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THE POWER-SERIES ALGORITHM APPLIED TO THE SHORTEST-QUEUE MODEL

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Abstract

An iterative numerical technique for the evaluation of queue length distributions is applied to multi-server systems with queues in parallel in which customers join (one of) the shortest queues upon arrival. The technique is based on power-series expansions of the state probabilities as functions of the load of the system. The convergence of the series is accelerated by applying a modified form of the epsilon algorithm. The shortest-queue model lends itself particularly well to a numerical analysis by means of the power-series algorithm due to a specific property of this model. Numerical values for the mean and the standard deviation of the total number of customers and of the waiting times in stationary symmetrical systems have been obtained for practically all values of the load for systems with up to 8 queues and for a load not exceeding 70% for systems with up to 25 queues. Further, data are presented for systems with 4 queues and unequal service rates.

Keywords: traffic intensity, waiting time, epsilon algorithm, memory space.

1. Introduction

The power-series algorithm is a numerical procedure for computing state probabilities and moments of joint queue-length distributions for systems with more than one waiting line, which can be modeled by multi-dimensional (quasi-) birth-and-death processes, cf. [2], [4], [10]. The algorithm is based on power-series expansions of state probabilities and moments as function of a parameter of the system, usually the traffic intensity ρ of the system. With the aid of limiting properties of the state probabilities in light traffic a recursive scheme is obtained for calculating the coefficients of their power-series expansions. It has appeared, however, that the state probabilities of many queueing models possess singularities inside the unit circle in the complex ρ-plane (it is assumed that the traffic intensity is defined in such a way that the system is stable iff $0 < \rho < 1$). In all cases considered a bilinear transformation of the traffic intensity is suitable for obtaining convergence of the power-series expansions of the state probabilities over the whole range of values of the traffic intensity for which the system is stable, i.e. the interval (0,1). In order to accelerate the convergence of the power-series expansions of the state probabilities and the moments of the queue-length distributions for systems in heavy traffic (ρ close to 1) extrapolation methods such as the epsilon algorithm, cf. [4], [5], [13], can be applied to partial sums of the series.

Still, the power-series algorithm is an experimental method. It is in general not possible to give upper bounds for errors. But experience has learned that the algorithm provides more accurate results for moderately sized models than simulation does, in less computing time (once the coefficients of the power-series expansions have been computed, queueing characteristics can be calculated for an arbitrary number of values of the traffic intensity in relatively negligible time).

The power-series algorithm applies in theory to systems with arbitrary numbers of queues, but in practice the size of the models is limited by the amount of memory space which is available in a particular computer for storing the coefficients of the power series. Procedures for economic use of memory space during the execution of the power-series algorithm have been developed in [4]. With these procedures it is possible to obtain numerical data for systems with, typically, up to 4 to 6 queues, depending

on the structure of the model, the load of the system, the desired accuracy, and the available memory space.
The power-series algorithm will be applied in this paper to the well-known 'shortest-queue'-problem. In this model there are several service units in parallel, each with its own queue in front of it, and there is one arrival stream of customers who join one of the shortest queues upon arrival and remain in the queue of their choice until they have been served. This model has been investigated by several authors, by means of analytical as well as numerical methods, but usually only in the case of two service units, cf., e.g., [6], [7], [8], [9], [11], [12]. The shortest-queue model lends itself particularly well to a numerical analysis with the aid of the power-series algorithm, especially when all service units are identical, because much more coefficients of the power-series expansions of the state probabilities vanish in this model than in other models. This is due to the property that a certain queue can only contain n customers if all queues contained at least n-1 customers at a previous instant during the current busy period (n = 2,3,..) We present in this paper numerical values for the mean and the standard deviation of the number of customers in the system and of the waiting times for symmetrical shortest queue models with up to 8 queues (for $\rho \leq 0.8$ up to 18 queues and for $\rho \leq 0.7$ even up to 25 queues). Further, we consider for these models other quantities of interest such as the amount of work in the system, the difference between the longest and the shortest queue, and the number of servers which are idle but could work if they were able to serve customers who are waiting in other queues. Finally, a system with 4 queues and unequal service rates is discussed.

The organisation of this paper is as follows. Section 2 contains definitions for the shortest-queue model, the balance equations for the state probabilities in this model, and some general relations between various queueing characteristics. In Section 3 the power-series algorithm is outlined; further, it is shown that the algorithm requires less computation time and storage capacity for the shortest queue model than it does for other models of comparable complexity. Section 4 contains asymptotic expansions in light traffic for several performance measures. Numerical results are presented and discussed in Section 5. Comparable results for M/M/s systems are summarized for easy reference in an Appendix.

2. General relations for shortest-queue systems

The system consists of s servers in parallel. Customers arrive to the system according to a Poisson process with rate λ. They require service from (any) one of the servers, and they form separate queues in front of each server. It will be assumed that the amount of service which is required by a customer is negative exponentially distributed with mean $1/\mu$. Server j serves the customers who have joined the queue in front of him in order of arrival with rate r_j, $j=1,\dots,s$. Without loss of generality it may be assumed that the total service capacity of the system is 1, i.e.

$$\sum_{j=1}^{s} r_j = 1. \tag{2.1}$$

The foregoing implies that the holding times of customers who join queue j are negative exponentially distributed with mean $1/(r_j\mu)$, $j=1,..s$. Arriving customers only observe the lengths of the s queues and have no knowledge about the service rates of the servers. Therefore, they join one of the shortest queues upon arrival. Due to physical circumstances it is not possible for customers to leave one queue and join another queue. Each queue may contain an unbounded number of jobs. When there is not a unique queue which is shorter than all other queues at the instant of an arrival of a new customer, this arriving customer chooses one of the shortest queues with probabilities which are proportional to weight factors which are attributed to the queues. This will be described more precisely below.

First the condition for ergodicity of the system will be considered. The rate of arrivals to the system is λ, and the maximal departure rate of the total system is μ. Hence, the load or traffic intensity ρ of the system is in a natural way defined by

$$\rho := \lambda/\mu, \tag{2.2}$$

and a necessary and sufficient condition for ergodicity of the system is

$$\rho < 1. \tag{2.3}$$

Throughout this paper it will be assumed that the system is in steady state and hence (2.3) will hold. Let N_j denote the number of customers in queue j (waiting or being served), j=1,..,s. Further, let C be a random variable indicating the number of the queue which an arriving customer joins. Clearly, the variable C is dependent of the vector $\bar{N} := (N_1,\dots,N_s)$. The conditional distribution of C given the vector $\bar{N}$ is by definition:

$$\Pr\{C=j\,|\,\forall i\; N_i \geq N_j\} = \psi_j / [\textstyle\sum_{i=1}^{s} \psi_i \; I\{N_i = N_j\}],$$

$$\Pr\{C=j\,|\,\exists i\; N_i < N_j\} = 0; \tag{2.4}$$

here and below, I{E} stands for the indicator function of the event E, and ψ_j is a fixed weight factor related to queue j, j=1,..,s. Let $\bar{n}=(n_1,\dots,n_s)$ be a vector with non-negative integer entries. The stationary state probabilities are defined as follows: for $0 \leq \rho < 1$,

$$p(\rho;\bar{n}) := \Pr\{\bar{N}=\bar{n};\text{ at load } \rho\}. \tag{2.5}$$

Let $\bar{e}_j$ be the vector with zero entries except an entry of one at the j^{th} position (j=1,..,s). The balance equations for the state probabilities (2.5) are readily verified to be:

$$[\rho + \sum_{j=1}^{s} r_j\; I\{n_j>0\}]\; p(\rho;\bar{n}) = \sum_{j=1}^{s} r_j\; p(\rho;\bar{n}+\bar{e}_j) +$$

$$+ \rho \sum_{j=1}^{s} I\{n_j>0\}\; p(\rho;\bar{n}-\bar{e}_j)\; \Pr\{C=j\,|\,\bar{N}=\bar{n}-\bar{e}_j\}. \tag{2.6}$$

The rest of this section concerns the derivation of general relations between the distributions of L, the total number of customers in the system, W, the waiting time of a customer, H, the holding time of a customer, S, the sojourn time of a customer, Y, the amount of work in the system, and those of C and $\bar{N}$. The following relations, which can be derived from the balance equations (2.6) by summation over all states $\bar{n}$ with $n_j=m$ for some fixed j and m and by means of induction, express the balance of flows between the hyperplanes $n_j=m$ and $n_j=m+1$: for j=1,..,s, m=0,1,..,

$$\rho\; \Pr\{C=j, N_j=m\} = r_j\; \Pr\{N_j=m+1\}. \tag{2.7}$$

Summation of these relations over m, m=0,1,.., leads to balance equations for the flows into and out of queue j:

$$\rho \Pr\{C=j\} = r_j \Pr\{N_j>0\}, \qquad \text{for } j = 1,..,s. \tag{2.8}$$

These relations determine the unconditional distribution of C, i.e. the proportion of the customers who join queue j, j=1,..,s, in terms of the queue length distribution. They lead further, with (2.1), to:

$$\sum_{j=1}^{s} r_j \Pr\{N_j=0\} = 1-\rho. \tag{2.9}$$

This relation is useful for checking the correctness of computations of the state probabilities. Note that (2.9) and (2.8) imply in the symmetrical case in which all servers have the same service rate and all queues have the same weight factor ($r_j = 1/s$, $\psi_j = 1$, j=1,..,s):

$$\Pr\{N_j=0\} = 1-\rho, \qquad \Pr\{C=j\} = \frac{1}{s}, \qquad \text{for } j = 1,..,s. \tag{2.10}$$

The following relations which can also be deduced from the balance equations (2.6) express the balance of flows between the diagonal hyperplanes $n_1+..+n_s=m$ and $n_1+..+n_s=m+1$: for m=0,1,..,

$$\rho \Pr\{L=m\} = \Pr\{L=m+1\} - \sum_{j=1}^{s} r_j \Pr\{L=m+1,N_j=0\}. \tag{2.11}$$

With induction it follows from (2.11) that for m=0,1,..,

$$\Pr\{L=m\} = \sum_{i=0}^{m} \rho^i \sum_{j=1}^{s} r_j \Pr\{L=m-i,N_j=0\}. \tag{2.12}$$

It is readily verified that (2.12), together with (2.9), implies that:

$$E\{L\} = \frac{1}{1-\rho} [\rho + \sum_{j=1}^{s} r_j E\{L(N_j=0)\}]. \tag{2.13}$$

In the case that all service rates are equal and all weight factors are equal (2.13) reduces with (2.9) to

$$E\{L\} = \frac{\rho}{1-\rho} + E\{L|N_j=0\}, \qquad \text{for } j = 1,\dots,s. \tag{2.14}$$

Next consider the waiting time of a customer. When a customer joins queue j and this queue contains m customers at the instant of his arrival, then his waiting time consists of m exponentially distributed phases each with mean length $1/(r_j\mu)$, $j=1,\dots,s$, $m=0,1,\dots$ This observation leads to the following expression for the waiting time distribution (with m* denoting an m-fold convolution):

$$\Pr\{W<t\} = \sum_{j=1}^{s} \sum_{m=0}^{\infty} [1 - e^{-r_j\mu t}]^{m*} \Pr\{C=j,N_j=m\}. \tag{2.15}$$

With the aid of relation (2.7) the distribution of the waiting time can be expressed in terms of the marginal queue-length distributions:

$$\Pr\{W<t\} = \frac{1}{\rho} \sum_{j=1}^{s} \sum_{m=0}^{\infty} r_j [1 - e^{-r_j\mu t}]^{m*} \Pr\{N_j=m+1\}. \tag{2.16}$$

From this relation the probability that a customer does not have to wait and the moments of the waiting time distribution are readily obtained:

$$\Pr\{W=0\} = \Pr\{\exists i \; N_i=0\} = \frac{1}{\rho} \sum_{j=1}^{s} r_j \Pr\{N_j=1\}, \tag{2.17}$$

$$E\{W\} = \frac{1}{\rho\mu} [E\{L\} - \sum_{j=1}^{s} \Pr\{N_j>0\}], \tag{2.18}$$

$$E\{W^2\} = \frac{1}{\rho\mu^2} \sum_{j=1}^{s} \frac{1}{r_j} [E\{N_j^2\} - E\{N_j\}]. \tag{2.19}$$

Similar expressions as for the waiting time distribution, cf. (2.15) and (2.16), can be given for the distributions of the holding time and of the sojourn time. The first two moments of these distribution are:

$$E\{H\} = \frac{1}{\rho\mu} \sum_{j=1}^{s} \Pr\{N_j>0\}, \qquad E\{H^2\} = \frac{2}{\rho\mu^2} \sum_{j=1}^{s} \frac{1}{r_j} \Pr\{N_j>0\}; \tag{2.20}$$

$$E\{S\} = \frac{1}{\rho\mu} E\{L\}, \qquad E\{S^2\} = \frac{1}{\rho\mu^2} \sum_{j=1}^{s} \frac{1}{r_j} [E\{N_j^2\} + E\{N_j\}]. \tag{2.21}$$

Finally, consider the amount of work Y in the system. It is readily verified that the distribution of Y satisfies:

$$\Pr\{Y<t\} = \sum_{n_1=0}^{\infty} .. \sum_{n_s=0}^{\infty} p(\rho;\bar{n})\, [1 - e^{-r_1\mu t}]^{n_1 *} * .. * [1 - e^{-r_s\mu t}]^{n_s *}. \tag{2.22}$$

From this relation the moments of the distribution of Y are found to be:

$$E\{Y\} = \sum_{j=1}^{s} \frac{1}{r_j\mu} E\{N_j\}, \tag{2.23}$$

$$E\{Y^2\} = (\frac{1}{\mu})^2 \left[\sum_{i=1}^{s} \sum_{j=1}^{s} \frac{1}{r_i r_j} E\{N_i N_j\} + \sum_{j=1}^{s} (\frac{1}{r_j})^2 E\{N_j\}\right]. \tag{2.24}$$

When the system is symmetrical, these expressions reduce to

$$E\{Y\} = \frac{s}{\mu} E\{L\}, \qquad E\{Y^2\} = (\frac{s}{\mu})^2 [E\{L^2\} + E\{L\}]. \tag{2.25}$$

3. The power-series algorithm

The power-series algorithm will be briefly discussed in this section. The reader is referred to [2], [4], [10] for more details and a motivation of the method. First, introduce the bilinear mapping of the interval [0,1] onto itself,

$$\rho = \rho(\vartheta) = \frac{\vartheta}{1 + G - G\vartheta} \quad (\vartheta = \frac{(1+G)\rho}{1 + G\rho}), \quad G \geq 0. \tag{3.1}$$

Then, introduce the following power-series expansions,

$$p(\rho(\vartheta);\bar{n}) = \vartheta^{n_1+...+n_s} \sum_{k=0}^{\infty} \vartheta^k\, b(k;\bar{n}). \tag{3.2}$$

Replace ρ by ϑ in the balance equations (2.6) according to (3.1), and substitute the power-series (3.2) into these equations. Equating the coefficients of corresponding powers of ϑ in the resulting equations leads to the following iterative scheme for computing the coefficients of the power-series (3.2): for k=0,1,...,

$$(1+G) \sum_{j=1}^{s} r_j \, I\{n_j>0\} \, b(k;\bar{n}) = [-1 + G \sum_{j=1}^{s} r_j \, I\{n_j>0\}] \, I\{k>0\} \, b(k-1;\bar{n})$$

$$+ (1+G) \sum_{j=1}^{s} r_j \, I\{k>0\} \, b(k-1;\bar{n}+\bar{e}_j) - G \sum_{j=1}^{s} r_j \, I\{k>1\} \, b(k-2;\bar{n}+\bar{e}_j)$$

$$+ \sum_{j=1}^{s} I\{n_j>0\} \, b(k;\bar{n}-\bar{e}_j) \, Pr\{C=j|\bar{N}=\bar{n}-\bar{e}_j\}. \tag{3.3}$$

Note that the lefthand side of equation (3.3) vanishes when $\bar{n}=\bar{0}$. To complete the recursive scheme the law of total probability is used to determine the coefficients of $p(\rho(\vartheta);\bar{0})$. Substituting (3.1) and (3.2) into the law of total probability gives:

$$b(0;\bar{0}) = 1,$$

$$b(k;\bar{0}) = - \sum_{0<n_1+..+n_s\leq k} .. \sum b(k-n_1-..-n_s;\bar{n}), \quad k = 1,2,... \tag{3.4}$$

There are several ways to compute the coefficients $b(k;\bar{n})$ recursively from (3.3) and (3.4), see [1], [4]. A property which can be derived from (3.3), (3.4) and (2.4) by means of induction is the following:

$$b(k;\bar{n})=0, \text{ if } k+n_1+..+n_s < s[\max\{n_1,...,n_s\}-1] + \#\{i;n_i=\max\{n_1,...,n_s\}\}. \tag{3.5}$$

This property can intuitively be explaned by noting that states $\bar{n}$ for which there exist i and j such that $n_i-n_j>1$ can only be reached from the empty state by passing through a state in which all coordinates are at least equal to $\max\{n_1,..,n_s\}-1$, because arriving customers join one of the shortest queues. For example, when s=4 then the state $\bar{n}=(3,3,0,1)$ cannot be reached from the empty state unless there have been at least 2 customers in queue 3 and 4 (otherwise the length of queue 1 and 2 could not have become 3); this implies that $p(\rho;3,3,0,1)=O(\rho^{10})$ as $\rho \downarrow 0$, this order being the minimum number of arrivals which are necessary to reach a state from the empty state; hence, $b(k;3,3,0,1)=0$ for $k<10-7=3$, cf. (3.2).

The amount of computations becomes much smaller for the shortest-queue model when compared with other models (with the same number of queues), cf. [3], [4]. A further reduction of computation time and of the amount of storage capacity which is needed for the coefficients can be realized when the model is symmetrical.

Suppose that performance measures for the shortest-queue model have to be calculated up to the M^{th} power of θ. With the aid of combinatorial arguments it can be deduced how many coefficients $b(k;\bar{n})$ are required for this purpose. When the model is symmetrical then $M+1-j$ coefficients are non-vanishing for the following number of states, cf. (3.5),

$$\binom{a+s-k}{a}, \qquad \text{if } j = as+k \leq M, \quad a=0,1,\dots, \quad k=1,\dots,s. \tag{3.6}$$

This implies that the total number of coefficients that have to be calculated is equal to, for $M=As+c$, $c=1,\dots,s$, $A=0,1,\dots,$

$$M+1+\sum_{a=0}^{A-1}\sum_{k=1}^{s}(c+1-k)\binom{a+s-k}{a}+\sum_{k=1}^{c}(c+1-k)\binom{A+s-k}{A}=$$

$$=1+\frac{A\ s}{A+s+1}\binom{A+s+2}{s+1}+c\binom{A+s}{s}+\sum_{k=1}^{c}(c+1-k)\binom{A+s-k}{A}. \tag{3.7}$$

When the model is not symmetrical then $M+1-j$ coefficients are non-vanishing for the following number of states, cf. (3.5),

$$\binom{s}{k}(a+1)^{s-k}, \qquad \text{if } j = as+k \leq M, \quad a=0,1,\dots, \quad k=1,\dots,s. \tag{3.8}$$

This implies that the total number of coefficients that have to be calculated is equal to, for $M=As+c$, $c=1,\dots,s$, $A=0,1,\dots,$

$$M+1+\sum_{a=0}^{A-1}\sum_{k=1}^{s}(c+1-k)\binom{s}{k}(a+1)^{s-k}+\sum_{k=1}^{c}(c+1-k)\binom{s}{k}(A+1)^{s-k}=$$

$$=(c+1)\ (A+1)^{s}+s\sum_{a=1}^{A}a(a+1)^{s-1}+\sum_{k=1}^{c}(c+1-k)\binom{s}{k}(A+1)^{s-k}. \tag{3.9}$$

Table 1. The number of coefficients $b(k;\bar{n})$ which are required to determine power-series expansions up to a certain power (M) of ϑ.

s	Asymmetrical General M=48	Asymmetrical Shortest Queue M=48	Asymmetrical Shortest Queue M=72	Symmetrical General M=48	Symmetrical Shortest Queue M=48	Symmetrical Shortest Queue M=96
2	20825	11025	35113	10725	5825	41601
3	270725	69785	315925	50625	14365	176121
4	2869685	352521	2236585	157977	24193	471745
5	25827165	1548460	13422565	368149	32671	926234
6	202927725	5676921	68191357	694497	36609	1444609
7	1.42 E+09	19953444	323368848	1121693	38611	1951568
8	9.00 E+09	59421537	1.30 E+09	1614444	36609	2273921
9	5.22 E+10	201270006	4.95 E+09	2131737	35938	2544608
10	2.80 E+11	529806725	1.91 E+10	2637924	32671	2620255
11	1.40 E+12	1.62 E+09	7.08 E+10	3107797	29947	2544606
12	6.57 E+12	2.74 E+09	1.80 E+11	3526974	24193	2273921
13	2.91 E+13	9.70 E+09	7.50 E+11	3889825	24094	2240166
14	1.22 E+14	2.85 E+10	1.78 E+12	4196781	21679	1951568
15	4.89 E+14	5.03 E+10	5.32 E+12	4451881	17539	1829125
16	1.87 E+15	5.63 E+10	1.99 E+13	4660969	14365	1444609

Table 1 gives an overview of these numbers of coefficients, cf. (3.9), (3.7), for some values of M, for the shortest queue model as well as for general multi-dimensional birth-and-death models. It can be seen that the number of coefficients which is required for the shortest queue model is considerably less than for other models due to property (3.5). It should be noted that more terms (M) of the power-series expansions are required with increasing number of queues (s) in order to reach a certain level of accuracy, because the power-series converge less quickly. This is related to the fact that the interaction between the queues increases with increasing number of queues. While for other models storage capacity is the main limiting factor for application of the algorithm, computation time becomes the main factor for the shortest queue model.

Once the coefficients of the power-series expansions of the state probabilities have been determined, those of the moments of the queue length distribution can be obtained as well, cf. [2]. In order to accelerate the convergence of the power series expansions the modified ε-algorithm, cf. [4], can be applied. It means that extrapolating terms which take into account the asymptotic behaviour of moments and probabilities as $\rho \uparrow 1$ are added to the partial sums of the power-series expansions. Then the ε-algorithm, described in [5], [13], is applied. Because the ε-algorithm transforms polynomials into rational functions, it is not necessary to choose the value of G in the transformation (3.1) so large that the power series are convergent for all required values of ϑ: the ε-algorithm transforms divergent series into convergent series as long as singularities are poles. Numerical experiments have shown that values of G for which the series are divergent, but not too strongly, give the best performance of the power-series algorithm together with the ε-algorithm (for G=0, when computations are less than for G>0, cf. (3.3), the series are in most cases so strongly divergent that numerical instabilities occur). We have used G=0.5 (s=2) up to G=1.5 (s=25) for symmetrical models, and G=1.5 up to G=2.5 for asymmetrical models to obtain the data presented in Section 5.

4. Light traffic behaviour

In this section we shall derive some light traffic limits for symmetrical systems, i.e., for systems with $r_j = 1/s$, $\psi_j = 1$, $j=1,\dots,s$, and for G=0, i.e. $\rho=\vartheta$, in (3.1)-(3.3). First, the following expression holds for coefficients $b(k;\bar{n})$ with $k+n_1+\dots+n_s \le s$ and, cf. (3.5), $|n_i-n_j| \le 1$ for all i and j, $i,j=1,\dots s$:

$$b(k;\bar{n}) = (-1)^k \frac{s^k}{k!} \frac{s^m}{m!} \binom{s}{m}^{-1}, \qquad \text{with } m:=n_1+\dots+n_s. \qquad (4.1)$$

These coefficients are the same as those for the corresponding M/M/s queueing system; as long as there are not more than s customers in the system during a busy period the behaviour of the shortest-queue model is the same as that of the M/M/s model. Also the first coefficient of the state probabilities can easily be obtained: for instance, for states $\bar{n}$ with, cf. (3.5), $|n_i-n_j| \le 1$ for all i and j, $i,j=1,\dots s$:

$$b(0;\bar{n}) = \frac{s^s}{s!} \binom{s}{m}^{-1}, \quad \text{with } n_1+\dots+n_s=as+m, \quad a=1,2,\dots, \quad 0 \le m < s. \qquad (4.2)$$

With the above and similar relations the following asymptotical expansions for light traffic can be derived: as $\rho \downarrow 0$,

$$P\{L=0\} = \sum_{k=0}^{s+1} \frac{(-s\rho)^k}{k!} - \frac{s^s \rho^{s+1}}{s!} \sum_{j=2}^{s+1} \frac{1}{j} + O(\rho^{s+2})], \tag{4.3}$$

$$E\{L\} = s\rho + \frac{(s\rho)^{s+1}}{s!} [1 - \frac{s^2-1}{s} \rho + O(\rho^2)], \tag{4.4}$$

$$\sigma^2\{L\} = s\rho + \frac{(s\rho)^{s+1}}{2(s!)} [s+3 - \frac{s^3+4s^2-3s-3}{s} \rho + O(\rho^2)], \tag{4.5}$$

$$P\{W=0\} = 1 - \frac{(s\rho)^s}{s!} [1 - \frac{s^2-1}{s} \rho + O(\rho^2)], \tag{4.6}$$

$$E\{W\} = \frac{s}{\mu} \frac{(s\rho)^s}{s!} [1 - \frac{s^2-1}{s} \rho + O(\rho^2)], \tag{4.7}$$

$$\sigma^2\{W\} = 2 \left(\frac{s}{\mu}\right)^2 \frac{(s\rho)^s}{s!} [1 - \frac{s^2-1}{s} \rho + O(\rho^2)], \tag{4.8}$$

$$R = \frac{1}{s-1} [\frac{\rho}{1-\rho} + \frac{(s\rho)^s}{2(s!)} \{s-3 - \frac{s^3-3s^2-10s+3}{s} \rho + O(\rho^2)\}]; \tag{4.9}$$

here R denotes the coefficient of correlation between the lengths of two arbitrarily chosen queues in symmetrical shortest-queue models; it is readily verified that R satisfies:

$$R = \text{cov}\{N_1, N_2\}/\sigma^2\{N_1\} = \frac{1}{s(s-1)} [\sigma^2\{L\}/\sigma^2\{N_1\}] - \frac{1}{s-1}. \tag{4.10}$$

From (4.7) it follows that the mean waiting time for the shortest-queue model is asymptotically s times as large as that for the corresponding M/M/s model in light traffic. This can be explained by noting that the waiting time of the first customer who has to wait in a busy period is equal to the minimum of the s remaining holding times of the customers in service for the M/M/s model, while it is equal to the remaining holding time of the customer in front of the waiting customer in the shortest-queue model.

5. Numerical results

This section contains numerical data for the shortest-queue model which have been obtained by means of the power-series algorithm together with the modified ε-algorithm. The moments of the waiting time distributions have been computed for $\mu = 1$ in all tables. Tables 2, 3 and 4 show numerical values for the zero-probabilities, averages and standard deviations of the total number L of customers in the system and of the waiting time W and for the correlation coefficient R in symmetrical shortest-queue models ($r_j = 1/s$, $\psi_j = 1$, $j=1,\ldots,s$). It is interesting to compare these data with data for corresponding M/M/s models, cf. the Appendix (see also [1]). While customers are taken into service in order of arrival in M/M/s systems, this is not the case in systems in which customers join one of the shortest queue upon arrival (although customers are taken into service in order of arrival in each individual queue). This feature gives rise to a larger standard deviation of the waiting times in systems with the "join-the-shortest-queue" discipline. Further, it may happen in the latter systems that some servers are idle while there are customers waiting in other queues. This inefficiency causes on the one hand a larger probability that a customer finds a free server upon arrival, but it leads on the other hand to, on the average, a larger number of customers in the system, and as a consequence also to a larger mean waiting time than in M/M/s systems.

Table 5 concerns, for symmetrical systems, the random variables D, the largest difference between the lengths of the various queues, i.e.,

$$D := \max\{N_1,\ldots,N_s\} - \min\{N_1,\ldots,N_s\}, \tag{5.1}$$

and J, the number of servers who are idle but could work if they were able to serve customers who are waiting in other queues; more formally:

$$J := \#\{j;N_j=0\} - [s-L]^+. \tag{5.2}$$

The tables show that in the shortest-queue model both the mean and the standard deviation of the waiting times are decreasing with s for low values of ρ (see table 2). For moderate values of ρ the standard deviation is first increasing with s, but after having reached a maximal value (at s=6 when ρ=0.5, at s=11 when ρ=0.6 and at s=21 when ρ=0.7) it becomes again a decreasing function of s, while the mean waiting time still decreases with s (see table 3). Finally, for high values of ρ both the mean and the standard deviation of the waiting time are increasing functions of s as far as they have been computed (see table 4). The behaviour of the mean waiting time as function of the number of queues is governed by two opposite forces: on the one hand the probability that a customer finds a free server upon arrival increases, but on the other hand the mean total number of customers in the system also increases with increasing number of queues. The mean waiting time stops being a monotonously decreasing function of s between $\rho=0.86$ and $\rho=0.87$. For $\rho=0.87$ it has a local minimum at $s=4$ and a local maximum at $s=9$. For $\rho=0.88$ these extremal points are located at $s=3$ and at $s=12$ respectively. Note that the mean and the standard deviation of the waiting times in the M/M/s model are decreasing functions of s for every value of ρ. Further, the mean values of the random variables D and J increase with increasing s for fixed ρ and small values of s (see table 5). This will be due to the effect that increasing the number of queues leads to more stochastic fluctuations between the lengths of the various queues. When the number of queues increases further, however, these fluctuations will fade, because the average lengths of the individual queues become smaller and smaller. Asymptotically as the number of queues tends to infinity there will be, with probability tending to one, no customers waiting for service (but their mean holding time tends to infinity); this suggests that $E\{D\}$ will tend to one and $E\{J\}$ to zero. We remark that $E\{D\}$ approaches one from below for low values of ρ. The above mentioned features will also contribute to the behaviour of the standard deviation of the waiting times.

Table 6 contains limits as $\rho \uparrow 1$ of the following functions:

$$\frac{P\{L=0\}}{1-\rho}, \qquad E\{L\} - \frac{1}{1-\rho}, \qquad \sigma^2\{L\} - \left(\frac{1}{1-\rho}\right)^2 + \frac{1}{1-\rho},$$

$$\frac{P\{W=0\}}{1-\rho}, \qquad E\{W\} - \frac{1}{1-\rho}, \qquad \sigma^2\{W\} - \left(\frac{1}{1-\rho}\right)^2 - \frac{s-1}{1-\rho},$$

$$[1-R]/(1-\rho)^2, \quad P\{D>1\}, \quad E\{D\}, \quad P\{J>0\}/(1-\rho), \quad E\{J\}/(1-\rho). \tag{5.3}$$

The limits in table 6 have been computed by fitting Laurent series expansions at $\rho=1$, with the aid of more data than is shown in the tables 4, 5. The most remarkable difference with the heavy traffic behaviour of M/M/s systems is the term $(s-1)/(1-\rho)$ in the expansion of $\sigma^2\{W\}$. It should be noted that the form of this term and of the other terms indicated in (5.3) has been deduced from numerical data. The heavy traffic behaviour of E{L} in the case s=2 has been found in [7] by means of an analytical method.

In the tables 7 and 8 values of performance measures are presented for asymmetrical shortest-queue systems with s=4 queues. The service rates at the queues are, cf. (2.1): r_1=0.16, r_2=0.24, r_3=0.24, r_4=0.36. Three sets of weight factors have been considered: $\psi_j:\psi_{j+1}$=100:1 for j=1,2,3 (set Ψ1), $\psi_j:\psi_{j+1}$=1:1 for j=1,2,3 (set Ψ2), and $\psi_j:\psi_{j+1}$=1:100 for j=1,2,3 (set Ψ3). The data in table 7 indicate that systems with unequal service rates perform only better than the corresponding symmetrical system if the fastest servers possess sufficiently high weight factors.
Table 8 also contains data for models with 4 queues and equal service rates (r_j=0.25 for j=1,..,4), with weight factors $\psi_j:\psi_{j+1}$=100:1 for j=1,2,3 (set Ψ1s); the distributions of L and W do not depend on the weight factors for models with equal service rates, so that data concerning these distributions can be found in the tables 2, 3 and 4 for this model.

Table 2. Queueing characteristics for lightly loaded symmetrical systems.

ρ	s	P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}	R
.1	2	.8175	.203543	.4571	.98246906	.03542858	.37675	.0998
.1	3	.7405	.301049	.5506	.99650615	.01049253	.25082	.0553
.1	4	.6703	.400296	.6333	.99925981	.00296105	.15389	.0371
.1	5	.6065	.500081	.7073	.99983770	.00081149	.09008	.0278
.1	6	.5488	.600022	.7747	.99996364	.00021818	.05117	.0222
.1	7	.4966	.700006	.8367	.99999173	.00005788	.02847	.0185
.1	8	.4493	.800002	.8944	.99999810	.00001520	.01560	.0159
.1	9	.4066	.900000	.9487	.99999956	.00000396	.00845	.0139
.1	10	.3679	1.000000	1.0000	.99999990	.00000103	.00453	.0123
.1	11	.3329	1.100000	1.0488	.99999998	.00000026	.00241	.0111
.1	12	.3012	1.200000	1.0954	.99999999	.00000007	.00128	.0101
.1	13	.2725	1.300000	1.1402	1.00000000	.00000002	.00067	.0093
.1	14	.2466	1.400000	1.1832	1.00000000	.00000000	.00035	.0085
.1	15	.2231	1.500000	1.2247	1.00000000	.00000000	.00018	.0079
.1	16	.2019	1.600000	1.2649	1.00000000	.00000000	.00010	.0074
.3	2	.53042	.68642	.914	.869582	.288050	1.0904	.3032
.3	3	.39828	.95787	1.038	.937598	.192908	1.0749	.1929
.3	4	.29747	1.23823	1.153	.968433	.127431	1.0063	.1424
.3	5	.22146	1.52493	1.262	.983427	.083098	.9091	.1090
.3	6	.16456	1.81609	1.366	.991068	.053638	.8008	.0874
.3	7	.12213	2.11031	1.465	.995094	.034354	.6927	.0726
.3	8	.09057	2.40656	1.559	.997266	.021871	.5912	.0620
.3	9	.06714	2.70416	1.650	.998460	.013857	.4992	.0540
.3	10	.04976	3.00262	1.736	.999126	.008745	.4181	.0479
.3	11	.03687	3.30165	1.819	.999500	.005501	.3478	.0430
.3	12	.02732	3.60104	1.899	.999712	.003450	.2877	.0391
.3	13	.02024	3.90065	1.976	.999834	.002159	.2369	.0358
.3	14	.01499	4.20040	2.050	.999904	.001348	.1942	.0330
.3	15	.01111	4.50025	2.122	.999944	.000840	.1597	.0306
.3	16	.00823	4.80016	2.191	.999967	.000522	.1293	.0286
.3	17	.00610	5.10010	2.258	.999981	.000324	.1050	.0268
.3	18	.00452	5.40006	2.324	.999989	.000201	.0851	.0252
.3	19	.00335	5.70004	2.388	.999993	.000125	.0688	.0238
.3	20	.00248	6.00002	2.450	.999996	.000078	.0555	.0226
.3	21	.00184	6.30001	2.510	.999998	.000048	.0447	.0214
.3	22	.00136	6.60001	2.569	.999999	.000029	.0360	.0204
.3	23	.00101	6.90001	2.627	.999999	.000018	.0289	.0195
.3	24	.00075	7.20000	2.683	1.000000	.000011	.0232	.0186
.3	25	.00055	7.50000	2.739	1.000000	.000007	.0186	.0179

Table 3. Queueing characteristics for moderately loaded symmetrical systems.

ρ	s	P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}	R
.5	2	.315967	1.426	1.541	.68403	.85264	1.9653	.5257
.5	3	.197604	1.867	1.665	.78821	.73451	2.1327	.3925
.5	4	.122744	2.316	1.782	.85297	.63260	2.2451	.3060
.5	5	.075867	2.772	1.891	.89532	.54321	2.3112	.2472
.5	6	.046715	3.232	1.995	.92400	.46476	2.3384	.2053
.5	7	.028679	3.698	2.095	.94394	.39626	2.3334	.1744
.5	8	.017565	4.168	2.190	.95810	.33685	2.3028	.1507
.5	9	.010738	4.643	2.283	.96834	.28564	2.2523	.1322
.5	10	.006555	5.121	2.374	.97586	.24171	2.1868	.1173
.5	11	.003996	5.602	2.462	.98145	.20420	2.1104	.1052
.5	12	.002434	6.086	2.549	.98565	.17228	2.0264	.0952
.5	13	.001481	6.573	2.633	.98883	.14518	1.9376	.0869
.5	14	.000901	7.061	2.717	.99127	.12223	1.8460	.0798
.5	15	.000548	7.551	2.799	.99315	.10283	1.7534	.0738
.5	16	.000333	8.043	2.879	.99460	.08644	1.6609	.0686
.5	17	.000202	8.536	2.959	.99573	.07261	1.5696	.0641
.5	18	.000123	9.030	3.036	.99661	.06096	1.4802	.0601
.5	19	.000075	9.526	3.113	.99731	.05116	1.3933	.0566
.5	20	.000045	10.021	3.188	.99785	.04291	1.3093	.0535
.5	21	.000027	10.518	3.262	.99829	.03597	1.2286	.0507
.5	22	.000017	11.015	3.335	.99863	.03014	1.1512	.0482
.5	23	.000010	11.513	3.407	.99890	.02524	1.0773	.0459
.5	24	.000006	12.011	3.477	.99912	.02114	1.0070	.0438
.5	25	.000004	12.509	3.547	.99929	.01769	.9403	.0420
.7	2	.156485	2.951	2.891	.4435	2.2164	3.553	.766
.7	3	.080853	3.589	3.000	.5440	2.1266	3.896	.656
.7	4	.041542	4.232	3.109	.6198	2.0458	4.209	.565
.7	5	.021259	4.878	3.214	.6791	1.9685	4.494	.490
.7	6	.010845	5.525	3.315	.7265	1.8930	4.751	.430
.7	7	.005518	6.173	3.412	.7652	1.8186	4.982	.381
.7	8	.002802	6.822	3.504	.7970	1.7450	5.189	.341
.7	9	.001420	7.471	3.593	.8234	1.6725	5.373	.307
.7	10	.000718	8.121	3.678	.8456	1.6011	5.535	.279
.7	11	.000363	8.772	3.759	.8644	1.5311	5.677	.255
.7	12	.000183	9.424	3.837	.8803	1.4628	5.798	.235
.7	13	.000092	10.077	3.913	.8940	1.3962	5.902	.217
.7	14	.000046	10.732	3.985	.9058	1.3317	5.989	.202
.7	15	.000023	11.389	4.056	.9159	1.2693	6.060	.188
.7	16	.000012	12.046	4.125	.9248	1.2091	6.117	.176
.7	17	.000006	12.706	4.192	.9325	1.1512	6.160	.165
.7	18	.000003	13.367	4.257	.9393	1.0956	6.192	.155
.7	19	.000001	14.030	4.322	.9453	1.0423	6.212	.147
.7	20	.000001	14.694	4.385	.9505	.9912	6.222	.139
.7	21	.000000	15.360	4.447	.9552	.9425	6.223	.132
.7	22	.000000	16.027	4.508	.9593	.8959	6.216	.125
.7	23	.000000	16.696	4.569	.9630	.8515	6.202	.119
.7	24	.000000	17.366	4.629	.9663	.8090	6.180	.114
.7	25	.000000	18.038	4.688	.9693	.7687	6.152	.108

Table 4. Queueing characteristics for heavily loaded symmetrical systems.

ρ	s	P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}	R
.8	2	.094165	4.729	4.552	.3058	3.911	5.331	.878
.8	3	.043921	5.485	4.643	.3863	3.857	5.753	.807
.8	4	.020387	6.250	4.738	.4516	3.813	6.160	.738
.8	5	.009433	7.018	4.833	.5064	3.772	6.552	.675
.8	6	.004355	7.785	4.928	.5534	3.732	6.928	.618
.8	7	.002006	8.553	5.022	.5940	3.691	7.288	.567
.8	8	.000923	9.319	5.113	.6296	3.648	7.633	.522
.8	9	.000424	10.083	5.202	.6609	3.604	7.962	.482
.8	10	.000195	10.847	5.289	.6888	3.558	8.277	.446
.8	11	.000089	11.608	5.373	.7135	3.510	8.577	.415
.8	12	.000041	12.369	5.455	.7357	3.461	8.862	.387
.8	13	.000019	13.128	5.534	.7556	3.410	9.134	.363
.8	14	.000009	13.886	5.610	.7735	3.358	9.393	.341
.8	15	.000004	14.643	5.684	.7897	3.304	9.637	.321
.8	16	.000002	15.400	5.756	.8044	3.250	9.870	.303
.8	17	.000001	16.156	5.826	.8178	3.195	10.091	.287
.8	18	.000000	16.911	5.893	.8299	3.139	10.296	.272
.9	2	.042246	9.855	9.534	.1578	8.950	10.43	.965
.9	3	.017664	10.753	9.593	.2049	8.947	10.91	.940
.9	4	.007352	11.667	9.657	.2457	8.963	11.40	.914
.9	5	.003052	12.589	9.726	.2822	8.988	11.88	.885
.9	6	.001265	13.515	9.797	.3153	9.016	12.36	.856
.9	7	.000524	14.441	9.871	.3456	9.046	12.84	.826
.9	8	.000217	15.368	9.945	.3737	9.076	13.31	.796
.9	9	.000089	16.294	10.021	.3997	9.104	13.78	.767
.9	10	.000037	17.218	10.098	.4241	9.131	14.25	.738
.9	11	.000015	18.141	10.174	.4468	9.156	14.70	.711
.95	2	.019955	19.92	19.52	.08004	18.97	20.47	.9906
.95	3	.007873	20.90	19.55	.10536	19.00	20.97	.9838
.95	4	.003091	21.91	19.59	.12798	19.06	21.47	.9759
.95	5	.001210	22.93	19.63	.14874	19.13	21.98	.9671
.95	6	.000473	23.95	19.68	.16809	19.22	22.50	.9575
.95	7	.000185	24.99	19.73	.18630	19.30	23.01	.9472
.95	8	.000072	26.02	19.78	.20357	19.39	23.53	.9368
.95	9	.000028	27.06	19.83	.22003	19.48	24.05	.9247
.98	2	.007707	49.97	49.51	.03229	48.99	50.49	.9984
.98	3	.002933	51.00	49.52	.04284	49.04	50.99	.9973
.98	4	.001110	52.07	49.54	.05243	49.13	51.50	.9960
.98	5	.000419	53.15	49.56	.06138	49.24	52.01	.9944
.98	6	.000158	54.25	49.58	.06985	49.36	52.52	.9927
.98	7	.000059	55.36	49.60	.07794	49.49	53.04	.9909
.98	8	.000022	56.47	49.63	.08573	49.62	53.56	.9889
.99	2	.003808	99.98	99.50	.01619	98.99	100.49	.9996
.99	3	.001431	101.04	99.51	.02154	99.06	101.00	.9993
.99	4	.000535	102.13	99.52	.02642	99.16	101.50	.9991
.99	5	.000199	103.23	99.53	.03101	99.28	102.01	.9986
.99	6	.000074	104.36	99.54	.03537	99.41	102.52	.9982
.99	7	.000028	105.49	99.55	.03956	99.56	103.03	.9977
.99	8	.000010	106.63	99.56	.04360	99.71	103.54	.9972

Table 5. Queueing characteristics for symmetrical systems.

s	ρ	P{D>1}	E{D}	P{J>0}	E{J}	ρ	P{D>1}	E{D}	P{J>0}	E{J}
2	.5	.06597	.5737	.05210	.05210	.7	.1235	.735	.0680	.0680
3	.5	.09104	.8008	.08161	.08236	.7	.1850	.990	.1263	.1284
4	.5	.09914	.9183	.09405	.09796	.7	.2214	1.111	.1713	.1811
5	.5	.09858	.9828	.09607	.10351	.7	.2445	1.176	.2050	.2261
6	.5	.09338	1.0178	.09220	.10252	.7	.2595	1.213	.2294	.2638
7	.5	.08583	1.0357	.08529	.09751	.7	.2691	1.234	.2467	.2948
8	.5	.07730	1.0435	.07706	.09024	.7	.2747	1.247	.2584	.3200
9	.5	.06862	1.0454	.06851	.08190	.7	.2773	1.254	.2655	.3398
10	.5	.06026	1.0441	.06022	.07326	.7	.2776	1.257	.2693	.3549
11	.5	.05250	1.0410	.05248	.06482	.7	.2762	1.258	.2702	.3660
12	.5	.04545	1.0371	.04544	.05687	.7	.2732	1.257	.2690	.3735
13	.5	.03916	1.0330	.03915	.04955	.7	.2691	1.254	.2662	.3779
14	.5	.03360	1.0290	.03360	.04293	.7	.2641	1.251	.2620	.3797
15	.5	.02873	1.0253	.02873	.03703	.7	.2583	1.246	.2569	.3793
16	.5	.02451	1.0219	.02451	.03182	.7	.2519	1.241	.2510	.3769
17	.5	.02085	1.0188	.02085	.02726	.7	.2452	1.235	.2445	.3730
18	.5	.01771	1.0161	.01771	.02328	.7	.2381	1.229	.2376	.3677
19	.5	.01501	1.0138	.01501	.01984	.7	.2308	1.222	.2304	.3613
20	.5	.01271	1.0117	.01271	.01688	.7	.2233	1.215	.2231	.3541
21	.5	.01074	1.0100	.01074	.01433	.7	.2158	1.208	.2156	.3461
22	.5	.00907	1.0084	.00907	.01214	.7	.2083	1.201	.2082	.3375
23	.5	.00765	1.0071	.00765	.01027	.7	.2008	1.194	.2007	.3285
24	.5	.00644	1.0060	.00644	.00868	.7	.1934	1.188	.1933	.3192
25	.5	.00542	1.0051	.00542	.00733	.7	.1861	1.181	.1860	.3097
2	.9	.1883	0.905	.0395	.0395	.98	.2154	0.980	.0095	.0095
3	.9	.2835	1.191	.0802	.0821	.98	.3211	1.282	.0197	.0202
4	.9	.3424	1.323	.1184	.1265	.98	.3846	1.421	.0297	.0318
5	.9	.3828	1.396	.1535	.1718	.98	.4272	1.498	.0392	.0439
6	.9	.4125	1.441	.1857	.2173	.98	.4578	1.545	.0483	.0565
7	.9	.4353	1.471	.2153	.2629	.98	.4808	1.577	.0570	.0694
8	.9	.4536	1.492	.2426	.3083	.98	.4988	1.598	.0653	.0826
9	.9	.4686	1.507	.2678	.3534					

Table 6. Heavy traffic limits for symmetrical systems.

s	P{L=0}	E{L}	σ^2{L}	P{W=0}	E{W}	σ^2{W}	R	P{D>1}	E{D}	P{J>0}	E{J}
2	.3763	0.00	1.14	1.62	-1.00	-0.86	4.0	.222	1.00	.49	.49
3	.1397	1.07	2.68	2.17	-0.93	0.36	6.9	.330	1.31	1.03	1.06
4	.0516	2.18	4.52	2.66	-0.82	2.64	10.3	.395	1.45	1.56	1.67
5	.0190	3.32	6.62	3.13	-0.68	5.96	14.2	.437	1.53	2.07	2.32
6	.0070	4.47	8.94	3.58	-0.53	10.3	18.6	.468	1.57	2.56	2.99
7	.0026	5.63	11.5	4.01	-0.37	15.7	23.4	.491	1.61	3.03	3.69
8	.0009	6.80	14.1	4.44	-0.20	21.9	28.8	.509	1.63	3.49	4.40

Table 7. Queueing characteristics for asymmetrical systems with 4 queues.

ρ		P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}	E{H}	E{D}
.1	Ψ1	.5694	.5433	.7135	.998700	.007814	.3083	5.426	.430
.1	Ψ2	.6494	.4294	.6523	.999155	.003665	.1855	4.291	.350
.1	Ψ3	.7382	.3098	.5676	.999527	.001387	.0912	3.096	.261
.5	Ψ1	.07619	2.681	1.847	.8280	.9240	3.114	4.439	1.034
.5	Ψ2	.10699	2.430	1.806	.8442	.7159	2.571	4.144	.952
.5	Ψ3	.15549	2.134	1.752	.8640	.5153	1.981	3.752	.858
.7	Ψ1	.02235	4.751	3.187	.5877	2.560	5.257	4.227	1.262
.7	Ψ2	.03383	4.427	3.146	.6059	2.234	4.728	4.090	1.163
.7	Ψ3	.05331	4.046	3.092	.6293	1.878	4.118	3.901	1.050
.9	Ψ1	.00330	12.475	9.725	.2264	9.792	12.96	4.069	1.530
.9	Ψ2	.00541	12.042	9.698	.2356	9.347	12.51	4.033	1.410
.9	Ψ3	.00933	11.519	9.661	.2482	8.818	11.99	3.982	1.269
.95	Ψ1	.00131	22.84	19.64	.1169	20.01	23.77	4.034	1.612
.95	Ψ2	.00220	22.37	19.62	.1220	19.53	23.39	4.017	1.485
.95	Ψ3	.00390	21.79	19.60	.1289	18.94	22.92	3.993	1.335

Table 8. The distribution of customers over the queues and the mean number of customers in the various queues in asymmetrical systems.

ρ		P{C=1}	P{C=2}	P{C=3}	P{C=4}	$E\{N_1\}$	$E\{N_2\}$	$E\{N_3\}$	$E\{N_4\}$
.1	Ψ1s	.7099	.2348	.0484	.0069	.2842	.0939	.0194	.0028
.1	Ψ1	.6125	.3014	.0737	.0124	.3836	.1256	.0307	.0034
.1	Ψ2	.2350	.2509	.2509	.2633	.1470	.1046	.1046	.0732
.1	Ψ3	.0043	.0335	.1851	.7771	.0027	.0140	.0771	.2160
.5	Ψ1s	.3574	.2908	.2135	.1384	.8743	.6688	.4728	.3004
.5	Ψ1	.2605	.3068	.2380	.1947	1.0721	.7555	.5588	.2950
.5	Ψ2	.1949	.2484	.2484	.3084	.7283	.6048	.6048	.4921
.5	Ψ3	.1005	.1871	.2634	.4490	.3423	.4300	.6261	.7354
.7	Ψ1s	.3002	.2718	.2353	.1927	1.3807	1.1594	.9466	.7453
.7	Ψ1	.2084	.2763	.2463	.2690	1.6605	1.2877	1.0731	.7298
.7	Ψ2	.1812	.2460	.2460	.3268	1.2741	1.1042	1.1042	.9447
.7	Ψ3	.1361	.2158	.2530	.3951	.8360	.8986	1.1126	1.1986
.9	Ψ1s	.2638	.2565	.2464	.2334	3.271	3.033	2.797	2.566
.9	Ψ1	.1738	.2514	.2439	.3308	3.684	3.240	3.003	2.549
.9	Ψ2	.1675	.2426	.2426	.3472	3.250	3.009	3.009	2.774
.9	Ψ3	.1557	.2335	.2440	.3668	2.743	2.752	2.990	3.035
.95	Ψ1s	.2566	.2531	.2483	.2419	5.841	5.598	5.355	5.116
.95	Ψ1	.1667	.2457	.2422	.3454	6.304	5.837	5.594	5.103
.95	Ψ2	.1639	.2414	.2414	.3533	5.857	5.590	5.590	5.328
.95	Ψ3	.1583	.2370	.2420	.3628	5.327	5.315	5.560	5.589

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Appendix.

This appendix contains numerical data for the classical M/M/s queueing system for comparison with the results for the symmetrical shortest queue model (tables A.1, A.2 and A.3). Below, asymptotic expressions for light traffic behaviour of several queueing characteristics are presented (cf. Section 4): as $\rho \downarrow 0$,

$$P\{L=0\} = \sum_{k=0}^{s+1} \frac{(-s\rho)^k}{k!} - \frac{s^s \rho^{s+1}}{(s+1)!} + O(\rho^{s+2})], \tag{A.1}$$

$$E\{L\} = s\rho + \frac{s^s \rho^{s+1}}{s!} [1 - (s-2)\,\rho + O(\rho^2)], \tag{A.2}$$

$$\sigma^2\{L\} = s\rho + \frac{s^s \rho^{s+1}}{s!} [s+1 - (s^2-4)\,\rho + O(\rho^2)], \tag{A.3}$$

$$P\{W=0\} = 1 - \frac{(s\rho)^s}{s!} [1 - (s-1)\,\rho + O(\rho^2)], \tag{A.4}$$

$$E\{W\} = \frac{1}{\mu} \frac{(s\rho)^s}{s!} [1 - (s-2)\,\rho + O(\rho^2)], \tag{A.5}$$

$$\sigma^2\{W\} = 2\,(\frac{1}{\mu})^2 \frac{(s\rho)^s}{s!} [1 - (s-3)\,\rho + O(\rho^2)]. \tag{A.6}$$

Table A.1. Queueing characteristics for lightly loaded M/M/s-systems.

ρ	s	P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}
.1	1	.90000	.111111	.3514	.90000000	.11111111	.48432
.1	2	.81818	.202020	.4540	.98181818	.02020202	.21092
.1	3	.74074	.300412	.5492	.99629630	.00411523	.09554
.1	4	.67031	.400088	.6328	.99920556	.00088271	.04428
.1	5	.60653	.500020	.7072	.99982450	.00019500	.02082
.1	6	.54881	.600004	.7746	.99996049	.00004390	.00988
.1	7	.49659	.700001	.8367	.99999098	.00001002	.00472
.1	8	.44933	.800000	.8944	.99999792	.00000231	.00226
.1	9	.40657	.900000	.9487	.99999952	.00000054	.00109
.1	10	.36788	1.000000	1.0000	.99999989	.00000013	.00053
.1	11	.33287	1.100000	1.0488	.99999997	.00000003	.00026
.1	12	.30119	1.200000	1.0954	.99999999	.00000001	.00012

Table A.2. Queueing characteristics for moderately loaded M/M/s-systems.

ρ	s	P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}
.5	1	.500000	1.000	1.414	.5000	1.0000	1.7321
.5	2	.333333	1.333	1.491	.6667	.6667	1.4907
.5	3	.210526	1.737	1.584	.7632	.4737	1.2924
.5	4	.130435	2.174	1.685	.8261	.3478	1.1271
.5	5	.080100	2.630	1.789	.8696	.2607	.9874
.5	6	.048960	3.099	1.893	.9010	.1983	.8682
.5	7	.029845	3.576	1.997	.9238	.1524	.7657
.5	8	.018163	4.059	2.100	.9410	.1181	.6771
.5	9	.011042	4.546	2.201	.9540	.0921	.5999
.5	10	.006708	5.036	2.299	.9639	.0722	.5326
.5	11	.004073	5.528	2.396	.9716	.0569	.4735
.5	12	.002473	6.022	2.490	.9775	.0449	.4216
.5	13	.001501	6.518	2.582	.9822	.0356	.3759
.5	14	.000911	7.014	2.672	.9858	.0283	.3355
.5	15	.000553	7.511	2.760	.9887	.0226	.2997
.5	16	.000335	8.009	2.846	.9910	.0180	.2680
.5	17	.000203	8.507	2.930	.9929	.0144	.2398
.5	18	.000123	9.006	3.012	.9942	.0116	.2148
.5	19	.000075	9.505	3.092	.9954	.0093	.1925
.5	20	.000045	10.004	3.170	.9963	.0075	.1726
.5	21	.000028	10.503	3.247	.9970	.0060	.1549
.5	22	.000017	11.002	3.322	.9976	.0048	.1390
.5	23	.000010	11.502	3.395	.9980	.0039	.1249
.5	24	.000006	12.002	3.468	.9984	.0031	.1122
.5	25	.000004	12.501	3.538	.9987	.0025	.1008
.7	1	.300000	2.333	2.789	.3000	2.3333	3.180
.7	2	.176471	2.745	2.832	.4235	1.9216	3.020
.7	3	.095694	3.249	2.885	.5077	1.6411	2.872
.7	4	.050212	3.800	2.944	.5713	1.4288	2.736
.7	5	.025898	4.382	3.007	.6222	1.2595	2.610
.7	6	.013221	4.984	3.072	.6640	1.1199	2.492
.7	7	.006705	5.602	3.140	.6993	1.0025	2.383
.7	8	.003384	6.231	3.209	.7294	.9020	2.280
.7	9	.001703	6.871	3.278	.7555	.8151	2.184
.7	10	.000855	7.517	3.349	.7783	.7391	2.093
.7	11	.000428	8.171	3.420	.7983	.6722	2.007
.7	12	.000214	8.829	3.492	.8161	.6130	1.926
.7	13	.000107	9.492	3.563	.8319	.5602	1.850
.7	14	.000053	10.159	3.635	.8461	.5129	1.777
.7	15	.000027	10.829	3.707	.8588	.4705	1.707
.7	16	.000013	11.503	3.778	.8703	.4323	1.642
.7	17	.000007	12.178	3.850	.8807	.3977	1.579
.7	18	.000003	12.856	3.921	.8901	.3663	1.519
.7	19	.000002	13.536	3.991	.8987	.3378	1.462
.7	20	.000001	14.218	4.062	.9064	.3119	1.408
.7	21	.000000	14.902	4.132	.9135	.2882	1.356
.7	22	.000000	15.587	4.202	.9200	.2666	1.306
.7	23	.000000	16.273	4.271	.9260	.2467	1.259
.7	24	.000000	16.960	4.340	.9314	.2286	1.213
.7	25	.000000	17.648	4.408	.9364	.2119	1.169

Table A.3. Queueing characteristics for heavily loaded M/M/s-systems.

ρ	s	P{L=0}	E{L}	σ{L}	P{W=0}	E{W}	σ{W}
.8	1	.200000	4.000	4.474	.2000	4.000	4.899
.8	2	.111111	4.444	4.500	.2889	3.556	4.787
.8	3	.056180	4.989	4.533	.3528	3.236	4.678
.8	4	.027303	5.586	4.570	.4036	2.982	4.575
.8	5	.012987	6.216	4.610	.4459	2.771	4.475
.8	6	.006096	6.871	4.651	.4822	2.589	4.380
.8	7	.002836	7.544	4.694	.5141	2.430	4.289
.8	8	.001311	8.231	4.738	.5424	2.288	4.201
.8	9	.000603	8.929	4.784	.5678	2.161	4.116
.8	10	.000276	9.637	4.830	.5908	2.046	4.034
.8	11	.000126	10.353	4.876	.6119	1.941	3.955
.8	12	.000058	11.075	4.924	.6312	1.844	3.878
.8	13	.000026	11.804	4.972	.6490	1.755	3.804
.8	14	.000012	12.538	5.020	.6655	1.673	3.732
.8	15	.000005	13.277	5.069	.6808	1.596	3.662
.8	16	.000002	14.020	5.118	.6951	1.524	3.594
.9	1	.100000	9.000	9.487	.1000	9.000	9.950
.9	2	.052632	9.474	9.500	.1474	8.526	9.891
.9	3	.024907	10.054	9.516	.1829	8.171	9.831
.9	4	.011256	10.690	9.533	.2122	7.878	9.772
.9	5	.004959	11.362	9.551	.2375	7.625	9.714
.9	6	.002149	12.061	9.570	.2599	7.401	9.656
.9	7	.000921	12.780	9.590	.2800	7.200	9.600
.9	8	.000392	13.514	9.610	.2985	7.015	9.544
.9	9	.000165	14.261	9.630	.3155	6.845	9.489
.9	10	.000070	15.019	9.651	.3313	6.687	9.435
.9	11	.000029	15.786	9.673	.3461	6.539	9.382
.9	12	.000012	16.560	9.694	.3600	6.400	9.330
.9	13	.000005	17.342	9.716	.3731	6.269	9.278
.9	14	.000002	18.130	9.738	.3855	6.145	9.227
.9	15	.000001	18.924	9.761	.3974	6.026	9.177
.9	16	.000000	19.722	9.783	.4087	5.913	9.127
.99	1	.010000	99.00	99.499	.01000	99.00	99.995
.99	2	.005025	99.50	99.500	.01497	98.50	99.989
.99	3	.002247	100.11	99.501	.01883	98.12	99.982
.99	4	.000954	100.77	99.503	.02209	97.79	99.976
.99	5	.000394	101.48	99.505	.02497	97.50	99.969
.99	6	.000159	102.21	99.506	.02758	97.24	99.962
.99	7	.000064	102.96	99.508	.02997	97.00	99.955
.99	8	.000025	103.73	99.510	.03220	96.78	99.948
.99	9	.000010	104.52	99.512	.03429	96.57	99.941
.99	10	.000004	105.31	99.514	.03626	96.37	99.934
.99	11	.000002	106.11	99.515	.03814	96.19	99.927
.99	12	.000001	106.93	99.517	.03993	96.01	99.920
.99	13	.000000	107.75	99.519	.04165	95.84	99.913
.99	14	.000000	108.57	99.521	.04330	95.67	99.906
.99	15	.000000	109.41	99.523	.04489	95.51	99.899
.99	16	.000000	110.24	99.525	.04643	95.36	99.892

Table A.4. Heavy traffic limits of queueing characteristics for M/M/s-systems.

s	P{L=0}	E{L}	σ^2 {L}	P{W=0}	E{W}	σ^2 {W}
1	1.000000	-1.000000	0.000000	1.000000	-1.000000	-1.000000
2	0.500000	-0.500000	0.250000	1.500000	-1.500000	-2.250000
3	0.222222	0.111111	0.543210	1.888889	-1.888889	-3.567901
4	0.093750	0.781250	0.858398	2.218750	-2.218750	-4.922852
5	0.038400	1.489600	1.187492	2.510400	-2.510400	-6.302108
6	0.015432	2.225309	1.526397	2.774691	-2.774691	-7.698912
7	0.006120	2.981861	1.872700	3.018139	-3.018139	-9.109161
8	0.002403	3.754982	2.224840	3.245018	-3.245018	-10.530142
9	0.000937	4.541684	2.581736	3.458316	-3.458316	-11.959948
10	0.000363	5.339784	2.942605	3.660216	-3.660216	-13.397179
11	0.000140	6.147628	3.306858	3.852372	-3.852372	-14.840770
12	0.000054	6.963926	3.674036	4.036074	-4.036074	-16.289891

Table A.4 contains values of the following heavy traffic limits:

$$\lim_{\rho\uparrow 1} \frac{P\{L=0\}}{1-\rho} = \frac{s!}{s^s}, \qquad \lim_{\rho\uparrow 1} \left[E\{L\} - \frac{1}{1-\rho}\right] = s - 1 - \Phi,$$

$$\lim_{\rho\uparrow 1} \left[\sigma^2\{L\} - \left(\frac{1}{1-\rho}\right)^2 + \frac{1}{1-\rho}\right] = 2s - \Phi - \Phi^2, \qquad \lim_{\rho\uparrow 1} \frac{P\{W=0\}}{1-\rho} = \Phi,$$

$$\lim_{\rho\uparrow 1} \left[E\{W\} - \frac{1}{1-\rho}\right] = -\Phi, \qquad \lim_{\rho\uparrow 1} \left[\sigma^2\{W\} - \left(\frac{1}{1-\rho}\right)^2\right] = -\Phi^2, \tag{A.7}$$

with

$$\Phi := \frac{s!}{s^s} \sum_{j=0}^{s-1} \frac{s^j}{j!}. \tag{A.8}$$

Finally, we note that the mean and the variance of the amount of work Y in an M/M/s system are given by

$$E\{Y\} = \frac{s}{\mu} E\{L\}, \qquad \sigma^2\{Y\} = \left(\frac{s}{\mu}\right)^2 [\sigma^2\{L\} + E\{L\}]. \tag{A.9}$$

Hence, data for these quantities can be easily derived from the distribution of L, for the M/M/s model as well as for the shortest-queue model, cf. (2.25).

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