of P^0. We need, however, expressions for the Radon-Nikodym derivatives. For $A \in \mathcal{I}$ we have

$$\begin{aligned} P^0(A) &= \frac{1}{\lambda} E_1\left(\frac{1}{\alpha_{-1}} 1_A\right) = \frac{1}{\lambda} E\left(\frac{1}{\alpha_0} 1_{\theta_1^{-1} A}\right) = \frac{1}{\lambda} E\left[1_A E\left(\frac{1}{\alpha_0} |\mathcal{I}\right)\right], \\ P(A) &= P_1(A) = \lambda E^0(\alpha_{-1} 1_A) = \lambda E^0(\alpha_0 1_A) = \lambda E^0[1_A E^0(\alpha_0 | \mathcal{I})]. \end{aligned}$$

Hence,

$$\frac{dP\,|_{\mathcal{I}}}{dP^0\,|_{\mathcal{I}}} = \lambda E^0(\alpha_0\,|\mathcal{I}) \qquad \text{and} \qquad \frac{dP^0\,|_{\mathcal{I}}}{dP\,|_{\mathcal{I}}} = \frac{1}{\lambda} E(\frac{1}{\alpha_0}\,|\mathcal{I})\,. \tag{2.12}$$

Another probability measure on $(M^\infty, \mathcal{M}^\infty)$ which is in some sense intermediate between P and P^0 is the measure P' defined by

$$P'(A) := \frac{1}{\lambda} E\left(\frac{1}{\alpha_0} 1_A\right), \quad A \in \mathcal{M}^\infty. \tag{2.13}$$

Note that P' is indeed a probability measure (see (2.4)), that $P' \perp P^0$, and that

$$P' \sim P \quad \text{with} \quad \frac{dP}{dP'} = \lambda \alpha_0. \tag{2.14}$$

By (2.13) and Theorem 2.1 we obtain for $A \in \mathcal{M}^\infty$ and $n \in \mathbf{Z}$ that

$$\begin{aligned} P'(\theta_n^{-1}A) &= \frac{1}{\lambda} E\left(\frac{1}{\alpha_0} 1_A(\theta_n \cdot)\right) = \frac{1}{\lambda} E_0\left(\frac{1}{\alpha_0} 1_A(\theta_n \cdot)\right) \\ &= E^0(1_A(\theta_n \cdot)) = P^0(A). \end{aligned}$$

Consequently,

$$P'\theta_n^{-1} = P^0, \quad n \in \mathbf{Z}. \tag{2.15}$$

This relation implies that random sequences on M^∞ generated by θ_1 are not only P^0-stationary but also P'-stationary. If Φ is a renewal process, then the sequence (α_i) is both iid under P^0 and under P' (note that the P'- and the $P'\theta_0^{-1}$-distribution of $(\alpha_1, \ldots, \alpha_n)$ are the same).

The following diagram comprises some of the above results.

$$\begin{array}{ccccc} & P & \sim & P' & \\ \theta_0 & \downarrow & & \downarrow & \theta_0 \\ & P_0 & \sim & P^0 & \end{array}$$

The Radon-Nikodym derivative of P with respect to P' is not affected by applying θ_0 (cf. Theorem 2.1 and (2.14)). By (2.15) and the above diagram it is obvious that the position of P' as intermediate probability measure between P and P^0 is similar to the position of P_0. Relations (2.1) and (2.2) can also be derived with P'. In Nieuwenhuis (1989; Th. 7.4) the measure P' has been used to prove that a functional central limit theorem holds equivalently under P and P^0.

In Section 4 some other intermediate probability measures will be considered.

3 Cross ergodic theorems

Birkhoff's ergodic theorem holds for stationary sequences. Although sequences (ξ_i) (generated by θ_1) and $(\Phi(i-1,i])$ are usually not stationary under P and P^0 respectively, we can derive strong laws also under these probability measures. In literature these so-called cross ergodic theorems are usually formulated under ergodicity conditions, see Franken et al. (1982; Th. 1.3.12), Baccelli & Brémaud (1987; p. 29/30), Rolski (1981; § 3.3). By applying Theorem 2.1 we can give simple proofs for more general results without assuming ergodicity.

We need some preliminaries first. Set

$$\mathcal{I}' := \{A \in \mathcal{M}^\infty : T_t^{-1}A = A \quad \text{for all} \quad t \in \mathbf{R}\} \tag{3.1}$$

and recall the definition of $\mathcal{I}$ in (2.10). For $A \in \mathcal{I}'$ we have $\varphi \in A$ iff $T_t\varphi \in A$ for all $t \in \mathbf{R}$. Consequently, $\varphi \in A$ iff $\theta_1\varphi \in A$. So, $A = \theta_1^{-1}A$ and $\mathcal{I}' \subset \mathcal{I}$.
A stationary point process Φ (or its distribution P) with $P(M^\infty) = 1$ is called *ergodic* if $P(A) \in \{0, 1\}$ for all $A \in \mathcal{I}'$ or, equivalently, if $E(f|\mathcal{I}') = Ef$ P-as for all $f : M^\infty \to \mathbf{R}$ with $E|f| < \infty$. P^0 is called *ergodic* if $P^0(A) \in \{0, 1\}$ for all $A \in \mathcal{I}$ or, equivalently, if $E^0(g|\mathcal{I}) = E^0 g$ P^0-as for all $g : M^0 \to \mathbf{R}$ with $E^0|g| < \infty$.

Recall the definition of N_t in (1.7). Let $g : M^\infty \to \mathbf{R}$ be P-integrable. By ergodic type theorems we have

$$P\left[\frac{1}{t}N_t \to E(N_1|\mathcal{I}')\right] = 1, \tag{3.2}$$

$$P\left[\frac{1}{t}\int_0^t g \circ T_s ds \to E(g|\mathcal{I}')\right] = 1, \tag{3.3}$$

and, if (ξ_i) is P^0-stationary (in particular if (ξ_i) is generated by θ_1) and $E^0|\xi_0| < \infty$,

$$P^0\left[\frac{1}{n}\sum_{i=1}^{n}\xi_i \to E^0(\xi_0|\mathcal{I})\right] = 1. \tag{3.4}$$

Set $U := E^0(\alpha_0|\mathcal{I})$ and $V' := E(N_1|\mathcal{I}')$. In the proof of the next theorem it will be used repeatedly that any $\mathcal{I}$- (or $\mathcal{I}'$-) measurable function $h : M^\infty \to \mathbf{R}$ satisfies $h \circ \theta_i = h$ for all $i \in \mathbf{Z}$.

Theorem 3.1.

(a) *If* (ξ_i) *is generated by* θ_1 *and* $E^0|\xi_0| < \infty$, *then* (3.4) *holds as well with* P *instead of* P^0.

(b) *Relations* (3.2) *and* (3.3) *hold as well with* P^0 *instead of* P.

Proof. Since $P_0 \sim P^0$, Relation (3.4) holds with P_0 as well. Part (a) follows immediately. For (b), consider

$$\begin{aligned} P_1\left[\frac{1}{t}N_t \to V'\right] &= P\left[\frac{1}{t}\varphi(X_1(\varphi), X_1(\varphi)+t] \to V'(\varphi)\right] \\ &= P\left[\frac{1}{t}\varphi(0, X_1(\varphi)+t] \to V'(\varphi)\right] \\ &= P\left[\frac{\varphi(0, X_1(\varphi)+t]}{X_1(\varphi)+t} \cdot \frac{X_1(\varphi)+t}{t} \to V'(\varphi)\right] = 1. \end{aligned}$$

Since $P_1 \sim P^0$, the first part of (b) follows. For $\varphi \in M^\infty$ we have

$$\frac{1}{t}\int_0^t g(T_s(\theta_0\varphi))ds = \frac{1}{t}\int_{X_0(\varphi)}^{t+X_0(\varphi)} g(T_s\varphi)ds.$$

By this observation it is obvious that (3.3) is also valid with P_0 and thus with P^0. □

Remarks. It is easy to prove that the events in (3.2)-(3.4) are elements of $\mathcal{I}$. This observation, combined with (2.11), leads to another proof of Theorem 3.1 (see Baccelli & Brémaud (1987; p. 29/30) for the ergodic case).
Application of Theorem 3.1(a) with $\xi_i = g \circ \theta_i$ for P^0- integrable functions $g : M^0 \to \mathbf{R}$ yields:

$$\frac{1}{n}\sum_{i=1}^n g \circ \theta_i \to E^0(g|\mathcal{I}) \quad P^0\text{- and } P\text{-as.}$$

See also Franken et al. (1982; (1.3.18)) for the ergodic case.

By conditioning on $\mathcal{I}$ we obtain

$$E^0\left[\alpha_0 1_{[U=0]}\right] = E^0\left[U 1_{[U=0]}\right] = 0.$$

Since $P^0[\alpha_0 = 0] = 0$, we have (apply Theorem 2.1)

$$U > 0 \quad P^0\text{- and } P\text{-as.} \tag{3.5}$$

Application of Theorem 3.1(a) with $\xi_i = \alpha_i$ yields

$$1 = P\left[\frac{1}{n}X_n \to U\right] = P\left[\frac{1}{N_t}X_{N_t} \to U\right] = P\left[\frac{1}{t}N_t \to \frac{1}{U}\right]. \tag{3.6}$$

(The last equality holds since

$$\frac{1}{N_t(\varphi)}X_{N_t(\varphi)}(\varphi) \le \frac{t}{N_t(\varphi)} \le \frac{1}{N_t(\varphi)}X_{N_t(\varphi)+1}(\varphi)$$

for all $\varphi \in M^\infty$ with $N_t(\varphi) > 0$. Use (3.5).). By (3.5), (2.12), (3.2) and Theorem 2.1(i) we have

$$E\left(\frac{1}{\alpha_0}|\mathcal{I}\right) = \frac{1}{E^0(\alpha_0|\mathcal{I})} = E(N_1|\mathcal{I}') \quad P^0\text{- and } P\text{-as.} \tag{3.7}$$

Note the resemblance between (3.7) and Relations (1.6) and (2.4).

A similar almost sure limit result for

$$I(t) := \frac{1}{t}\int_0^t g \circ T_s ds$$

can be derived directly from (a) and the first part of (b). $I(t)$ can be decomposed as follows:

$$\frac{N(t)}{t}\frac{1}{N(t)}\sum_{i=1}^{N(t)}\int_{X_{i-1}}^{X_i} g \circ T_s ds + \frac{1}{t}\int_{X_{N(t)}}^t g \circ T_s ds - \frac{1}{t}\int_{X_0}^0 g \circ T_s ds. \tag{3.8}$$

Note that the sequences $\left(\int_{X_{i-1}}^{X_i} g \circ T_s ds\right)$ and $\left(\int_{X_{i-1}}^{X_i} |g \circ T_s| ds\right)$ are both generated by θ_1. By Theorem 3.1(a), the first part of (b) and (3.7) we have

$$\left|\frac{1}{t}\int_{X_{N(t)}}^{t} g \circ T_s ds\right| \leq \frac{N(t)+1}{t}\frac{1}{N(t)+1}\int_{X_{N(t)}}^{X_{N(t)+1}} |g \circ T_s| ds \;\to 0 \quad \text{as } t \to \infty,$$

P^0- and P-as, and

$$I(t) \to \frac{1}{E^0(\alpha_0|\mathcal{I})} E^0\left(\int_0^{\alpha_0} g \circ T_s ds|\mathcal{I}\right) \quad P^0\text{- and } P\text{-as.} \tag{3.9}$$

Combining the limit results in (3.9) and the second part of Theorem 3.1(b) yields

$$E(g|\mathcal{I}') = \frac{1}{E^0(\alpha_0|\mathcal{I})} E^0\left(\int_0^{\alpha_0} g \circ T_s ds|\mathcal{I}\right) \quad P^0\text{- and } P\text{-as.} \tag{3.10}$$

This relation is a conditional version of the inversion formula (1.5) (replace 1_A in (1.5) by g and apply Fubini's theorem). Conditional versions of (2.1) and (2.2) can be derived from (3.10). For $f : M^0 \to \mathbf{R}$ with $E^0|f| < \infty$ we have

$$E^0(f|\mathcal{I}) = E^0\left(\frac{1}{\alpha_0}\int_0^{\alpha_0} f \circ \theta_0 \circ T_s ds|\mathcal{I}\right) \quad P^0,\ P\text{-as.}$$

By (3.10) and (3.7) we obtain (take $g = f \circ \theta_0/\alpha_0$)

$$E^0(f|\mathcal{I}) = \frac{1}{E(N_1|\mathcal{I}')} E\left(\frac{1}{\alpha_0} f \circ \theta_0|\mathcal{I}'\right) \quad P^0, P\text{ -as.} \tag{3.11}$$

If $g : M^\infty \to \mathbf{R}$ is such that $E|g| < \infty$ and $E(g|\mathcal{I}') = E(g \circ \theta_0|\mathcal{I}')$ P-as, then (3.10) implies

$$E(g|\mathcal{I}') = \frac{E^0(\alpha_0 g|\mathcal{I})}{E^0(\alpha_0|\mathcal{I})} \quad P^0\text{- and } P\text{-as.} \tag{3.12}$$

By Relation (3.11) it can easily be proved that P-ergodicity implies P^0-ergodicity, since

$$E^0(f|\mathcal{I}) = \frac{1}{\lambda} E\left(\frac{1}{\alpha_0} f \circ \theta_0\right) = E^0 f \quad P^0\text{- and } P\text{-as}$$

for any P^0-integrable $f : M^0 \to \mathbf{R}$, provided that P is ergodic. Since $P\,|_{\mathcal{I}} \sim P^0|_{\mathcal{I}}$ (see (2.11)) and $\mathcal{I}' \subset \mathcal{I}$, this implication may also be reversed.

With this uncommon proof we have established the following well-known result (cf. e.g. Franken et al. (1982; Th. 1.3.9) or Baccelli & Brémaud (1987; p. 28/29)):

$$P \text{ is ergodic} \quad \text{iff} \quad P^0 \text{ is ergodic.} \tag{3.13}$$

The choice $g = 1/\alpha_0$ in (3.9) yields

$$\frac{1}{t}\int_0^t \frac{1}{\alpha_0 \circ T_s} ds \to \lambda \quad P^0\text{- and } P\text{-as,} \tag{3.14}$$

provided that $E^0(\alpha_0|\mathcal{I}) = 1/\lambda$ P^0-as. This condition is weaker than ergodicity of Φ; see also Section 4.

4 Approximations of P^0

In this section we will consider several expressions tending in some sense to P^0 as $n \to \infty$. For this purpose a notion is introduced which is weaker than ergodicity of Φ. Several new intermediate probability measures are defined, all equivalent to P^0. The corresponding Radon-Nikodym derivatives are used to approximate P^0 starting from P.

The following theorem is a generalization of Franken et al. (1982; (1.3.20)). See also Matthes, Kerstan & Mecke (1978; Th. 9.4.5) and Miyazawa (1977; Th. 3.2′).

Theorem 4.1. *The following statements are equivalent:*

(i) $P^0[\alpha_0(\varphi) \in B \text{ and } \theta_n\varphi \in A] \to P^0[\alpha_0 \in B]P^0(A)$ *for all* $B \in Bor\mathbf{R}^+$ *and* $A \in \mathcal{M}^0$,

(ii) $P[\alpha_0(\varphi) \in B \text{ and } \theta_n\varphi \in A] \to P[\alpha_0 \in B]P^0(A)$ *for all* $B \in Bor\mathbf{R}^+$ *and* $A \in \mathcal{M}^0$.

Proof. Assume (i). For all $B \in \mathrm{Bor}\mathbf{R}^+$ and $A \in \mathcal{M}^0$ we have (cf. Theorem 2.1)

$$
\begin{aligned}
&P[\alpha_0 \in B] \cap [\theta_n\varphi \in A] = \lambda E^0 \left[\alpha_0 1_{[\alpha_0 \in B]} 1_{[\theta_n\varphi \in A]}\right] \\
&= \lambda \int_0^\infty P^0[\alpha_0 > x] \cap [\alpha_0 \in B] \cap [\theta_n\varphi \in A] dx \\
&\to \lambda \int_0^\infty P^0[\alpha_0 > x \text{ and } \alpha_0 \in B] dx P^0(A) \quad \text{as } n \to \infty
\end{aligned}
$$

because of (i) and dominated convergence. This limit is equal to

$$
\lambda E^0 \left[\alpha_0 1_{[\alpha_0 \in B]}\right] P^0(A) = P[\alpha_0 \in B] P^0(A),
$$

which proves (ii). The implication (ii) $\Longrightarrow$ (i) can be proved the same way. □

Hypothesis (i) is weaker than the mixing (ergodic-sense) property for P^0 (cf. e.g. Franken et al. (1982; p. 37)); hypothesis (ii) could equivalently be formulated as (cf. Nieuwenhuis (1989; Section 5))

$$
P_n = P[\theta_n\varphi \in \cdot] \to P^0 \text{ pointwise, independently of } \sigma(\alpha_0). \tag{4.1}
$$

Next we consider strong approximation of P^0. For $n \in \mathbf{N}$ the empirical distribution $\hat{P}_n$ is defined by

$$
\hat{P}_n(A, \varphi) := \frac{1}{n} \sum_{i=1}^{n} 1_A(\theta_i\varphi), \quad A \in \mathcal{M}^\infty \text{ and } \varphi \in M^\infty. \tag{4.2}
$$

Since the sequence $(1_A \circ \theta_i)$ is generated by θ_1, we obtain by (3.4) and Theorem 3.1(a) that

$$
\hat{P}_n(A) \to E^0(1_A | \mathcal{I}) \quad P^0\text{- and } P\text{-as.} \tag{4.3}
$$

Note that for each $\varphi \in M^\infty$ $\hat{P}_n(\cdot, \varphi)$ is a probability measure on $(M^\infty, \mathcal{M}^\infty)$ and that $\hat{P}_n(A)$ is a P^0-unbiased estimator of $P^0(A)$. The next statement follows immediately from (3.13) and (4.3). It characterizes strong approximation of P^0 by $\hat{P}_n$ under P.

$$\Phi \text{ is ergodic iff } \hat{P}_n(A) \to P^0(A) \ P\text{-as for all} A \in \mathcal{M}^\infty. \tag{4.4}$$

Starting with (4.3) under P we obtain:

$$E\hat{P}_n(A) = \frac{1}{n}\sum_{i=1}^{n} P_i(A) \to E[E^0(1_A|\mathcal{I})] =: Q^0(A), \quad A \in \mathcal{M}^\infty. \tag{4.5}$$

Q^0 is a probability measure on $(M^\infty, \mathcal{M}^\infty)$ having $Q^0(M^0) = 1$, since $E^0(1_{M^0}|\mathcal{I}) = 1$ P^0- and P-as (cf. Th. 2.1(i)).

Lemma 4.2. *Q^0 and P^0 are equivalent. The Radon-Nikodym derivative of Q^0 with respect to P^0 is:*

$$\frac{dQ^0}{dP^0} = \lambda E^0(\alpha_0|\mathcal{I}). \tag{4.6}$$

Proof. By Theorem 2.1 we have:

$$\begin{aligned} Q^0(A) &= \lambda E^0[\alpha_0 E^0(1_A|\mathcal{I})] = \lambda E^0[E^0(\alpha_0|\mathcal{I})E^0(1_A|\mathcal{I})] \\ &= \lambda E^0[E^0(1_A E^0(\alpha_0|\mathcal{I})|\mathcal{I})] = \lambda E^0[1_A E^0(\alpha_0|\mathcal{I})]. \end{aligned}$$

In the second equality we conditioned on $\mathcal{I}$. Since $P^0[E^0(\alpha_0|\mathcal{I}) = 0] = 0$ by (3.5), the conclusions of the lemma follow immediately. □

By (4.6) we obtain

$$Q^0 = P^0 \quad \text{iff} \quad E^0(\alpha_0|\mathcal{I}) = \frac{1}{\lambda} \ P^0\text{-as}. \tag{4.7}$$

If Φ is ergodic, then $E^0(\alpha_0|\mathcal{I}) = E^0\alpha_0 = \lambda^{-1}$ P^0-as. Relation (4.5) could then be taken as a definition of P^0. If, however, Φ is not ergodic, then it is possible that $Q^0 \neq P^0$.

Example 4.3. Set $\varphi_k := \#(\cdot \cap k\mathbf{Z}),\ \ k \in \{1,2\}$. Let Φ^0 be a random element in M^0 such that $\mathbf{P}[\Phi^0 = \varphi_1] = p$ and $\mathbf{P}[\Phi^0 = \varphi_2] = 1 - p$, $p \in (0,1)$. Then $\mathbf{E}(\alpha_i(\Phi^0)) = 2 - p$ for all $i \in \mathbf{Z}$ and $(\alpha_i(\Phi^0))$ is stationary. According to Franken et al. (1982; Th. 1.3.4) there exists exactly one distribution P of a stationary point process Φ such that its Palm distribution P^0 equals the distribution of Φ^0. For $B_1 := [\alpha_i(\varphi) = 1 \text{ for all } i \in \mathbf{Z}]$ and $B_2 := [\alpha_i(\varphi) = 2 \text{ for all } i \in \mathbf{Z}]$ it can easily be proved that $P^0(B_1) = p, P^0(B_2) = 1 - p$, that $B_1, B_2 \in \mathcal{I}$, and that $E^0(\alpha_0|\mathcal{I}) = 1_{B_1} + 21_{B_2}$ P^0-as. Consequently, Φ is not ergodic and $Q^0 \neq P^0$.

Definition 4.4. *A stationary point process Φ with $\mathbf{P}[\Phi \in M^\infty] = 1$ and $\lambda \in (0,\infty)$ is called* **pseudo-ergodic** *if $E^0(\alpha_0|\mathcal{I}) = \lambda^{-1}$ P^0-as.*

An ergodic point process is pseudo-ergodic. A pseudo-ergodic point process need not be ergodic.

Example 4.5. Let φ_1 be as in Example 4.3, $A_1 := [\alpha_i = 1 \text{ for all } i \in \mathbf{Z}]$, and $A_2 := [\alpha_i \in \{1/2,\ 3/2\} \text{ for all } i \in \mathbf{Z}]$. Consider the following experiment. A fair coin is tossed. If head appears, then φ_1 is taken as outcome of Φ^0. If, however, tail appears, then we let for each $i \in \mathbf{Z}$ the coin decide whether α_i equals $1/2$ or $3/2$, and take the resulting $\varphi \in A_2 \cap M^0$ as outcome for Φ^0. Note that $(\alpha_i(\Phi^0))$ is stationary and that $\mathbf{E}(\alpha_i(\Phi^0)) = 1$. Let Φ (with distribution P) be the stationary point process for which the corresponding P^0 equals the distribution of Φ^0. Then Φ is not ergodic, since $P^0(A_1) = P^0(A_2) = \frac{1}{2}$ and $A_1, A_2 \in \mathcal{I}$. Since $P^0[E^0(\alpha_0|\mathcal{I}) = 1] = 1$, Φ is pseudo-ergodic.

Since $E\hat{P}_n \ll P^0$ with Radon-Nikodym derivative $\lambda n^{-1} \sum_{i=1}^n \alpha_{-i}$ (see Theorem 2.1), we obtain by (4.6) that $d(E\hat{P}_n, Q^0) = \lambda E^0|n^{-1}\sum_{i=1}^n \alpha_{-i} - E^0(\alpha_0|\mathcal{I})|$ (recall the definition of d in (1.8)). We want to prove that this last expression tends to 0 as $n \to \infty$.
A sequence $(Y_n)_{n\in\mathbf{N}}$ of integrable rv's is *uniformly integrable* if

$$\lim_{a\to\infty} \sup_{n\in\mathbf{N}} \mathbf{E}|Y_n| 1_{[|Y_n|>a]} = 0, \tag{4.8}$$

or, equivalently,

$$\begin{aligned}&\sup\nolimits_{n\in\mathbf{N}} \mathbf{E}|Y_n| = M < \infty \text{ and for every } \varepsilon > 0 \text{ there exists } \delta > 0\\ &\text{such that for all events } A \text{ with } \mathbf{P}(A) < \delta \text{ we have:}\\ &\sup\nolimits_{n\in\mathbf{N}} \mathbf{E}|Y_n| 1_A < \varepsilon.\end{aligned} \tag{4.9}$$

If $(Y_n)_{n\in\mathbf{N}}$ is uniformly integrable, then so is $(n^{-1}\sum_{i=1}^n Y_i)_{n\in\mathbf{N}}$ as is obvious by (4.9). A random sequence with identically distributed elements is uniformly integrable. Consequently, $(n^{-1}\sum_{i=1}^n \alpha_{-i})_{n\in\mathbf{N}}$ is uniformly P^0-integrable. Since $n^{-1}\sum_{i=1}^n \alpha_{-i} \to E^0(\alpha_0|\mathcal{I})$ P^0-as, we obtain that $d(E\hat{P}_n, Q^0) \to 0$ as $n \to \infty$ (cf. e.g. Th. T26 in Brémaud (1981)). We conclude that the convergence in (4.5) is uniform in A:

$$\sup_{A\in\mathcal{M}^\infty} |\frac{1}{n}\sum_{i=1}^n P_i(A) - Q^0(A)| \to 0. \tag{4.10}$$

The consequences of this observation for the Palm distribution are explained in the next theorem.

Theorem 4.6. *For stationary point processes with $P(M^\infty) = 1$ and $\lambda \in (0,\infty)$ the following statements are equivalent:*

(i) $\frac{1}{n}\sum_{i=1}^n P_i(A) \to P^0(A)$ *for all* $A \in \mathcal{M}^\infty$,

(ii) $\sup_{A\in\mathcal{M}^\infty} |\frac{1}{n}\sum_{i=1}^n P_i(A) - P^0(A)| \to 0$,

(iii) Φ *is pseudo-ergodic,*

(iv) $P^0[\frac{1}{n}\sum_{i=1}^n \alpha_i \to \frac{1}{\lambda}] = 1$,

(v) $P[\frac{1}{t}N_t \to \lambda] = 1$,

(vi) $P^0 = P$ *on* $\mathcal{I}$.

Proof. Relations (4.5), (4.10), (4.7), and (2.12) imply (i) $\iff$ (ii), (i) $\iff$ (iii), and (iii) $\iff$ (vi). The equivalence of (iii) and (iv) is an immediate consequence of Birkhoff's ergodic theorem. The implication (iv) $\Longrightarrow$ (v) is a corollary of Theorem 3.1(a) and observations as in (3.6), with U replaced by λ^{-1}. Theorem 3.1(b) and

$$P^0\left[\frac{1}{t}N_t \to \lambda\right] \le P^0\left[\frac{1}{X_n}N_{X_n} \to \lambda\right] = P^0\left[\frac{1}{n}X_n \to \frac{1}{\lambda}\right]$$

yield the implication (v) $\Longrightarrow$ (iv). □

The main conclusion of Theorem 4.6 is that it is not always correct to define P^0 as the limit of $n^{-1}\sum_{i=1}^n P_i$, attractive as it may be. It is, however, possible to obtain $P^0(A)$ as another limit without any restraint, uniformly in $A \in \mathcal{M}^\infty$. Note that

$$\begin{aligned} P^0(A) &= E^0[E^0(1_A|\mathcal{I})] = \lambda^{-1}E[\alpha_0^{-1}E^0(1_A|\mathcal{I})] \\ &= \lambda^{-1}E[E(\alpha_0^{-1}|\mathcal{I})E^0(1_A|\mathcal{I})] = \lambda^{-1}E[E^0(1_A E(\alpha_0^{-1}|\mathcal{I})|\mathcal{I})]. \end{aligned} \tag{4.11}$$

Since the sequence $(\lambda^{-1}1_A(\theta_i\cdot)E(\alpha_0^{-1}|\mathcal{I}))_{i\in\mathbb{Z}}$ is generated by θ_1, we obtain by Theorem 3.1(a) that

$$\frac{1}{\lambda}E(\frac{1}{\alpha_0}|\mathcal{I})\hat{P}_n(A) \to \frac{1}{\lambda}E^0(1_A E(\frac{1}{\alpha_0}|\mathcal{I})|\mathcal{I}) \ \ P\text{-as.} \tag{4.12}$$

So (cf. (4.11)),

$$Q_n(A) := \frac{1}{\lambda}E[E(\frac{1}{\alpha_0}|\mathcal{I})\hat{P}_n(A)] \to P^0(A), \ \ A \in \mathcal{M}^\infty. \tag{4.13}$$

By Relation (2.4) Q_n is a probability measure. By Theorem 2.1, (3.7), (1.4) and the observation preceeding Theorem 3.1 we have

$$Q_n(A) = E^0\left[\frac{\alpha_0}{E^0(\alpha_0|\mathcal{I})}\hat{P}_n(A)\right] = \frac{1}{n}\sum_{i=1}^n E^0\left[\frac{\alpha_{-i}}{E^0(\alpha_0|\mathcal{I})}1_A\right].$$

Hence, $Q_n \sim P^0$ and

$$\frac{dQ_n}{dP^0} = \frac{1}{n}\sum_{i=1}^{n} \frac{\alpha_{-i}}{E^0(\alpha_0|\mathcal{I})} \to 1 \;\; P^0\text{-as.} \tag{4.14}$$

For $B \in \mathrm{Bor}\mathbf{R}^+$ we have for $k \in \mathbf{Z}$

$$P^0\left[\frac{\alpha_{-k}}{E^0(\alpha_0|\mathcal{I})} \in B\right] = P^0\left[\frac{\alpha_0}{E^0(\alpha_0|\mathcal{I})} \in B\right].$$

So, the random sequence $(\alpha_{-i}/E^0(\alpha_0|\mathcal{I}))$ is identically P^0-distributed and hence $(n^{-1}\sum_{i=1}^n \alpha_{-i}/E^0(\alpha_0|\mathcal{I}))_{n\in\mathbf{N}}$ is uniformly P^0-integrable (cf. the arguments preceeding (4.10)). By (4.14) it is obvious that the convergence in (4.13) is uniform in $A \in \mathcal{M}^\infty$. Note that $Q_n = n^{-1}\sum_{i=1}^n P_i = E\hat{P}_n$ iff Φ is pseudo- ergodic.

According to (4.4) the sequence $(\hat{P}_n)$, considered as a sequence of estimators of P^0, is strongly P-consistent iff Φ is ergodic. By Theorem 4.6 it is asymptotically P-unbiased iff Φ is pseudo-ergodic. It is an easy exercise to prove that $E(\hat{P}_n(A) - P^0(A))^2$, the mean squared error under P, tends to 0 iff Φ is ergodic.

In the next theorem we examine for sequences (ξ_i) generated by θ_1 the asymptotic P-unbiasedness of the estimator $n^{-1}\sum_{i=1}^n \xi_i$ of $E^0\xi_0$.

Theorem 4.7. *Suppose that (ξ_i) is generated by θ_1 and that $E^0\alpha_0^2 \vee E^0\xi_0^2 < \infty$. Then*

$$\frac{1}{n}\sum_{i=1}^{n} E\xi_i \to \lambda E^0[\alpha_0 E^0(\xi_0|\mathcal{I})] \quad \text{as} \quad n \to \infty. \tag{4.15}$$

If Φ is pseudo-ergodic, then $n^{-1}\sum_{i=1}^n \xi_i$ is asymptotically P-unbiased for $E^0\xi_0$.

Proof. By Theorem 2.1 we have

$$\frac{1}{n}\sum_{i=1}^{n} E\xi_i = \lambda E^0\left[\frac{1}{n}\sum_{i=1}^{n}\alpha_0\xi_i\right]. \tag{4.16}$$

Since

$$\begin{aligned} E^0|\alpha_0\xi_n|1_{[|\alpha_0\xi_n|>a]} &\leq E^0|\alpha_0\xi_n|1_{[\alpha_0^2>a]} + E^0|\alpha_0\xi_n|1_{[\xi_n^2>a]} \\ &\leq \left(E^0\left[\alpha_0^2 1_{[\alpha_0^2>a]}\right]E^0\xi_0^2\right)^{\frac{1}{2}}\left(E^0\left[\xi_0^2 1_{[\xi_0^2>a]}\right]E^0\alpha_0^2\right)^{\frac{1}{2}} \end{aligned}$$

and since this upper bound tends to zero, it is obvious that $(\alpha_0\xi_n)_{n\in\mathbf{N}}$ and $(n^{-1}\sum_{i=1}^n \alpha_0\xi_i)_{n\in\mathbf{N}}$ are uniformly P^0-integrable (see (4.8) and the arguments following (4.9)). Note also that by (3.4)

$$\frac{1}{n}\sum_{i=1}^n \alpha_0\xi_i \to \alpha_0 E^0(\xi_0|\mathcal{I}) \quad P^0\text{-as.}$$

By (4.16) and Brémaud (1981; T26) Relation (4.15) follows immediately. The limit in (4.15) is equal to

$$\lambda E^0[E^0(\alpha_0|\mathcal{I})E^0(\xi_0||\mathcal{I})] = E^0\xi_0,$$

provided that Φ is pseudo-ergodic. □

Corollary 4.8. *Suppose that $E^0\alpha_0^2 < \infty$. The estimator $n^{-1}\sum_{i=1}^n \alpha_i$ of $E^0\alpha_0 = \lambda^{-1}$ is asymptotically P-unbiased iff Φ is pseudo-ergodic.*

Proof. The if-part is a consequence of Theorem 4.7. If $n^{-1}\sum_{i=1}^n \alpha_i$ is asymptotically P-unbiased, then we obtain by (4.15) that $E^0[\alpha_0 E^0(\alpha_0|\mathcal{I})] = (E^0\alpha_0)^2$. Consequently, $\mathrm{Var}_{P^0} E^0(\alpha_0|\mathcal{I}) = 0$ and $E^0(\alpha_0|\mathcal{I}) = 1/\lambda$ P^0-as. □

The point process in Example 4.3 satisfies $\lambda^{-1} = E^0(1_{B_1} + 21_{B_2}) = 2 - p$ and (cf. (4.15))

$$\frac{1}{n}\sum_{i=1}^n E\alpha_i \to \frac{E^0(E^0(\alpha_0|\mathcal{I}))^2}{2-p} = \frac{4-3p}{2-p}.$$

This limit is indeed not equal to $\lambda^{-1} = 2 - p$.

It is well known that P^0 can also be approximated by the probability measures $P_{1,n}$, $n \in \mathbf{N}$, defined by

$$P_{1,n}(A) := P[\theta_1\varphi \in A | X_1(\varphi) \leq \frac{1}{n}], \;\; A \in \mathcal{M}^\infty. \tag{4.17}$$

Franken et al. (1982; Th. 1.3.7) prove that $d(P^0, P_{1,n}) \to 0$ as $n \to \infty$. We will, however, express $d(P^0, P_{1,n})$ in terms of F, the distribution function of X_1 under P.

Theorem 4.9. *Let Φ be a stationary point process with $P(M^\infty) = 1$ and $\lambda \in (0, \infty)$. Then*

(i) $P_{1,n} \sim P^0$ *and* $\frac{dP_{1,n}}{dP^0} = \frac{\lambda}{F(1/n)}(\frac{1}{n} \wedge \alpha_{-1}) =: \sigma_n$,

(ii) $\sup_{A \in \mathcal{M}^\infty} |P[\theta_1\varphi \in A | X_1(\varphi) \leq \frac{1}{n}] - P^0(A)| = \frac{1}{2}E^0|\sigma_n - 1| = 1 - \frac{F(F(\frac{1}{n})/\lambda)}{F(\frac{1}{n})} \to 0.$

Proof. By (1.5) we obtain

$$\begin{aligned} P[\theta_1\varphi \in A; X_1(\varphi) \leq \frac{1}{n}] &= \lambda \int_{M^0} \int_0^{\alpha_0(\varphi)} 1_{[\theta_1\varphi \in A; \alpha_0(\varphi) - u \leq \frac{1}{n}]} du dP^0(\varphi) \\ &= \lambda E^0 \left[1_{[\theta_1\varphi \in A]}(\frac{1}{n} \wedge \alpha_0) \right] = \lambda E^0[1_A(\frac{1}{n} \wedge \alpha_{-1})]. \end{aligned}$$

Hence $P_{1,n}(A) = \lambda E^0[1_A(\frac{1}{n} \wedge \alpha_{-1})]/F(\frac{1}{n})$, which proves (i). By (1.8) it is obvious that $d(P_{1,n}, P^0) = E^0|\sigma_n - 1|$. We will express this P^0-expectation in terms of F.
First we note that (cf. Theorem 2.1)

$$P[\alpha_0 \leq x] = \lambda E^0[\alpha_0 1_{[\alpha_0 \leq x]}] \leq \lambda x P^0[\alpha_0 \leq x], \;\; x \in [0, \infty),$$

and (cf. (2.9))

$$F(\frac{1}{n}) = P[\alpha_0 \leq \frac{1}{n}] + \frac{\lambda}{n} - \frac{\lambda}{n} P^0[\alpha_0 \leq \frac{1}{n}] \;\leq\; \frac{\lambda}{n}. \tag{4.18}$$

Set $h(n) := F(1/n)/\lambda$. By (4.18), Theorem 2.1 and (2.9) we obtain

$$\begin{aligned}
E^0|\sigma_n - 1| &= \frac{1}{h(n)} E^0 |\frac{1}{n} \wedge \alpha_0 - h(n)| \\
&= \Big(E^0(h(n) - \alpha_0) 1_{[\alpha_0 \le h(n)]} + E^0(\alpha_0 - h(n)) 1_{[h(n) < \alpha_0 \le \frac{1}{n}]} \\
&\quad + E^0(\frac{1}{n} - h(n)) 1_{[\alpha_0 > \frac{1}{n}]} \Big) / h(n) \\
&= 2P^0[\alpha_0 \le h(n)] - \frac{1/n}{h(n)} P^0[\alpha_0 \le \frac{1}{n}] + \frac{1/n - h(n)}{h(n)} \\
&\quad - \frac{2}{\lambda h(n)} P[\alpha_0 \le h(n)] + \frac{1}{\lambda h(n)} P[\alpha_0 \le \frac{1}{n}] \\
&= 2 - 2\frac{F(h(n))}{\lambda h(n)} + \frac{\lambda h(n) - \lambda/n}{\lambda h(n)} + \frac{1/n - h(n)}{h(n)} \\
&= 2 - 2\frac{F(h(n))}{\lambda h(n)} = 2 - 2\frac{F(F(1/n)/\lambda)}{F(1/n)}.
\end{aligned}$$

The convergence to 0 follows immediately since $F(x) = \lambda x + o(x)$ as $x \to 0$, cf. e.g. Franken et al. (1982; Th. 1.2.12). □

Because of (ii) it is possible to determine in many situations the rate at which $P_{1,n}$ tends to P^0. If Φ is a Poisson process, then it is an easy exercise to prove that $d(P_{1,n}, P^0) = \frac{1}{2}F(\frac{1}{n}) + o(F(\frac{1}{n})) = \frac{1}{2}\lambda/n + o(\frac{1}{n})$ as $n \to \infty$. This rate $1/n$ is not universal; it turns out that the renewal process with $P^0[\alpha_0 \le x] = x^{1-p}$ for $0 < x < 1$, $p \in (0,1)$, satisfies $d(P_{1,n}, P^0) = cn^{-(1-p)} + o(n^{-(1-p)})$ as $n \to \infty$. (Here $c \in (0,\infty)$ is some constant, not depending on n.) We can, however, give conditions such that the rate $1/n$ is satisfied. Set $G(x) := P^0[\alpha_0 \le x]$, $x \in [0,\infty)$.

Corollary 4.10. *Suppose that G is differentiable on $(0,\varepsilon)$ for some $\varepsilon > 0$ with bounded derivative $g := G'$. Then*

$$\sup_{A \in \mathcal{M}^\infty} |P[\theta_1\varphi \in A | X_1(\varphi) \le \frac{1}{n}] - P^0(A)| = \mathcal{O}(\frac{1}{n}) \quad \text{as } n \to \infty.$$

Proof. Because of the continuity of G it is obvious (see (2.7)) that F is differentiable on

$(0,\varepsilon)$ with $F' = \lambda(1-G)$. By the mean value theorem we have for n sufficiently large:

$$F(F(\frac{1}{n})/\lambda) = F(0) + F(\frac{1}{n})(1 - G(\eta_n))$$

for some $\eta_n \in (0, F(1/n)/\lambda)$. Since $F(0) = 0$ and $F(1/n) \le \lambda/n$, see (4.18), we obtain by Theorem 4.9:

$$E^0|\sigma_n - 1| = 2G(\eta_n) \le 2G(\frac{1}{n}).$$

Another application of the mean value theorem yields for n sufficiently large:

$$E^0|\sigma_n - 1| \le 2G(\delta_n) \le \frac{2c}{n}$$

for some $\delta_n \in (0, \frac{1}{n})$. Here $c := \sup\{g(x) : x \in (0,\varepsilon)\}$, not depending on n. □

Remark. The condition in Corollary 4.10 may equivalently be replaced by

F is twice differentiable on $(0,\varepsilon)$ for some $\varepsilon > 0$ (4.19)
with bounded second derivative F''.

5 Generalization to marked point processes

The results of Sections 1 to 4 can be generalized to marked point processes. We briefly consider this extension.

Let K be a complete and separable metric space. A *marked point process on* $\mathbf{R}$ *with mark space* K is a random element Φ in the class of all integer-valued measures φ on the σ-field Bor $\mathbf{R} \times$ Bor K such that:

$\varphi(A \times K) < \infty$ for all bounded $A \in$ Bor $\mathbf{R}$.

Let M_K be this class and endow it with the σ-field $\mathcal{M}_K$ generated by the sets $[\varphi(A \times L) = k] := \{\varphi \in M_K : \varphi(A \times L) = k\}$, $k \in \mathbf{N}_0$, $L \in$ Bor K and $A \in$ Bor $\mathbf{R}$.

Here are some further notations and definitions. For $\varphi \in M_K$ and $L \in$ Bor K we define $\tilde{\varphi}_L \in M_K$ and $\varphi_L \in M$ by $\tilde{\varphi}_L(B) := \varphi(B \cap (\mathbf{R} \times L))$ and $\varphi_L(A) := \varphi(A \times L)$, $B \in$ Bor $\mathbf{R} \times$ Bor K and $A \in$ Bor $\mathbf{R}$. Note that $\tilde{\varphi}_L(\mathbf{R} \times L^c) = 0$ and $\tilde{\varphi}_K = \varphi$. Furthermore, set

$$\begin{aligned}
&\mathcal{M}_L^\infty := \{\varphi \in M_K : \varphi_L(-\infty, 0) = \varphi_L(0, \infty) = \infty;\ \varphi_K(\{s\}) \le 1 \text{ for all } s \in \mathbf{R}\},\\
&M_L^0 := \{\varphi \in M_L^\infty : \varphi_L(\{0\}) = 1\},\\
&\mathcal{M}_L^\infty := M_L^\infty \cap \mathcal{M}_K \text{ and } \mathcal{M}_L^0 := M_L^0 \cap \mathcal{M}_K,
\end{aligned}$$

$L \in$ Bor K. Let $T_t : M_K \to M_K$, $t \in \mathbf{R}$, be the time shifts determined by $T_t\varphi(A \times L) = \varphi((t + A) \times L)$. We will assume that Φ (or its distribution P) is stationary with respect to these time shifts (cf. Section 1). We also assume that $\lambda := \mathbf{E}\Phi((0,1] \times K) < \infty$, so that $\lambda(L) := \mathbf{E}\Phi((0,1] \times L) < \infty$ for all $L \in$ Bor K. We will confine our attention to L with $P(M_L^\infty) = 1$.

The atoms of $\varphi \in M_K^\infty$ are denoted by $(X_i(\varphi), k_i(\varphi))$, $i \in \mathbf{Z}$, enumerated such that $(X_i(\varphi))_{i \in \mathbf{Z}}$ represents φ_K as indicated in Section 1. For $\varphi \in M_L^\infty$ we write $X_i^L(\varphi) := X_i(\tilde{\varphi}_L)$, the '$i$th L-point of φ', and $k_i^L(\varphi) := k_i(\tilde{\varphi}_L))$, the 'mark of the ith L-point of φ'. Note that $T_t\varphi =: \varphi(t + \cdot)$ can be represented by $\{(X_i(\varphi) - t, k_i(\varphi))\}$. Some other notations:

$$\begin{aligned}
\alpha_i &:= X_{i+1} - X_i,\\
\alpha_i^L &:= X_{i+1}^L - X_i^L,\\
\theta_{n,L} &: M_L^\infty \to M_L^0 \text{ with } \theta_{n,L}\varphi := \varphi(X_n^L(\varphi) + \cdot),\\
P_{n,L} &:= P\theta_{n,L}^{-1},\\
\mathcal{I}' &:= \{A \in \mathcal{M}_K^\infty : T_t^{-1}A = A \text{ for all } t \in \mathbf{R}\},\\
\mathcal{I}'_L &:= \{A \in \mathcal{M}_L^\infty : T_t^{-1}A = A \text{ for all } t \in \mathbf{R}\},\\
\mathcal{I}_L &:= \{A \in \mathcal{M}_L^\infty : \theta_{1,L}^{-1}A = A\},
\end{aligned}$$

where $i, n \in \mathbf{Z}$ and $L \in$ Bor K. $P_{n,L}$ is obtained from P by shifting the origin to the nth L-point. Note that $\mathcal{I}'_L \subset \mathcal{I}'$, $\mathcal{I}'_L \subset \mathcal{I}_L$ and $\mathcal{I}' \cap M_L^\infty = \mathcal{I}'_L$.

The *Palm distribution* P_L^0 *of* P *with respect to* L is defined by:

$$P_L^0(A) := \frac{1}{\lambda(L)} \mathbf{E} \left[\sum_{i=1}^{\Phi((0,1]\times L)} 1_A(\theta_{i,L}\Phi) \right], \quad A \in \mathcal{M}_K,$$

which intuitively arises from P by shifting the origin to an arbitrary L-point. Now P_L^0 is a probability measure on $(M_K, \mathcal{M}_K)$ with $P_L^0(M_L^0) = 1$ and having the following properties (cf. (1.4) and (1.5)):

(i) $P_L^0 \theta_{n,L}^{-1} = P_L^0$ for all $n \in \mathbf{Z}$,

(ii) $P(A) = \lambda(L) \int_0^\infty P_L^0[X_1^L(\varphi) > u;\ \varphi(u + \cdot) \in A]du,\ \ A \in \mathcal{M}_K$,

see e.g. Franken et al. (1982).

We now generalize the results of Sections 1 to 4. Our emphasis is on conditioning on L-points in the origin with $L \in$ Bor K such that $P(M_L^\infty) = 1$. Hence, we must replace $M, \mathcal{M}, M^\infty, \mathcal{M}^\infty, M^0, \mathcal{M}^0, \lambda, \alpha_i, X_i, P^0, P_n, P', \theta_n, \mathcal{I}, \mathcal{I}', U, V', N(t)$ by $M_K, \mathcal{M}_K, M_L^\infty, \mathcal{M}_L^\infty, M_L^0, \mathcal{M}_L^0, \lambda(L), \alpha_i^L, X_i^L, P_L^0, P_{n,L}, P'_L, \theta_{n,L}, \mathcal{I}_L, \mathcal{I}'_L, U_L, V'_L, N_L(t)$ respectively. (The definitions of P'_L, U_L and V'_L are clear by (2.13) and the definitions following (3.4); $N_L(t,\varphi) := \varphi_L(0,t]$ if $t \geq 0$ and $N_L(t,\varphi) := -\varphi_L(t,0]$ if $t < 0$, see (1.7).) We must replace 'Φ pseudo-ergodic' by 'Φ pseudo-L-ergodic'.

With these modifications all results remain true. In fact only some of the proofs need an argument. Since $P(M_L^\infty) = 1$ and $\mathcal{I}' \cap M_L^\infty = \mathcal{I}'_L$, it is obvious that ergodicity of Φ can (indeed) equivalently be defined with $\mathcal{I}'_L$ instead of $\mathcal{I}'$ and that $E(g|\mathcal{I}') = E(g|\mathcal{I}'_L)$ P-as for all P-integrable functions $g : M_L^\infty \to \mathbf{R}$. With this in mind the generalized results of Section 3 follow immediately.

Examples of sequences (ξ_i) generated by $\theta_{1,L}$ (see Section 1) are given by

$\alpha_i^L(\varphi)$,

$\varphi_{L'}(X_i^L(\varphi) + t_1, X_i^L(\varphi) + t_2]$,

$\varphi_{L'}(X_{i-1}^L(\varphi), X_i^L(\varphi)]$,

$k_i^L(\varphi)$.

Here $L, L' \in$ Bor K with $P(M_L^\infty) = P(M_{L'}^\infty) = 1$ and $t_1 < t_2$. The third sequence is interesting. If $N_{L'}(t_1, t_2] := N_{L'}(t_2) - N_{L'}(t_1)$, it can be defined as $(N_{L'}(X_{i-1}^L, X_i^L])$. By (the generalization of) Theorem 3.1 we obtain

$$\frac{1}{n} N_{L'}(0, X_n^L] \to E_L^0(N_{L'}(0, X_1^L] | \mathcal{I}_L) \quad \text{as } n \to \infty \quad P_L^0\text{- and } P\text{-as.} \tag{5.1}$$

Since

$$\frac{N_L(t)}{t} \frac{N_{L'}(0, X_{N_L(t)}]}{N_L(t)} \leq \frac{1}{t} N_{L'}(0, t] \leq \frac{N_{L'}(0, X_{N_L(t)+1}^L]}{N_L(t)+1} \frac{N_L(t)+1}{t},$$

it follows from (5.1) and Theorem 3.1(b) that

$$\frac{1}{t} N_{L'}(t) \to E(N_L(1) | \mathcal{I}_L') E_L^0(N_{L'}(0, X_1^L] | \mathcal{I}_L) \quad \text{as } t \to \infty \quad P_L^0\text{- and } P\text{-as.} \tag{5.2}$$

Set $M_{L,L'}^\infty := M_L^\infty \cap M_{L'}^\infty$, $\mathcal{M}_{L,L'}^\infty := M_{L,L'}^\infty \cap \mathcal{M}_K$ and $\mathcal{I}_{L,L'}' := \{A \in \mathcal{M}_{L,L'}^\infty : T_t^{-1} A = A$ for all $t \in \mathbb{R}\}$. Note that $\mathcal{I}_{L,L'}' \subset \mathcal{I}_L$, $\mathcal{I}_{L,L'}' = \mathcal{I}' \cap M_{L,L'}^\infty$ and $P(M_{L,L'}^\infty) = 1$. By arguments as in the proof of the first part of Theorem 3.1(b) we have

$$\frac{1}{t} N_{L'}(t) \to E(N_{L'}(1) | \mathcal{I}_{L,L'}') \quad P_L^0\text{- and } P\text{-as.} \tag{5.3}$$

Combining (5.2) and (5.3) yields

$$E_L^0(N_{L'}(0, X_1^L] | \mathcal{I}_L) = \frac{E(N_{L'}(1) | \mathcal{I}_{L,L'}')}{E(N_L(1) | \mathcal{I}_L')} \quad P_L^0\text{- and } P\text{-as,} \tag{5.4}$$

which is a generalization of Relation (3.4.2) in Baccelli & Brémaud (1987).

Parts of this paper are also valid for special classes of non-stationary point processes. We are preparing a publication on these matters.

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