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LOCAL TIMES OF BERNOULLI WALK

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Abstract The joint probability generating functions of local times of Bernoulli walk at various stopping times are determined by simple equations. The results can be interpreted by means of branching processes with immigration satisfying the same equations. Some of the results are obtained also by martingale methods.

Keywords & Phrases: local time, Bernoulli walk, ascending local time, descending local time, branching process with immigration, martingales.

1. Introduction

Let $(S_n)_{n=0}^{\infty}$ be Bernoulli random walk starting at 0, i.e., $S_n = \sum_{k=0}^{n} X_k$ with iid X_k such that $p := P[X_k=1] = 1 - P[X_k=0]$. In the present paper we investigate the ascending local time

$$L^{\uparrow}(n,y) := \sum_{k=1}^{n} 1_{(y-1,y)}(S_{k-1},S_k) , \tag{1.1}$$

the descending local time

$$L^{\downarrow}(n,y) := \sum_{k=1}^{n} 1_{(y+1,y)}(S_{k-1},S_k) , \tag{1.2}$$

and the local time

$$L(n,y) := \sum_{k=1}^{n} 1_y(S_k) , \tag{1.3}$$

considered as infinite random integer-valued vectors

$$\underline{L}^{*}(n) := (L^{*}(n,y))_{y\in Z} ,$$

with * denoting $\uparrow$, $\downarrow$ or nothing. Rather than $\underline{L}^{*}(n)$ we will consider $\underline{L}^{*}(T)$ for specific stopping times T.

The main analytic tool for this investigation is the joint probability generating function (pgf) of infinite random vectors with nonnegative integer components. Set

$$\underline{s}^{\underline{m}} := \Pi_{k\in Z} s_k^{m_k}$$

for infinite vectors $\underline{m} = (m_k)_{k\in Z}$ with nonnegative integer components and $\underline{s} = (s_k)_{k\in Z}$ with $s_k \in [0,1]$. We define the joint pgf of a random vector $\underline{M} = (M_k)_{k\in Z}$ with nonnegative integer components by

$$E\underline{s}^{\underline{M}} = E\Pi_{k\in Z} s_k^{M_k}$$

as function of $\underline{s}$. Obviously, the joint pgf of $\underline{M}$ determines the distribution of $\underline{M}$.

We establish a calculus with joint pgf's, which resembles strongly Feller's (1971) classical treatment of Bernoulli random walk by (one-dimensional) pgf's. As an application we identify $\underline{L}(T)$ for stopping times T with $S_T = 0$ or $T = \infty$ as a simple functional of certain branching processes with immigration. The branching processes themselves can be identified as processes of ascents or processes of descents in the seperate excursions of Bernoulli walk.

The essence of these results is already known. The purpose of this paper is to streamline, complement and unify previous work by Dwass (1975), Rogers (1984) and parts of Cohen & Hooghiemstra (1981), Gerl (1984) and Woess (1985).

2. The pgf's of the local times

As in the introduction, $(S_n)_{n=0}^{\infty}$ is Bernoulli walk starting at 0, with local time $\underline{L}$ defined bij (1.3). Recall that $p := P[S_n - S_{n-1} = 1]$. We will take the following properties of Bernoulli walk for granted:

(a) the strong Markov property;

(b) the Bernoulli walk is recurrent if $p = \frac{1}{2}$, i.e., if $p = \frac{1}{2}$ then $L(\infty,y) := \lim_{n\to\infty} L(n,y) = \infty$ wp1 for all y;

(c) $\lim_{n\to\infty} S_n = \infty.\mathrm{sgn}(p-\frac{1}{2})$ wp1 for $p \neq \frac{1}{2}$, so $L(\infty,y) < \infty$ wp1 for $p \neq \frac{1}{2}$.

In the introduction we defined $\underline{s}^{\underline{m}}$ and pgf's $E\underline{s}^{\underline{m}}$ for $\underline{s} \in [0,1]^{\mathbf{Z}}$. Let S be the set of $\underline{s}$ in $[0,1]^{\mathbf{Z}}$ with all but finitely many components equal to 1. Then the pgf $E\underline{s}^{\underline{m}}$ restricted to S already determines all finite-dimensional distributions of $\underline{M}$, so in fact the distribution of $\underline{M}$. For this reason we will often consider pgf's only on S. We write $\underline{1}$ for $\underline{s} \in S$ with all components equal to 1.

The <u>shift</u> θ: $\mathbf{R}^{\mathbf{Z}} \to \mathbf{R}^{\mathbf{Z}}$ is defined by $(\theta\underline{x})_k := x_{k+1}$ for $k \in \mathbf{Z}$ and $\underline{x} = (x_k)_{k\in\mathbf{Z}} \in \mathbf{R}^{\mathbf{Z}}$. Note that θ maps S onto S, and that $\underline{s}^{\theta\underline{m}} = (\theta^{-1}\underline{s})^{\underline{m}}$.

From the definitions it is clear that

$L(n,y) = L^{\uparrow}(n,y) + L^{\downarrow}(n,y)$, and that

$L^{\uparrow}(n,y) = L^{\downarrow}(n,y-1) + 1_{\mathbf{N}_0}(S_n - y) - 1_{\mathbf{N}_0}(-y)$.

Thus we have

$$L(n,y) = L^{\downarrow}(n,y) + L^{\downarrow}(n,y-1) + 1_{\mathbf{N}_0}(S_n - y) - 1_{\mathbf{N}_0}(-y) . \tag{2.1}$$

Thus $\underline{L}(n)$ is determined by $(\underline{L}^{\downarrow}(n),S_n)$.

We will now derive recursive relations for the pgf of $(\underline{L}^{\downarrow}(n),S_n)$

$$\psi_n(\underline{s},t) := E\underline{s}^{\underline{L}^{\downarrow}(n)}t^{S_n} = E\Pi_{y\in\mathbf{Z}}s_y^{L^{\downarrow}(n,y)}t^{S_n} .$$

Splitting according to the values of S_1 and using the Markov property we find, with $q := 1-p$ and $\psi_0(\underline{s},t) := 1$,

$$\psi_n(\underline{s},t) = qs_{-1}t^{-1}\psi_{n-1}(\theta^{-1}\underline{s},t) + pt\psi_{n-1}(\theta\underline{s},t) .$$

This enables us, at least in principle, to calculate the pgf ψ_n of $(\underline{L}^{\downarrow}(n),S_n)$, but the results do not seem to allow successful analysis. We will do much better if we consider $\underline{L}(T)$ for stopping times T that imply fixed values for S_T.

3. Local times up to stopping times when finite

Let $T_y := \inf\{n\geq 1\colon S_n=y\}$, where $\inf\emptyset := \infty$, and consider the (defective) pgf

$$\phi^*_y(\underline{s}) := E1_{[T_y<\infty]}\underline{s}^{\underline{L}^*(T_y)} . \tag{3.1}$$

We will see that ϕ^*_{-1} and ϕ^*_1 determine ϕ^*_y for all other y.

First, let $y > 1$. By the strong Markov property, $\underline{L}^*(T_y) - \underline{L}^*(T_1)$ is independent of $\underline{L}^*(T_1)$ and distributed as $\theta^{-1}\underline{L}^*(T_{y-1})$. Consequently,

$$\begin{aligned}\phi^*_y(\underline{s}) &= E1_{[T_1<\infty]}\underline{s}^{\underline{L}^*(T_1)}1_{[T_y-T_1<\infty]}\underline{s}^{\underline{L}^*(T_y)-\underline{L}^*(T_1)}\\ &= E1_{[T_1<\infty]}\underline{s}^{L^*(T_1)}E1_{[T_{y-1}<\infty]}\underline{s}^{\theta^{-1}\underline{L}(T_{y-1})}\\ &= \phi^*_1(\underline{s})\phi^*_{y-1}(\theta\underline{s}) .\end{aligned}$$

By induction we find

$$\phi^*_y(\underline{s}) = \phi^*_1(\underline{s})\phi^*_1(\theta\underline{s})\ldots\phi^*_1(\theta^{y-1}\underline{s}) \text{ for } y > 1 , \tag{3.2}$$

and similarly

$$\phi^*_y(\underline{s}) = \phi^*_{-1}(\underline{s})\phi^*_{-1}(\theta^{-1}\underline{s})\ldots\phi^*_{-1}(\theta^{y+1}\underline{s}) \text{ for } y < -1 . \tag{3.3}$$

We can calculate $\phi^*_0(\underline{s})$ by splitting according to the values of S_1, finding

$$\begin{aligned}\phi^{\downarrow}_0(\underline{s}) &= E\underline{s}^{\underline{L}^{\downarrow}(1)}1_{[T_0<\infty]}\underline{s}^{\underline{L}^{\downarrow}(T_0)-\underline{L}^{\downarrow}(1)}\\ &= pE1_{[T_{-1}<\infty]}\underline{s}^{\theta^{-1}\underline{L}^{\downarrow}(T_{-1})} + qs_{-1}E1_{[T_1<\infty]}\underline{s}^{\theta\underline{L}^{\downarrow}(T_1)}\\ &= p\phi^{\downarrow}_{-1}(\theta\underline{s}) + qs_{-1}\phi^{\downarrow}_1(\theta^{-1}\underline{s}) ,\end{aligned} \tag{3.4}$$

and similarly

$$\phi^{\uparrow}_0(\underline{s}) = ps_1\phi^{\uparrow}_{-1}(\theta\underline{s}) + q\phi^{\uparrow}_1(\theta^{-1}\underline{s}) , \text{ and} \tag{3.5}$$

$$\phi_0(\underline{s}) = ps_1\phi_{-1}(\theta\underline{s}) + qs_{-1}\phi_1(\theta^{-1}\underline{s}) . \tag{3.6}$$

It remains to calculate ϕ^*_{-1} and ϕ^*_1. Splitting according to the values of S_1 we find

$$\phi^{\downarrow}_{-1}(\underline{s}) = qs_{-1} + pE1_{[T_{-2}<\infty]}\underline{s}^{\theta^{-1}\underline{L}^{\downarrow}(T_{-2})}$$

$$= qs_{-1} + p\phi^{\downarrow}_{-2}(\theta\underline{s}) \; .$$

With (3.3) for $y = -2$ this combines into $\phi^{\downarrow}_{-1}(\underline{s}) = qs_{-1} + p\phi^{\downarrow}_{-1}(\theta\underline{s})\phi^{\downarrow}_{-1}(\underline{s})$, from which we obtain

$$\phi^{\downarrow}_{-1}(\underline{s}) = \frac{qs_{-1}}{1 - p\phi^{\downarrow}_{-1}(\theta\underline{s})} \; . \tag{3.7}$$

Iterating (3.7) we can express $\phi^{\downarrow}_{-1}(\underline{s})$ into $\phi^{\downarrow}_{-1}(\theta^n\underline{s})$ by the continued fraction

$$\phi^{\downarrow}_{-1}(\underline{s}) = qs_{-1}/(1-pqs_0/(1-\ldots/(1-pqs_{n-2}/(1-p\phi^{\downarrow}_{-1}(\theta^n\underline{s})))\ldots)) \; . \tag{3.8}$$

Let $\underline{s} \in S$. Then $s_k = 1$ for $k \geq k_0(\underline{s})$, so $(\theta^n\underline{s})_k = 1$ for $k \geq -1$ if $n \geq k_0 + 1$. As $\underline{L}^{\downarrow}(T_{-1},y) = 0$ for $y < -1$, we see that $(\theta^n\underline{s})^{\underline{L}^{\downarrow}(T_{-1})} = \underline{1}$ for $n \geq k_0+1$, so $\phi^{\downarrow}_{-1}(\theta^n\underline{s}) = \phi^{\downarrow}_{-1}(\underline{1})$ for these n. Hence $\phi^{\downarrow}_{-1}$ can be calculated explicitly on S by expanding (3.8) for sufficiently large n and using

$$\phi^{\downarrow}_{-1}(\underline{1}) = 1 \wedge \frac{q}{p} \; . \tag{3.9}$$

(We have $\phi^{\downarrow}_{-1}(\underline{1}) = 1$ or $\frac{p}{q}$ by substituting $\underline{s} = \underline{1}$ in (3.7) and arive at (3.9) by properties (b) and (c)).

Although the continued fraction in (3.8) may look appealing (similar expressions have been observed by Gerl (1984) and Woess (1985)), it does not seem tractable for further analysis. Considering $\phi^{\downarrow}_{-1}$ as determined by (3.7) and (3.9) turns out to be more productive.

We will call equations like (3.7) <u>shift equations</u>. So $\phi^{\downarrow}_{-1}$ is determined by shift equation (3.7) with <u>boundary condition</u> (3.9). We will see that many other pgf's are determined this way. For example $\phi^{\downarrow}_{1}$. It is not hard to see that $\phi^{\downarrow}_{1}$ is determined by the shift equation

$$\phi^{\downarrow}_{1}(\underline{s}) = \frac{p}{1 - qs_{-1}\phi^{\downarrow}_{1}(\theta^{-1}\underline{s})} \; , \tag{3.10}$$

with boundary condition

$$\phi^{\downarrow}_{1}(\underline{1}) = 1 \wedge \frac{p}{q} \; . \tag{3.11}$$

The corresponding formulas for ϕ_1 can be derived in the same way as (3.7) and (3.9). By interchanging the roles of +1 and -1, p and q, $\downarrow$ and $\uparrow$, θ and θ^{-1} we can also find the formulas that determine $\phi^{\uparrow}_{1}$, $\phi^{\uparrow}_{-1}$, and ϕ_{-1}.

4. Local times of finite excursions

To allow a better interpretation of the results yet to come, we will make use of the term 'excursion'. The <u>first excursion</u> is defined as the sequence $(S_n)_{n=1}^{T_0}$. Depending on the value of T_0, the excursion is said to be <u>finite</u> or <u>infinite</u>. In Section 9 we will give the general definition of the term 'excursion'. Until then we will concentrate on the first excursion.

Let us consider ϕ_0^*, which is expressed into known functions by (3.4), (3.5) and (3.6). To do a more pleasant analysis we will consider the terms on the right hand side of these equations seperately. So we set

$$\phi_{0+}^*(\underline{s}) := E1_{[T_0<\infty]}1_{[S_1>0]}\underline{s}^{\underline{L}^*(T_0)} , \qquad (4.1)$$

and

$$\phi_{0-}^*(\underline{s}) := E1_{[T_0<\infty]}1_{[S_1<0]}\underline{s}^{\underline{L}^*(T_0)} . \qquad (4.2)$$

The pgf's $\phi_{0\pm}^*$ denote the pgf's of the local times of finite excursions. Note that $\phi_0^* = \phi_{0+}^* + \phi_{0-}^*$.

Combining the identities $\phi_{0+}^{\downarrow}(\theta\underline{s}) = p\phi_{-1}^{\downarrow}(\theta^2\underline{s})$ and

$$\phi_{0+}^{\downarrow}(\underline{s}) = p\phi_{-1}^{\downarrow}(\theta\underline{s}) = p.\frac{qs_0}{1 - p\phi_{-1}^{\downarrow}(\theta^2\underline{s})}$$

we find for $\phi_{0+}^{\downarrow}$ the shift equation

$$\phi_{0+}^{\downarrow}(\underline{s}) = \frac{pqs_0}{1 - \phi_{0+}^{\downarrow}(\theta\underline{s})} , \qquad (4.3)$$

which determines $\phi_{0+}^{\downarrow}$ with boundary condition $\phi_{0+}^{\downarrow}(\underline{1}) = p\wedge q$. (Substitution of $\underline{s} = \underline{1}$ in (4.3) yields $\phi_{0+}^{\downarrow}(\underline{1}) = p$ or q, use properties (b) and (c)).

Combining $\phi_{0-}^{\downarrow}(\theta^{-1}\underline{s}) = qs_{-2}\phi_1^{\downarrow}(\theta^{-2}\underline{s})$ and

$$\phi_{0-}^{\downarrow}(\underline{s}) = qs_{-1}\phi_1^{\downarrow}(\theta^{-1}\underline{s}) = qs_{-1}\cdot\frac{p}{1 - qs_{-2}\phi_1^{\downarrow}(\theta^{-2}\underline{s})}$$

we find for $\phi_{0-}^{\downarrow}$ the shift equation

$$\phi_{0-}^{\downarrow}(\underline{s}) = \frac{pqs_{-1}}{1 - \phi_{0-}^{\downarrow}(\theta^{-1}\underline{s})} , \qquad (4.4)$$

with boundary condition $\phi_{0-}^{\downarrow}(\underline{1}) = p\wedge q$.

In a similar way we can derive the formulas that determine ϕ_{0-}. By interchanging the roles of +1 and -1, p and q, $\uparrow$ and $\downarrow$, θ and θ^{-1}, we can also find the formulas that determine $\phi^{\uparrow}_{0-}$, $\phi^{\uparrow}_{0+}$ and ϕ_{0+}.

5. Applications

We now indicate some applications of the results in the previous sections. As these results are recursive relations rather than explicit expressions, they lead in the applications to more specific recursive relations, which then have to be solved by adhoc methods. Applications of a different character are given in Sections 10 and 11, where the structure of $\underline{L}^{*}(T_0)$ as a stochastic process is identified by its pgf.

5.1 Reobtaining the pgf of local time

Since there is a simple relation between descending local time and local time, given by (2.1), we can derive a shift equation for ϕ_{-1} directly from (3.7).

Let $\underline{s}\theta\underline{s}$ denote the infinite vector with $(\underline{s}\theta\underline{s})_k := s_k s_{k+1}$. We can rewrite $\phi_{-1}(\underline{s})$ as

$$\begin{aligned}
\phi_{-1}(\underline{s}) &= E\Pi_{k\in Z} 1_{[T_{-1}<\infty]} s_k^{L(T_{-1},k)} \\
&= E\Pi_{k\in Z} 1_{[T_{-1}<\infty]} s_k^{L^{\downarrow}(T_{-1},k) + L^{\downarrow}(T_{-1},k-1) - 1_{\{0\}}(k)} \\
&= E\Pi_{k\in Z} 1_{[T_{-1}<\infty]} (s_k s_{k+1})^{L^{\downarrow}(T_{-1},k)} s_0^{-1} \\
&= s_0^{-1}\phi^{\downarrow}_{-1}(\underline{s}\theta\underline{s}) .
\end{aligned}$$

Combining this result with (3.7) yields

$$\begin{aligned}
\phi_{-1}(\underline{s}) &= s_0^{-1}\cdot\frac{qs_{-1}s_0}{1 - p\phi^{\downarrow}_{-1}(\theta\underline{s}\theta^2\underline{s})} \\
&= \frac{qs_{-1}}{1 - ps_1\phi_{-1}(\theta\underline{s})} . \qquad (5.1)
\end{aligned}$$

Using $\phi_{0+}(\underline{s}) = \phi^{\downarrow}_{0+}(\underline{s}\theta\underline{s})$ and (4.3) we find for ϕ_{0+} the shift equation

$$\phi_0\underline{s}) = \frac{pqs_1s_0}{1 - \phi_{0+}(\theta\underline{s})} . \qquad (5.2)$$

Using the symmetry in Bernoulli walk, we also find

$$\phi_1(\underline{s}) = \frac{ps_1}{1 - qs_{-1}\phi_1(\theta^{-1}\underline{s})}, \text{ and} \tag{5.3}$$

$$\phi_{0-}(\underline{s}) = \frac{pqs_{-1}s_0}{1 - \phi_{0-}(\theta^{-1}\underline{s})}. \tag{5.4}$$

Equations (3.2) and (3.3), together with shift equations (5.1) and (5.3), together with boundary conditions (3.9) and (3.11) (which also hold for * instead of $\downarrow$) determine $\phi_y(\underline{s})$ for all y. Furthermore, ϕ_{0-} and ϕ_{0+} are determined by (5.2), (5.4) and $\phi_{0\pm}(\underline{1}) = p \wedge q$.

5.2 Specializing the pgf's

5.2.1 For random times T we obtain the defective (one-dimensional) pgf of T by

$$E1_{[T<\infty]}(s\underline{1})^{\underline{L}(T)} = E1_{[T<\infty]}s^{\sum_{y\in Z}L(T,y)} = E1_{[T<\infty]}s^{T}.$$

Consequently,

$$A_y(s) := E1_{[T_y<\infty]}s^{T_y} = \phi_y(s\underline{1}).$$

Specializing equations (3.2), (3.3), (3.6), (5.1) and (5.3) we reobtain the equations by which Feller (1971) obtains the defective pgf's of T_y:

$$A_{\pm y} = (A_{\pm 1})^y \text{ for } y \geq 1,$$
$$A_0(s) = psA_{-1}(s) + qsA_1(s),$$

$$A_{-1}(s) = \frac{qs}{1 - psA_{-1}(s)},$$

$$A_1(s) = \frac{ps}{1 - qsA_1(s)}.$$

Analogous results can of course be derived for the total number of descents $\sum_{k\in Z}L^{\downarrow}(T,k)$ and for the total number of ascents $\sum_{k\in Z}L^{\uparrow}(T,k)$.

5.2.2 For subsets A of Z Let us define the <u>restriction operator</u> R_A: $S \to S$ by $(R_A\underline{s})_k := s_k$ if $k \in A$, $:= 1$ else. Write $R_k := R_{\{k\}}$ for $k \in Z$. The (one-dimensional) pgf of $L(T,y)$ is obtained from the pgf of $\underline{L}(T)$ by substituting $\underline{s} = R_y s\underline{1}$.

We now calculate the defective pgf

$$\Lambda_y(s) := E1_{[T_0<\infty]}1_{[S_1>0]}s^{L(T_0,y)}.$$

We have $\lambda_y(s) = \Phi_{0+}(R_y s\underline{1})$. From (5.2) we obtain, as $\theta R_y s\underline{1} = R_{y-1}s\underline{1}$,

$$\lambda_y(s) = \frac{pq}{1 - \lambda_{y-1}(s)} \quad \text{for } y \geq 2 ,$$

$$\lambda_1(s) = \frac{pqs}{1 - \lambda_0(s)} ,$$

and by direct interpretation

$\lambda_0(s) = sP[T_0<\infty,\ S_1>0] = s\Phi_{0+}(\underline{1}) = (p\wedge q)s$, and

$\lambda_y(s) = p \wedge q$ for $y \leq -1$.

We obtain explicit expressions for λ_y by induction, or by standard difference equation techniques, yielding

$$\lambda_y(s) = \frac{pqP_{y-1}(s)}{P_y(s)} \quad \text{for } y \geq 1,$$

where, with $v := p\wedge q$, $w := p\vee q$,

$$P_y(s) = \begin{cases} w^y - v^y + (v^y - 2vw^y)s & \text{for } p \neq \frac{1}{2} \\ 2y(\frac{1}{2})^y + (1-2y)(\frac{1}{2})^y s & \text{for } p = \frac{1}{2} \end{cases} .$$

Analogous results can be derived for ascending and descending local time.

5.2.3 Let $M := \sup_{0\leq n\leq T_0} S_n$. Then $[M<y] = [L(T_0,y)=0]$ for $y \geq 1$, so

$$P[M<y,\ T_0<\infty,\ S_1>0] = \lambda_y(0) = \begin{cases} pq(p^{y-1}-q^{y-1})/(p^y-q^y) & \text{for } p \neq \frac{1}{2} \\ (y-1)/2y & \text{for } p = \frac{1}{2} \end{cases} .$$

5.2.4 Let graphs of Bernoulli walks be represented by polygons with slopes $\pm\pi/4$ as in Feller (1971). Then $\Delta := \sum_{y\in Z} yL(T_0,y)$ is the signed area between this polygon and the time axis restricted to $[0,T_0]$. The defective pgf of Δ on $[T_0<\infty,\ S_1>0]$ is obtained by

$$\delta(s) := \Phi_{0+}((s^y)_{y\in Z}) = E1_{[T_0<\infty]}1_{[S_1>0]}\Pi_{y=0}^{\infty} s^{yL(T_0,y)}$$

$$= E1_{[T_0<\infty]}1_{[S_1>0]}s^{\Delta} .$$

From (5.2) and $\theta^n(s^y)_{y\in Z} = s^n(s^y)_{y\in Z}$ we obtain

$$\Phi_{0+}(s^n(s^y)_{y\in Z}) = \frac{pqs^{2n+1}}{1 - \Phi_{0+}(s^{n+1}(s^y)_{y\in Z})} .$$

Hence

$\delta(s) = pqs/(1 - pqs^3/(1 - pqs^5/(1 - \ldots$.

From Perron (1954) we know that this infinite continued fraction converges.

6. Local times up to stopping times when infinite

We define

$$\chi^*_y(\underline{s}) := E1_{[T_y=\infty]}\underline{s}^{\underline{L}^*(\infty)} , \tag{6.1}$$

$$\chi^*_{0+}(\underline{s}) := E1_{[T_0=\infty]}1_{[S_1>0]}\underline{s}^{\underline{L}^*(\infty)} , \text{ and} \tag{6.2}$$

$$\chi^*_{0-}(\underline{s}) := E1_{[T_0=\infty]}1_{[S_1<0]}\underline{s}^{\underline{L}^*(\infty)} . \tag{6.3}$$

We can calculate these pgf's by splitting according to the values of T_1, or to the values of S_1, and by using symmetry.

Splitting according to T_1 being finite or infinite yields

$$\chi^*_y(\underline{s}) = \phi^*_{-1}(\underline{s})\chi^*_{y+1}(\theta^{-1}\underline{s}) + \chi^*_{-1}(\underline{s}) \text{ for } y < -1 . \tag{6.4}$$

Splitting according to S_1 being 1 or -1 yields (especially for $y \in \{-1,0,1\}$):

$$\chi^{\downarrow}_y(\underline{s}) = (1 - 1_{\{1\}}(y))p\chi^{\downarrow}_{y-1}(\theta\underline{s}) + (1 - 1_{\{-1\}}(y))qs_{-1}\chi^{\downarrow}_{y+1}(\theta^{-1}\underline{s}) , \tag{6.5}$$

$$\chi_y(\underline{s}) = (1 - 1_{\{1\}}(y))ps_1\chi_{y-1}(\theta\underline{s}) + (1 - 1_{\{-1\}}(y))qs_{-1}\chi_{y+1}(\theta^{-1}\underline{s}) , \tag{6.6}$$

$$\chi^{\uparrow}_y(\underline{s}) = (1 - 1_{\{1\}}(y))ps_1\chi^{\uparrow}_{y-1}(\theta\underline{s}) + (1 - 1_{\{-1\}}(y))q\chi^{\uparrow}_{y+1}(\theta^{-1}\underline{s}) , \tag{6.7}$$

and likewise

$$\chi^{\downarrow}_{0+}(\underline{s}) = p\chi^{\downarrow}_{-1}(\theta\underline{s}) , \tag{6.8}$$

$$\chi_{0+}(\underline{s}) = ps_1\chi_{-1}(\theta\underline{s}) . \tag{6.9}$$

Equation (6.9) also holds for $\chi^{\uparrow}$ instead of χ.

Using the symmetry in Bernoulli walk we can find similar equations for χ^*_y with $y > 1$ and for χ^*_{0-}.

From (6.4) and its counterpart for $y > 1$, and from (6.5), (6.6), (6.7) for $y = 0$, we see that χ^*_{-1} and χ^*_1 determine χ^*_y for all other y. Thus it remains to calculate χ^*_{-1} and χ^*_1. Specializing (6.5) for $y = -1$ yields $\chi^{\downarrow}_{-1}(\underline{s}) = p\chi^{\downarrow}_{-2}(\theta\underline{s})$. With (6.4) for $y = -2$ this combines into $\chi^{\downarrow}_{-1}(\underline{s}) = p\phi^{\downarrow}_{-1}(\theta\underline{s})\chi^{\downarrow}_{-1}(\underline{s}) + p\chi^{\downarrow}_{-1}(\theta\underline{s})$, from which we find the shift equation

$$\chi^{\downarrow}_{-1}(\underline{s}) = \frac{p\chi^{\downarrow}_{-1}(\theta\underline{s})}{1 - p\phi^{\downarrow}_{-1}(\theta\underline{s})} . \tag{6.10}$$

Similar shift equations can be found for $\chi_1^{\downarrow}$ and χ_1. Shift equations for $\chi_1^{\uparrow}$, $\chi_{-1}^{\uparrow}$ and χ_{-1} can then be found by the symmetry in Bernoulli walk. Here we only mention

$$\chi_{-1}(\underline{s}) = \frac{ps_1\chi_{-1}(\theta\underline{s})}{1 - ps_1\phi_{-1}(\theta\underline{s})} . \tag{6.11}$$

The corresponding boundary conditions follow by direct interpretation:

$$\chi^*_{-1}(\underline{1}) = P[T_{-1}=\infty] = 1 - \phi^*_{-1}(\underline{1}) = 0 \vee (1 - \tfrac{q}{p}) , \tag{6.12}$$

$$\chi^*_{1}(\underline{1}) = P[T_{1}=\infty] = 1 - \phi^*_{1}(\underline{1}) = 0 \vee (1 - \tfrac{p}{q}) . \tag{6.13}$$

7. Branching processes

In Sections 10 and 11 we are going to compare local times of Bernoulli walk with branching processes. To this end we study here the joint pgf's of branching processes on the whole time domain.

A branching process with or without immigration is denoted by $(Z_n)_{n=0}^{\infty}$, where Z_n is the number of individuals in the nth generation. The (n+1)st generation consists of the total progeny of the nth generation, the sum of Z_n iid random variables with pgf π, together with the immigrants at time (n+1), whose number is an independent random variable with pgf β. The pgf of the zeroth generation or 'patriarchate' Z_0 is denoted by α.

The distribution of $(Z_n)_{n=0}^{\infty}$, or equivalently, the joint pgf $\zeta(\underline{s}) := E\underline{s}^{\underline{Z}}$ (with $Z_n := 0$ for $n < 0$) is determined by the pgf's α, β and π. When α and β or α, β and π vary, we will write $\zeta_{\alpha,\beta}$ or ${}_{\pi}\zeta_{\alpha,\beta}$. Fundamental choises for α and β are $\alpha = 1$ and $\alpha = \iota$ with $\iota(s) := s$ corresponding to empty and unit patriarchate, and $\beta = 1$ corresponding to no immigration. So $\zeta_{\iota,1}$ is the joint pgf of the monopatriarchal branching process without immigration, and $\zeta_{1,\beta}$ that of the branching process with immigration starting with empty population.

The $\zeta_{\alpha,\beta}$ are determined by the mutual relations

$$\zeta_{\alpha,1}(\underline{s}) = \alpha(\zeta_{\iota,1}(\underline{s})) , \tag{7.1}$$

$$\zeta_{\alpha,\beta}(\underline{s}) = \zeta_{\alpha,1}(\underline{s})\zeta_{1,\beta}(\underline{s}) , \tag{7.2}$$

the shift equations

$$\zeta_{\iota,1}(\underline{s}) = s_0\zeta_{\pi,1}(\theta\underline{s}) = s_0\pi(\zeta_{\iota,1}(\theta\underline{s})) , \tag{7.3}$$

$$\zeta_{1,\beta}(\underline{s}) = \zeta_{\beta,\beta}(\theta\underline{s}) = \zeta_{\beta,1}(\theta\underline{s})\zeta_{1,\beta}(\theta\underline{s}) , \tag{7.4}$$

and the boundary condition

$$\zeta_{\alpha,\beta}(\underline{1}) = 1 . \tag{7.5}$$

The first four relations are obtained by the following observations:

(7.1): the Z_0 patriarchs generate Z_0 independent monopatriarchal branching processes without immigration;
(7.2): split the process into the process of descendants of the patriarchs and the independent process of descendants of immigrants;
(7.3): $Z_0 = 1$ and $\theta\underline{Z}$ is a branching process with patriarchate Z_1 (with pgf π) without immigration; apply (7.1);
(7.4): $Z_0 = 0$ and $\theta\underline{Z}$ is a branching process with patriarchate Z_1 (with pgf β) and immigration generating function β; apply (7.2).

We see that $\zeta_{\iota,1}$ is determined on S by (7.3) and (7.5), as $\zeta_{\iota,1}(\theta^n\underline{s}) = 1$ for all sufficiently large n. After this, $\zeta_{\alpha,1}$ is determined by (7.1), $\zeta_{1,\beta}$ by (7.4) and (7.5), and finally $\zeta_{\alpha,\beta}$ by (7.2).

8. Conditioning on extinction

We concentrate on $\zeta := \zeta_{\iota,1}$, being determined by the shift equation

$$\zeta(\underline{s}) = s_0\pi(\zeta(\theta\underline{s})) \tag{8.1}$$

plus boundary condition $\zeta(\underline{1}) = 1$. Other boundary conditions $\zeta(\underline{1}) = x$ combine with (8.1) iff $x = \pi(x)$. It is well-known that other solutions than $x = 1$ exist in [0,1] iff $\pi = \iota$ or the average progeny $\pi'(1) > 1$, and that the smallest solution (of at most two in case $\pi \neq \iota$) is the probability of <u>extinction</u>, the event $\dagger := \bigcup_{n=1}^{\infty}[Z_n=0]$. Define the defective pgf $\zeta^\dagger$ as

$$\zeta^\dagger(\underline{s}) := E1_\dagger\underline{s}^{\underline{Z}} .$$

It is not hard to see that also $\zeta^\dagger$ satisfies (8.1), with boundary condition $\zeta^\dagger(\underline{1}) = P\dagger$. Furthermore, if $P\dagger > 0$ then

$$\frac{\zeta^\dagger(\underline{s})}{P\dagger} = E(\underline{s}^{\underline{Z}}|\ \dagger)$$

is the (nondefective) pgf of the branching process conditioned on extinction.

One easily verifies from $\zeta^\dagger$ satisfying (8.1) that

$$\frac{\zeta^\dagger(\underline{s})}{P\dagger} = s_0\pi^\dagger\left(\frac{\zeta^\dagger(\theta\underline{s})}{P\dagger}\right)$$

with

$$\pi^\dagger(s) := \frac{\pi(sP\dagger)}{P\dagger} . \tag{8.2}$$

So $\zeta^\dagger/P\dagger$ satisfies (8.1) with π replaced by $\pi^\dagger$. We have obtained the following lemma in the monopatriarchal case, which easily extends to processes with more general Z_0.

Lemma

The branching process without immigration with progeny generating function π conditioned on extinction is a branching process without immigration with progeny generating function $\pi^\dagger$ given by (8.2), provided that $P\dagger > 0$.

Example. Consider the pgf of the geometric distribution

$$\gamma_p(s) := \frac{q}{1 - ps} \text{ for } s \in [0,1) , \tag{8.3}$$

and define $\gamma_p(1) := \lim_{s\uparrow 1}\gamma_p(s)$. With $\pi = \gamma_p$ the above specializes to $P\dagger = 1 \wedge \frac{q}{p}$ and $\gamma_p^\dagger = \gamma_{p\wedge q}$. To limit the complexity of our notations we often denote $\gamma_{p\wedge q}$ by γ.

9. Decomposition of the random walk

In the last four sections we discussed only defective pgf's of the local times. In fact, we have only calculated the pgf's of the local times in the first excursion. To extend the calculations to the whole time domain we also have to consider the other excursions, if any.

Define $T_0^{(0)} := 0$ and, for $k > 0$, $T_0^{(k)} := \inf\{n \geq T_0^{(k-1)}+1 \colon S_n = 0\}$. For $k > 0$ $T_0^{(k)}$ denotes the waiting time for the kth return to zero. For $i \in \mathbf{N}$, the ith <u>excursion</u> E_i is defined by $E_i := (S_{T_0^{(i-1)}+j})_{j=1}^{T_0^{(i)}-T_0^{(i-1)}}$ if $T_0^{(i-1)} < \infty$, $E_i := \emptyset$ else. Nonempty excursions E_i we will call <u>positive</u> or <u>negative</u>, depending on $S_{T_0^{(i-1)}+1}$ being 1 or -1. Depending on the value of $T_0^{(i)}$ these excursions are said to be <u>finite</u> or <u>infinite</u>. From property (a) we know that these excursions are independent, and that the finite excursions are even iid. The random walk $(S_n)_{n=0}^\infty$ thus can be decomposed into a sequence $(E_i)_{i=1}^\infty$ of independent excursions.

This decomposition allows us to calculate the pgf's of local times up to infinity. From property (b) we have $\underline{L}^*(\infty) = \underline{\infty}$ if $p = \frac{1}{2}$, so let us define for $p \neq \frac{1}{2}$

$$\psi_\infty^*(\underline{s}) := E\underline{s}^{\underline{L}^*(\infty)} . \tag{9.1}$$

By splitting ψ_∞^* according to $T_0 = \infty$ or $T_0 < \infty$ we find

$$\psi_\infty^*(\underline{s}) = \chi_0^*(\underline{s}) + \phi_0^*(\underline{s})\psi_\infty^*(\underline{s}) ,$$

so

$$\overset{*}{\psi}_{\infty}(\underline{s}) = \frac{\overset{*}{\chi}_0(\underline{s})}{1 - \overset{*}{\phi}_0(\underline{s})} \quad . \tag{9.2}$$

10. Ascents, descents and branching processes

In this and the next section we will compare local times with branching processes by means of the relations that exist between the respective pgf's. To avoid trivialities we will asume that $p\wedge q \neq 0$.

Consider shift equations (4.3) and (8.1). From (4.3) we have

$$\frac{\phi^{\downarrow}_{0+}(\underline{s})}{p\wedge q} = s_0 \cdot \frac{p\vee q}{1 - (p\wedge q) \cdot \dfrac{\phi^{\downarrow}_{0+}(\theta \underline{s})}{p\wedge q}} \quad .$$

By substituting Υ for π in (8.1) we see that ${}_{\Upsilon}\zeta_{\iota,1}$ and $\phi^{\downarrow}_{0+}/(p\wedge q)$ satisfy the same shift equation. Furthermore, both ${}_{\Upsilon}\zeta_{\iota,1}(\underline{1})$ and $\phi^{\downarrow}_{0+}(\underline{1})/(p\wedge q)$ are equal to 1. So we have

$$\frac{\phi^{\downarrow}_{0+}(\underline{s})}{p\wedge q} = {}_{\Upsilon}\zeta_{\iota,1}(\underline{s}) \quad . \tag{10.1}$$

Note that $\phi^{\downarrow}_{0+}(\underline{s})/(p\wedge q) = \phi^{\downarrow}_{0+}(\underline{s})/P[T_0<\infty,\ S_1>0] = E(\underline{s}^{\underline{L}^{\downarrow}(T_0)} | T_0<\infty,\ S_1>0)$. We have obtained the following theorem.

Theorem 1

The conditional joint distribution of descending local time $\underline{L}^{\downarrow}(T_0)$, given that the first excursion is positive and finite, is equal to the joint distribution of the monopatriarchal branching process without immigration with progeny generating function Υ.

With 'positive' replaced by 'negative', this theorem also holds for ascending local time.

Descending and ascending local time in an infinite excursion can also be described in terms of a branching process. For this purpose, consider ${}_{\Upsilon}\zeta_{1,\Upsilon}$.

Using successively equations (7.4), (7.1), (10.1) and the definition of Υ, we find

$$\begin{aligned}
{}_{\Upsilon}\zeta_{1,\Upsilon}(\underline{s}) &= {}_{\Upsilon}\zeta_{\Upsilon,1}(\theta\underline{s}) \cdot {}_{\Upsilon}\zeta_{1,\Upsilon}(\theta\underline{s}) \\
&= \Upsilon({}_{\Upsilon}\zeta_{\iota,1}(\theta\underline{s})) \cdot {}_{\Upsilon}\zeta_{1,\Upsilon}(\theta\underline{s}) \\
&= \Upsilon\left(\frac{\phi^{\downarrow}_{0+}(\theta\underline{s})}{p\wedge q}\right) \cdot {}_{\Upsilon}\zeta_{1,\Upsilon}(\theta\underline{s})
\end{aligned}$$

$$= \frac{pvq}{1 - \phi^{\downarrow}_{0+}(\theta\underline{s})} \cdot {}_{\Upsilon}\zeta_{1,\Upsilon}(\theta\underline{s}) \; .$$

Before comparing this with the shift equation for $\chi^{\downarrow}_{0+}$, we note that $\chi^{\downarrow}_{0+} = 0$ if $p \le \frac{1}{2}$. So we asume $p > \frac{1}{2}$. As $\chi^{\downarrow}_{0+}(\underline{1}) = p\chi^{\downarrow}_{-1}(\underline{1}) = p-q$ and ${}_{\Upsilon}\zeta_{1,\Upsilon}(\underline{1}) = 1$, consider the shift equation for $\chi^{\downarrow}_{0+}(\underline{s})/(p-q)$

$$\frac{\chi^{\downarrow}_{0+}(\underline{s})}{p-q} = \frac{p.\dfrac{\chi^{\downarrow}_{0+}(\theta\underline{s})}{p-q}}{1 - \phi^{\downarrow}_{0+}(\theta\underline{s})} \; ,$$

which follows easily from equations (6.8) and (6.10). So

$$\frac{\chi^{\downarrow}_{0+}(\underline{s})}{p-q} = {}_{\Upsilon}\zeta_{1,\Upsilon}(\underline{s}) \; . \tag{10.2}$$

Note that $\chi^{\downarrow}_{0+}(\underline{s})/(p-q) = \chi^{\downarrow}_{0+}(\underline{s})/P[T_0=\infty,\ S_1>0] = E(\underline{s}^{\underline{L}^{\downarrow}(\infty)}|T_0=\infty,\ S_1>0)$. We have obtained the following theorem.

Theorem 2

If $p > \frac{1}{2}$, then the conditional joint distribution of descending local time $\underline{L}^{\downarrow}(T_0)$, given that the first excursion is positive and infinite, is equal to the joint distribution of the branching process with empty patriarchate, progeny generating function Υ and immigration generating function Υ.

With 'positive' replaced by 'negative', and '$p > \frac{1}{2}$' by '$p < \frac{1}{2}$', Theorem 2 also holds for ascending local time.

Theorems 1 and 2 generalize similar results found by Dwass (1975).

11. Local time as a branching process

Also local time can be described as a branching process. In the case of symmetric Bernoulli random walk ($p = \frac{1}{2}$), this has already been done by Rogers (1984). Because of the simple relationship (2.1) between descending local time and local time, we can extend Rogers's theorem to the general case $p \in (0,1)$.

Combination of (2.1) and Theorem 1 gives us

Corollary 1

The conditional distribution of the sequence $(L(T_0,k))_{k=0}^{\infty}$, given that the first excursion is positive and finite, is equal to the distribution of the sequence $(Z_k + Z_{k-1})_{k=0}^{\infty}$, where $(Z_n)_{n=0}^{\infty}$ is the monopatriarchal branching process without immigration with progeny generating function γ.

This corollary also holds for $L((T_0,-k))_{k=0}^{\infty}$ instead of $((T_0,k))_{k=0}^{\infty}$, if 'positive' is replaced by 'negative'.

From (2.1) and Theorem 2 we find

Corollary 2
If $p > \frac{1}{2}$, then the conditional distribution of the sequence $((T_0,k))_{k=1}^{\infty}$, given that the first excursion is positive and infinite, is equal to the distribution of the sequence $((1 + Z_k + Z_{k-1}))_{k=1}^{\infty}$, where $(Z_n)_{n=0}^{\infty}$ is the branching process with empty patriarchate, progeny generating function γ and immigration generating function γ.

Corollary 2 also holds for $(L(T_0,-k))_{k=1}^{\infty}$ instead of $(L(T_0,k))_{k=1}^{\infty}$, if 'positive' is replaced by 'negative', and '$p > \frac{1}{2}$' by '$p < \frac{1}{2}$'.

12. Another derivation via martingales

Inspired by the proof of Rogers (1984) for $p = \frac{1}{2}$ we rederive some of the results of Sections 10 and 11 by martingale arguments. Let F_n be the σ-field generated by $X_1, X_2, \ldots, X_n$, and let $f: \mathbf{Z} \to \mathbf{R}$ be a function with $f(0) = 1$, to be determined further below. Define for $\underline{s} \in S$

$$M_n := f(S_{n\wedge T_0})\underline{s}^{\underline{L}^{\downarrow}(n\wedge T_0)} 1_{[S_1>0]} .$$

Then we have

$$M_1 = f(1)1_{[S_1>0]}$$

and

$$E(M_{n+1}|F_n) = \frac{pf(S_n+1) + qf(S_n-1)s_{S_n-1}}{f(S_n)} . M_n .$$

So, if we pick f such that

$$pf(k+1) + qf(k-1)s_{k-1} = f(k) \text{ for } k \in \mathbf{N} , \tag{12.1}$$

then $(M_n)_{n=1}^{\infty}$ (not $(M_n)_{n=0}^{\infty}$) is a martingale relative to $(F_n)_{n=1}^{\infty}$. Note that all functions f satisfying the previous conditions form a one-dimensional affine space. The extra condition $f(k) = f(k+1)$ for $k \geq k_0$ (with k_0 satisfying $s_k = 1$ for $k \geq k_0$) determines f completely, and makes

$(M_n)_{n=1}^{\infty}$ a bounded martingale. In this case, $M_\infty := \lim_{n\to\infty} M_n$ exists wp1, and we have

$$E(M_\infty|F_1) = M_1 .$$

As $M_\infty = f(0)\underline{s}^{\underline{L}^{\downarrow}(T_0)} 1_{[S_1>0,\ T_0<\infty]} + f(k_0)\underline{s}^{\underline{L}^{\downarrow}(\infty)} 1_{[S_1>0,\ T_0=\infty]}$, the previous identity is equivalent to

$$1_{[S_1>0]}\left(E\underline{s}^{\underline{L}^{\downarrow}(T_0)} 1_{[T_0<\infty]} + f(k_0)Es^{\underline{L}^{\downarrow}(\infty)} 1_{[T_0=\infty]}\right) = 1_{[S_1>0]}f(1) .$$

This equation is not particularly tractable in case there are two nonzero terms on the left-hand side, but in case $p \le \frac{1}{2}$ (so $T_0 < \infty$ wp1) it simpifies to

$$E1_{[S_1>0]}\underline{s}^{\underline{L}^{\downarrow}(T_0)} = 1_{[S_1>0]}f(1) . \tag{12.2}$$

In order to calculate $f(1)$ we define $\rho_k := f(k+1)/f(k)$, so that $\rho_k = 1$ for $k \ge k_0$ and $\rho_0 = f(1)$. We now can rewrite the recursive relation (12.1) as

$$p\rho_k\rho_{k-1} + qs_{k-1} = \rho_{k-1} ,$$

or equivalently,

$$\rho_{k-1} = s_{k-1}\gamma(\rho_k) . \tag{12.3}$$

Iterating (12.3) we find

$$\begin{aligned} f(1) &= \rho_0 \\ &= s_0\gamma(s_1\gamma(\ldots s_{k_0-2}\gamma(s_{k_0-1})\ldots)) \\ &= {}_{\gamma}\zeta_{\iota,1}(\underline{s}) . \end{aligned} \tag{12.4}$$

Combining (12.2) and (12.4) we reobtain Theorem 1 and Corollary 1, this time restricted to the special case $p \le \frac{1}{2}$.

A similar martingale argument proves the version of Theorem 1 and Corollary 1 for ascending local time in a negative first excursion, in case $p \ge \frac{1}{2}$.

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