NERSITY
KE
UNIVERSITEIT
BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

PARALLEL NORMREDUCING TRANSFORMATIONS FOR THE ALGEBRAIC EIGENVALUE PROBLEM

M.H.C. Paardekooper

FEW 423

PARALLEL NORMREDUCING TRANSFORMATIONS FOR THE ALGEBRAIC EIGENVALUE PROBLEM

M.H.C. Paardekooper

Department of Econometrics
Tilburg University
5000 LE Tilburg

This research is part of the VF-program "Parallelle Algoritmiek", TD-WI-08185-25, which has been approved by the Netherlands Ministery of Education and Sciences.

PARALLEL NORMREDUCING TRANSFORMATIONS FOR THE ALGEBRAIC EIGENVALUE PROBLEM

ABSTRACT

This article presents a unified approach for parallel normreducing methods for the algebraic eigenproblem. The so-called Euclidean parameters presents the problem, to minimize the Frobenius norm of the transform matrix, in a simple form. The use of appropriate preprocessing unitary transforms together with an appropriate pivot strategy leads to convergence to formality.

Keywords: Jacobi methods, parallel transformations, eigenvalues, convergence to normality, Euclidean parameters, normreduction, commutator.

INTRODUCTION

In 1971 Sameh [5] proposed a Jacobi-like eigenvalue algorithm for a parallel computer. Sameh's method is a parallel elaboration of Eberlein's sequentially normreducing transformation procedure [1,4].

In this paper we present a unified approach to Jacobi-like normreducing tranformations and we apply it to elucidate and improve Sameh's method. In a sequentially Jacobi-like procedure for the computation of the eigenvalues $\lambda_1,\lambda_2,\ldots,\lambda_n$ of a real or complex matrix $A = A^{(0)}$ (order n) a sequence $\{A^{(j)}\}$ is constructed recursively:

$$A^{(j+1)} = T^{(j)^{-1}} A^{(j)} T^{(j)} , \quad j \geqq 0. \qquad (1.1)$$

In (1.1) $T^{(j)}$ is a unimodular shear matrix with *Jacobi parameter* $(p_j,q_j,r_j,s_j) \in \mathbb{C}^4$ in (ℓ,m)-*restriction* $\hat{T}_{\ell_j,m_j}$ of $T_{\ell_j,m_j} = T^{(j)}$

$$\hat{T}_{\ell_j,m_j} = \begin{array}{cc} \begin{bmatrix} p_j & q_j \\ r_j & s_j \end{bmatrix} & \begin{array}{l} \longleftarrow \ell_j \\ \longleftarrow m_j \end{array} \\ \begin{array}{cc} \uparrow & \uparrow \\ \ell_j & m_j \end{array} & \end{array} \qquad (1.2)$$

In Eberlein-like normreducing processes [1,3,6] the aim is to construct $\{A^{(j)}\}$ so that

$$\lim_{k\to\infty} \| A^{(j)} \|_F = \sum_{j=1}^{n} |\lambda_j|^2 , \qquad (1.3)$$

that means $\{A^{(j)}\}$ *converges to normality* [2]. Eberlein [1] gives for each iteration an approximation of the optimal normreducing $T_{\ell,m}$. These choices of the Jacobi parameters, together with a well-defined pivot strategy $\{(\ell_j,m_j)\}$ brings about convergence to normality.

Since the Frobenius norm is invariant under unitary transformations, the optimal norm-reducing shear $T_{\ell,m}$ is determined except for a unitary factor. Hence we discuss the normreduction in theoretical terms that are invariant under unitary transformations.

Matrices $S,P \in \mathbb{C}^{n\times n}$ will be called *row-congruent* if and only if $S = PU$ for some unitary U, notation $S \sim P$. It is easy to see that $S \sim P$ if and only if $SS^* = PP^*$. Now for shear $T_{\ell,m}$, with $\hat{T}_{\ell,m} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$

$$\hat{T}_{\ell,m}\hat{T}^*_{\ell,m} \begin{bmatrix} |p|^2 + |q|^2 & p\bar{r} + q\bar{s} \\ \bar{p}r + \bar{q}s & |r|^2 + |s|^2 \end{bmatrix} = \begin{bmatrix} x & z \\ \bar{z} & y \end{bmatrix} \tag{1.3}$$

The quantities

$$x = x(T_{\ell,m}) = |p|^2 + |q|^2, \; y = y(T_{\ell,m)} = |r|^2 + |s|^2 ,$$

$$z = z(T_{\ell,m}) = u + iv = p\bar{r} + q\bar{s} \tag{1.4}$$

will be called the *Euclidean parameters* of $T_{\ell,m}$ [3]. These parameters are pre-eminent appropriate for the formulation of the Frobenius norm of $T^{-1}AT$.

We assume $T_{\ell,m}$ to be unimodular, so

$$x,y > 0 \text{ and } |ps - qr|^2 = xy - |z|^2 = 1 \tag{1.4}$$

With $z = u + iv$

$$\mathcal{H} = \{(x,y,u,v) \in \mathbb{R}^4 | xy-u^2-v^2 = 1, \; x,y > 0\} \tag{1.5}$$

is the *positive sheet* of the elliptic hyperboloid $xy-u^2-v^2 = 1$. In case of real norm-reducing shears, $\hat{T}_{\ell,m} \in \mathbb{R}^{2\times 2}$,

$$x = p^2 + q^2, \; y = r^2 + s^2, \; z = pr + qs, \; xy - z^2 = 1. \tag{1.6}$$

The Euclidean paramters x, y and z of that unimodular shear $T_{\ell,m}$ correspond with the points in the positive sheet of $xy - z^2 = 1$ in $\mathbb{R}^3$.

In the *parallel normreduction* $A^{(j)}$ is transformed by a direct sum of *identical unimodular* 2×2 *matrices*:

$$A^{(j+1)} = W_j^{-1}A^{(j)}W_j\ , \quad j \geqq 0\ , \tag{1.7}$$

where $w_j = \mathrm{diag}(T_{1,j},\ldots,T_{k,j})$, $n = 2k$ and

$$T = T_{i,j} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}\ , \quad i = 1,\ldots,k. \tag{1.8}$$

Then $\|A^{(j+1)}\|_F^2$ is a quadratic function of the Euclidean parameters x, y, u and v ($z = v + iv$) of the k matrices $T_{i,j}$ in W_j. The minimization of that quartic on $\mathcal{H}$ leads to a generalized eigenvalue problem in four dimensions. Section 2 describes the first step $A^{(1)} = W^{-1}AW$ of the normreducing process for real matrices, and there is shown the *commutator* $(c_{i,j}^{(1)}) = C^{(1)} = A^{(1)^T}A^{(1)} - A^{(1)}A^{(1)^T}$ in relation to parallel shear transformations: $c_{2\ell-1,2\ell}^{(1)} = 0$ and $c_{2\ell-1,2\ell-1}^{(1)} = c_{2\ell,2\ell}^{(1)}$, $\ell = 1,\ldots,k$ iff transformation W minimizes $\|A^{(1)}\|_F$. This section gives also the construction of the optimal W. A special analysis is given to the step in which $\inf\{\|W_j^{-1}A^{(j)}W_j\|_F \mid W = T_{1,j}\oplus\ldots\oplus T_{k,j}\}$ is not assumed for unimodular 2×2 shear $T_{i,j}$. Section 3 describes the same problems for complex matrices. In section 4 will be shown that a well-chosen pivot strategy $\{(\ell_{i,j},m_{i,j}) \mid i = 1,\ldots,k\}_{j=0}^{\infty}$ together with an appropriate preprocessing sequence of unitary matrices $U^{(j)}$ results in sequence $\{A^{(j)}\}$ that converges to normality.

2. PARALLEL NORMREDUCTION: REAL MATRICES

Let the matrix S be real and of even order n = 2k. Then it can be partitioned as follows

$$A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1k} \\ A_{21} & A_{22} & \cdots & A_{2k} \\ \vdots & & & \\ A_{k1} & A_{k2} & \cdots & A_{kk} \end{bmatrix} \tag{2.1}$$

where each submatrix is given by

$$A_{\ell,m} = \begin{bmatrix} a_{2\ell-1,2m-1} & a_{2\ell-1,2m} \\ a_{2\ell,2m-1} & a_{2\ell,2m} \end{bmatrix}, \quad \ell,m = 1,\dots,k. \tag{2.2}$$

For convenience define

$$\begin{gathered} \sigma_{\ell,m} = a_{2\ell,2m-1},\ \mu_{\ell,m} = a_{2\ell-1,2m},\ \alpha_{\ell,m} = a_{2\ell-1,2m-1}, \\ \beta_{\ell,m} = a_{2\ell,2m},\ \nu_{\ell,m} = \alpha_{\ell,m} - \beta_{\ell,m}, \quad \ell,m = 1,\dots,k. \end{gathered} \tag{2.3}$$

Let

$$A' = W^{-1}AW, \tag{2.4}$$

where

$$W = \mathrm{diag}(S_1,S_2,\dots,S_k)$$

with

$$S_j = \begin{bmatrix} p_j & q_j \\ r_j & s_j \end{bmatrix}, \ p_js_j - q_jr_j = 1, \quad j = 1,\dots,k.$$

We introduce

$$x_j = p_j^2 + q_j^2,\ y_j = x_j^2 + s_j^2,\ z_j = p_j r_j + q_j s_j\ ,\qquad j = 1,\dots,k;$$

they are the Euclidean parameters of the unimodular S_j. So

$$x_j y_j - z_j^2 = 1\ ,\qquad i = 1,\dots,k.$$

Then

$$A'_{\ell,m} = S_\ell^{-1} A_{\ell,m} S_m\ .$$

Easy calculations give the following results.

THEOREM 2.1. For each (ℓ,m), $\ell,m = 1,\dots,k$, $\|A'_{\ell,m}\|_F^2$ is a bilinear function of the Euclidean parameters (x_ℓ,y_ℓ,z_ℓ) of S_ℓ and (x_m,y_m,z_m) of S_m:

$$\|A'_{\ell,m}\|_F^2 = (x_\ell,y_\ell,z_\ell)\begin{bmatrix} \sigma_{\ell,m}^2 & \beta_{\ell,m}^2 & 2\sigma_{\ell,m}\beta_{\ell,m} \\ \alpha_{\ell,m}^2 & \mu_{\ell,m}^2 & 2\alpha_{\ell,m}\mu_{\ell,m} \\ -2\alpha_{\ell,m}\sigma_{\ell,m} & -2\beta_{\ell,m}\mu_{\ell,m} & -2(\alpha_{\ell,m}\beta_{\ell,m}+\sigma_{\ell,m}\mu_{\ell,m}) \end{bmatrix}\begin{bmatrix} x_m \\ y_m \\ z_m \end{bmatrix}. \ \square$$

THEOREM 2.2. For each $j = 1,\dots,k$ let be

$$w_j = (x_j-y_j)/2,\ t_j = (x_j+y_j)/2 = (1+w_j^2+z_j^2)^{\frac{1}{2}}. \tag{2.5}$$

Then $\|W^{-1}AW\|_F^2$ is a quadratic function $(\mathbf{w},\mathbf{z}) \longmapsto g(\mathbf{w},\mathbf{z};A)$ where $\mathbf{w} = (w_1,\dots,w_k)^\top$ and $\mathbf{z} = (z_1,\dots,z_k)^\top$. Moreover

$$\frac{\partial g}{\partial w_\ell}(0,0;A) = c_{2\ell-1,2\ell+1} - c_{2\ell,2\ell},\ \frac{\partial g}{\partial z_\ell}(0,0;A) = c_{2\ell-1,2\ell}\ ,$$

where $(c_{i,j}) = A^\top A - AA^\top$. $\square$

The complexity of the unconstrained minimization of g forces the restriction to a problem with fewer degrees of freedom. Therefore, we consider, as in [5]

$$W = \mathrm{diag}(S_1,\dots,S_k) \tag{2.6}$$

with

$$S_j = S = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \; ps - qr = 1 \;, \quad j = 1,\dots,k. \tag{2.7}$$

Such a matrix W will be called a *diagonal of shears*. Now

$$x = p^2 + q^2, \; y = r^2 + s^2, \; z = pr - qs. \tag{2.8}$$

The unimodularity of S implies

$$xy - z^2 = 1. \tag{2.9}$$

THEOREM 2.3. If W is a diagonal of shears with Euclidean parameters (x,y,z) then

$$\|W^{-1}AW\|_F^2 = \kappa + \sum_{\ell,m=1}^{k} (-\sigma_{\ell,m}x + \mu_{\ell,m}y + \nu_{\ell,m}z)^2 \;, \tag{2.10}$$

where $\kappa = \sum_{\ell,m=1}^{k} (\mathrm{tr}(A_{\ell,m})^2 - 2\det(A_{\ell,m}))$. □

As in (2.5) we define

$$w := (x-y)/2, \; t := (x+y)/2 = (1+w^2+z^2)^{\frac{1}{2}}.$$

Then, as follows from (2.10), $\|W^{-1}AW\|_F^2$ is a function of w and z:

$$g(w,z;A) := \|W^{-1}AW\|_F^2. \tag{2.11}$$

With simple but cumbersome calculations one proves the following lemma.

LEMMA 2.4. Let be $(c'_{i,j}) = (A')^{\mathsf{T}}A' - A'(A')^{\mathsf{T}}$, where $A' = W^{-1}AW$ with W as defined in (2.6) and (2.7). Then

$$\sum_{\ell=1}^{k}\begin{bmatrix} c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ \\ 2c'_{2\ell-1,2\ell-1} \end{bmatrix} = \begin{bmatrix} (p^2+s^2-q^2-r^2)/2 & pr-qs \\ \\ pq - rs & ps+qr \end{bmatrix}\begin{bmatrix} \frac{\partial g}{\partial w}\ (w,z;A) \\ \\ \frac{\partial g}{\partial z}\ (w,z;A) \end{bmatrix}, \tag{2.12}$$

where g as defined in (2.11). □

THEOREM 2.5. The function g is stationary in $(w,z) \in \mathbf{R}^2$ iff

$$\sum_{\ell=1}^{k}(c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = 0 \text{ and } \sum_{\ell=1}^{k}(c'_{2\ell-1,2\ell}) = 0.$$

PROOF. The determinant of the coefficientmatrix in (2.11) equals $\frac{1}{2}(x+y)(ps-qr) \neq 0$. □

Theorem 2.3 implies that the determination of the optimale normreducing diagonal of shears requires the minimization of a quadratic function subject to $xy - z^2 = 1$. Let be

$$\mathbf{d} = (d_1,d_2,d_3)^T = (x,y,z),$$

$$H := \begin{bmatrix} 0 & \frac{1}{2} & 0 \\ \frac{1}{2} & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}. \tag{2.13}$$

and

$$\mathcal{H} := \{\mathbf{d} \in \mathbf{R}^3 | \mathbf{d}^T H\mathbf{d} = 1,\ d_1 > 0\ \}. \tag{2.14}$$

Further

$$B = (\mathbf{b}_1,\mathbf{b}_2,\mathbf{b}_3) \in \mathbf{R}^{k^2\times 3}$$

with

$$\mathbf{b}_1 = (\sigma_{1,1},\sigma_{1,2},\dots,\sigma_{k,k})^\top \in \mathbb{R}^{k^2}$$

$$\mathbf{b}_2 = (\mu_{1,1},\mu_{1,2},\dots,\mu_{k,k})^\top \in \mathbb{R}^{k^2}$$

$$\mathbf{b}_3 = (\nu_{1,1},\nu_{1,2},\dots,\nu_{k,k})^\top \in \mathbb{R}^{k^2}$$

The Euclidean parameters $\mathbf{d} = (x,y,z)$ of an optimal normreducing diagonal of shears solve the problem

$$\min\{\|B\mathbf{d}\| \,|\, \mathbf{d}^\top H\mathbf{d} = 1\}. \tag{2.15}$$

(2.15) leads to a gerenalized eigenproblem in three dimensions. The analysis for the three cases rank(B) equals 3, 2 or 1 is summarized in the following theorems.

THEOREM 2.6. Let be rank(B) = 3, and $B = QR$ with $Q \in \mathbb{R}^{k^2\times 3}$ orthogonal and $R \in \mathbb{R}^{3\times 3}$ uppertriangular. Then $\|B\mathbf{d}\|$ assumes its minimum on $\mathcal{H}$ in $\tilde{\mathbf{d}}$, where $R\tilde{\mathbf{d}}$ is an eigenvector corresponding with the unique positive eigenvector of $R^{-1}HR$.

PROOF. The existence of a minimum follows from compactness and continuity arguments. $(B^\top B-\lambda H)\mathbf{d} = \mathbf{0}$ implies $(R^{-\top}HR^{-1}-\rho^{-1}I)R\mathbf{d} = \mathbf{0}$. The eigenvalue ρ^{-1} corresponding with the minimum is positive on base of convexity arguments. The positive eigenvalue of $R^{-\top}HR^{-1}$ is unique. □

THEOREM 2.7. Let be rank(B) = 1 and $\text{range}(B^\top) = \text{span}(-\sigma,\mu,\nu)^\top$. Then

(i) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0,\ \nu^2 + 4\sigma\mu > 0$ or $\nu = \sigma = \mu = 0$;

(ii) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = |\nu^2+4\sigma\mu|,\ \nu^2 + 4\sigma\mu < 0$;

(iii) $\inf\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0,\ \nu^2 + 4\sigma\mu = 0 \wedge |\sigma| + |\mu| \neq 0,$

in this case the infimum is not assumed.

PROOF. In A each block $A_{\ell,m}$ gives a similar contribution to $B\mathbf{d}$; hence $B\mathbf{d}$ can be considered to come from one block, say $C = \begin{bmatrix} \alpha & \mu \\ \sigma & \beta \end{bmatrix}$.

(i) C is diagonalizable with a real shear. The equation $-\sigma d_1 + \mu d_2 + \nu d_3 = 0$ determines a solution curve Γ in $\mathcal{H}$. The parametric form of Γ is

$$\mathbf{p}(\tau) = (\tau, \tau^{-1}, 0)^{\mathsf{T}} \qquad , \sigma = \mu = 0 \qquad (p_1 > 0)$$

$$\mathbf{p}(\tau) = (\nu\tau/\sigma,\ \sigma(1+\tau^2)(\nu\tau)^{-1},\ \tau)^{\mathsf{T}} \quad , \sigma\nu \neq 0,\ \mu = 0 \qquad (p_1 > 0)$$

$$\mathbf{p}(\tau) = (-\mu(1+\tau^2)(\nu\tau)^{-1}\ , -\nu\tau/\mu,\ \tau)^{\mathsf{T}} \ , \mu\nu \neq 0,\ \lambda = 0 \qquad (p_1 > 0)$$

$$\mathbf{p}(\tau) = (\frac{\nu\tau \pm D}{2\sigma}, \frac{-\nu\tau \mp D}{2\mu}, \tau)^{\mathsf{T}} \qquad , D^2 = (\nu^2+4\sigma\mu)\tau^2 + 4\sigma\mu > 0,\ \lambda\mu \neq 0.$$

(ii) Transform C in a Murnaghan form. The positive minimum is assumed for

$$\mathbf{d} = -|4\sigma\mu + \nu^2|^{\frac{1}{2}}\sigma/|\sigma|(-2\sigma, 2\mu, -\nu)^{\mathsf{T}}.$$

(iii) C is not diagonalizable. $B^{\mathsf{T}}B$ has no eigenvector in $\mathcal{H}$. The plane $-\sigma d_1 + \mu d_2 + \nu d_3 = 0$ contacts the recession cone $\{\mathbf{d} | d_1 d_2 - d_3^2 = 0,\ d_1 > 0\}$ of $\mathcal{H}$ along the line $\mathcal{L} : \mathbf{p}(\tau) = \tau(2\mu, -2\sigma, -\nu)^{\mathsf{T}}$. Hence $\|B\mathbf{d}\| > 0$ on $\mathcal{H}$, for $\mathcal{L} \cap \mathcal{H} = \emptyset$. Now we describe a curve Γ on $\mathcal{H}$ such that $\mathcal{L}$ is its asymptote:

$$\Gamma : \mathbf{d}(\tau) = \tfrac{1}{2}\tau(\mu-\sigma)^{-1}(2\mu, -2\sigma, -\nu) + \tfrac{1}{2}\tau^{-1}(\mu-\sigma)^{-1}(-2\sigma, 2\mu, \nu),\ \tau > 0. \tag{2.16}$$

On Γ we find, using the fact that $\nu^2 + 4\sigma\mu = 0$

$$-\sigma d_1(\tau) + \mu d_2(\tau) + \nu d_3(\tau) = (\mu-\sigma)/\tau \to 0,\ \tau \to \infty.$$

The infimum zero of $\|B\mathbf{d}\|$ on $\mathcal{H}$ is not assumed. □

EXAMPLE 2.1.

$$A = \begin{bmatrix} -1 & 1 & -2 & 2 \\ -1 & 1 & -2 & 2 \\ -3 & 3 & -4 & 4 \\ -3 & 3 & -4 & 4 \end{bmatrix}$$

Then

$$B^{\top} = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \\ -2 & -4 & -6 & -8 \end{bmatrix} \text{ and } H^{-1}B^{\top}B = 60 \begin{bmatrix} 1 & 1 & -2 \\ 1 & 1 & -2 \\ 1 & 1 & -2 \end{bmatrix}$$

Remark that $\mathrm{tr}(H^{-1}B^{\top}B) = -(4\sigma\mu+\nu^2) = 0$.
The threefold eigenvalue 0 of nondiagonalizable $H^{-1}B^{\top}B$ gives eigenvectors $(-\gamma_1+2\gamma_2,\gamma_1,\gamma_2)^{\top}$, none in $\mathcal{H}$, for $(-\gamma_1+2\gamma_2)\gamma_1 - \gamma_2^2 = -(\gamma_1-\gamma_2)^2 \leqq 0$. With (2.16) we get, since $\sigma = -1$, $\mu = 1$, $\nu = -2$.

$$\Gamma : \mathbf{d}(\tau) = \tfrac{1}{2}(\tau+\tau^{-1},\ \tau+\tau^{-1},\ t-\tau^{-1}) \in \mathcal{H},\ \tau > 0.$$

Along Γ holds $\|B\mathbf{d}(\tau)\| \to 0$ $(,\tau \to 0)$. □

In a similar way one derives the next theorem for the case $\mathrm{rank}(B) = 2$.

THEOREM 2.8. Let be $\mathrm{rank}(B) = 2$ and $\mathrm{nullity}(B) = \mathrm{span}(t_1,t_2,t_3)^{\top}$. Then

(i) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0$, $t_1t_2 > t_3^2$;

(ii) $\min\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} > 0$, $t_1t_2 < t_3^2$;

(iii) $\inf\{\|B\mathbf{d}\| \,|\, \mathbf{d} \in \mathcal{H}\} = 0$, $t_1t_2 = t_3^2$. This infimum is not assumed for finite $\mathbf{d} \in \mathcal{H}$. □

REMARK. In case (iii) $H^{-1}B^{\top}B$ is not diagonalizable. The algebraic multiplicity of eigenvalue zero of $H^{-1}B^{\top}B$ equals two as can be seen from the coefficient of the first grade term in the characteristic polynomial of $H^{-1}B^{T}B$, being

$$(t_3^2-t_1t_2)\left[\left[\sum_{i=1}^{k}\mu_i\nu_i\right]^2 - \sum_{i=1}^{k}\mu_i\sum_{i=1}^{k}\nu_i\right]/t_1^2 = 0. \ \square$$

EXAMPLE 2.2.

$$A = \begin{bmatrix} 2 & -2 & 5 & -4 \\ -1 & 1 & -2 & 3 \\ 4 & -4 & 9 & -8 \\ -3 & 0 & -6 & 1 \end{bmatrix}$$

Then

$$B^{T} = \begin{bmatrix} 1 & 2 & 3 & 6 \\ -2 & -4 & -4 & -8 \\ 1 & 2 & 4 & 8 \end{bmatrix}; \ \mathrm{rank}(B) = 2,\ H^{-1}B^{T}B = \begin{bmatrix} -140 & 200 & -180 \\ 100 & -140 & 130 \\ -65 & 90 & -85 \end{bmatrix}.$$

$\mathcal{N}(B) = \mathrm{span}(4,1,-2)^{T}$. Along curve Γ in $\mathcal{H}$,

$$\Gamma : \mathbf{d}(\tau) = \tfrac{1}{2}\tau(3 + 5\sqrt{1+4\tau^{-2}/25},\ -3+5\sqrt{1+4\tau^{-2}/25},\ -4\tau)^{T},\ \tau > 0$$

holds: $\lim_{\tau\to\infty} \tau^{-1}\mathbf{d}(\tau) = (4,1,-2)^{T}$ and

$$\|B\mathbf{d}(\tau)\|^2 = \frac{32}{5}\,(1 + \sqrt{1+4\tau^{-2}/25})^{-2}\,\tau^{-4} \to 0 \ (\tau \to \infty).$$

$\mathcal{N}(B)$ is the eigenspace of the eigenvalue zero of $H \ \ B^{T}B$. □

3. PARALLEL NORMREDUCTION: COMPLEX MATRICES

Let be A a complex matrix of even order $n = 2k$. The partitioning of A and the notations are as in (2.1) until (2.4). Now we denote

$$x_j = |p_j|^2+|q_j|^2,\ y_j = |r_j|^2+|s_j|^2,\ z_j = p_j\bar{r}_j+q_j\bar{s}_j,\ j = 1,\dots,k\ , \tag{3.1}$$

where p_j, q_j, r_j, s_j are the Jacobi parameters of the complex shear S_j in W. As a consequence of the unimodularity of S_j:

$$x_jy_j - |z_j|^2 = 1\ ,\quad j = 1,\dots,k\ . \tag{3.2}$$

By simple calculations one derives the following theorem.

THEOREM 3.1. Let be $A' = W^{-1}AW$, $W = \mathrm{diag}(S_1,\dots,S_k)$. Then $\|A'\|_F^2$ is a quadratic function of x_j, y_j, z_j and $\bar{z}_j$, namely

$$\|A'\|_F^2 = \sum_{\ell,m=1}^{k} (x_\ell, y_\ell, z_\ell, \bar{z}_\ell)B_{\ell,m}(x_m, y_m, z_m, \bar{z}_m)^T =$$

where

$$B_{\ell,m} = \begin{bmatrix} |\sigma_{\ell,m}|^2 & |\beta_{\ell,m}|^2 & \sigma_{\ell,m}\bar{\beta}_{\ell,m} & \bar{\sigma}_{\ell,m}\beta_{\ell,m} \\ |\alpha_{\ell,m}|^2 & |\mu_{\ell,m}|^2 & \alpha_{\ell,m}\bar{\mu}_{\ell,m} & \bar{\alpha}_{\ell,m}\mu_{\ell,m} \\ -\bar{\alpha}_{\ell,m}\sigma_{\ell,m} & -\bar{\mu}_{\ell,m}\beta_{\ell,m} & -\sigma_{\ell,m}\bar{\mu}_{\ell,m} & -\bar{\alpha}_{\ell,m}\beta_{\ell,m} \\ -\alpha_{\ell,m}\bar{\sigma}_{\ell,m} & -\mu_{\ell,m}\bar{\beta}_{\ell,m} & -\alpha_{\ell,m}\beta_{\ell,m} & -\bar{\sigma}_{\ell,m}\mu_{\ell,m} \end{bmatrix}. \tag{3.3}$$

□

Let be

$$x_j = t_j + w_j,\quad y_j = t_j - w_j,\ j = 1,\dots,k \tag{3.4}$$

Then, as follows from the unimodularity of S_j,

$$t_j = (1 + w_j^2 + z_j\bar{z}_j)^{\frac{1}{2}}.$$

Hence $\|A'\|_F^2$ is a function of the k triples $(w_j, z_j, \bar{z}_j)$, $j = 1,\dots,k$:

$$\|A'\|_F^2 = g(w_1, z_1, \bar{z}_1, \dots, w_k, z_k, \bar{z}_k).$$

The relation between the commutar $C^{(')} = A^{(')*}A^{(')} - A^{(')}A^{(')*}$ and g is mentioned in

THEOREM 3.2. The partial derivatives of g in $\mathbf{0} = (0,0,0,\dots,0,0,0) \in \mathbb{C}^{3k}$ satisfy the following densities

$$\begin{cases} \dfrac{\partial g}{\partial w_j}(0) = c_{2j-1,2j-1} - c_{2j,2j}\,, \\ \dfrac{\partial g}{\partial z_j}(0) = c_{2j,2j-1}\,, \qquad\qquad j = 1,\dots,k \\ \dfrac{\partial g}{\partial \bar{z}_j}(0) = c_{2j-1,2j}\,. \end{cases} \tag{3.5}$$

PROOF. Use the properties

$$\frac{\partial x_j}{\partial w_j} = 1 + w_j/t_j = x_j/t_j,\ \frac{\partial y_j}{\partial w_j} = 1 - w_j/t_j = y_j/t_j,\ \frac{\partial x_j}{\partial z_j} = \bar{z}_j/t_j$$

and

$$c_{2j-1,2j-1} - c_{2j,2j} = \sum_{m=1}^{k} (|\alpha_{m,j}|^2 + |\lambda_{m,j}|^2 - |\mu_{m,j}|^2 - |\beta_{m,j}|^2 +$$

$$|\lambda_{j,m}|^2 + |\beta_{j,m}|^2 - |\alpha_{j,m}|^2 - |\mu_{j,m}|^2),$$

$$c_{2j-1,2j} = \sum_{m=1}^{k} (\bar{\alpha}_{m,j}\mu_{m,j} + \bar{\lambda}_{m,j}\beta_{m,j} - \alpha_{j,m}\bar{\lambda}_{j,m} - \mu_{j,m}\bar{\beta}_{j,m}).$$

With (3.3), and (3.4) and easy but cumbersome calculation one finds (3.5). □

As in the preceding section we restrict ourselves to a *diagonal of shears*:

$$W = \mathrm{diag}(S_1,\dots,S_k) \tag{3.6}$$

with

$$S_j = S = \begin{bmatrix} p & q \\ r & s \end{bmatrix}, \; ps - qr = 1, \; j = 1,\dots,k. \qquad (3.7)$$

Similar to (2.8) we define common Euclidean parameters

$$x = |p|^2 + |q|^2, \; y = |r|^2 + |s|^2, \; z = p\bar{r} + q\bar{s} = u + iv \qquad (3.8)$$

with

$$xy - |z|^2 = 1 \, , \qquad (3.9)$$

where $|z|^2 = u^2 + v^2$. Further

$$\mathcal{H} = \{(x,y,u,v) \,|\, x > 0, \; xy - u^2 - v^2 = 1\} \, .$$

By rather simple calculations $\|W^{-1}AW\|_F^2$ appears to be a quadratic function of x, y, u and v.

THEOREM 3.3. If W is a diagonal of shears with common Euclidean parameters $x,y,z = u + iv$ then $\|W^{-1}AW\|_F^2$ is expressible in terms of these parameters, viz.

$$\|W^{-1}AW\|_F^2 = f(x,y,z) = \sum_{\ell,m=1}^{k} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2 + |\mu_{\ell,m}y + \nu_{\ell,m}z|^2$$

$$+ |\sigma_{\ell,m}x - \nu_{\ell,m}\bar{z}|^2 - |\nu_{\ell,m}z|^2 - 2\,\mathrm{Re}(\bar{\sigma}_{\ell,m}\mu_{\ell,m}z^2) =$$

$$= \sum_{\ell,m=1}^{k} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2) + (x,y,u,v) \sum_{\ell,m=1}^{k} P_{\ell,m}(x,y,u,v)^{\mathsf{T}}, \qquad (3.10)$$

where

$$\begin{bmatrix} |\sigma_{\ell,m}|^2 & 0 & -\mathrm{Re}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) & -\mathrm{Im}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) \\ 0 & |\mu_{\ell,m}|^2 & \mathrm{Re}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) & \mathrm{Im}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) \\ -\mathrm{Re}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) & \mathrm{Re}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) & |\nu_{\ell,m}|^2-2\mathrm{Re}(\sigma_{\ell,m}\bar{\mu}_{\ell,m}) & 2\mathrm{Im}(\sigma_{\ell,m}\bar{\mu}_{\ell,m}) \\ -\mathrm{Im}(\bar{\sigma}_{\ell,m}\nu_{\ell,m}) & \mathrm{Im}(\mu_{\ell,m}\bar{\nu}_{\ell,m}) & 2\mathrm{Im}(\sigma_{\ell,m},\bar{\mu}_{\ell,m}) & |\nu_{\ell,m}|^2+2\mathrm{Re}(\sigma_{\ell,m}\bar{\mu}_{\ell,m}) \end{bmatrix} \quad \square \quad (3.11)$$

Analogously to section 2 we define

$$w = (x-g)/2, \; t = (x+y)/2 = \sqrt{1+u^2+v^2+w^2}\;. \qquad (3.12)$$

Then, as follows from (3.10), $\|W^{-1}AW\|_F^2$ is a function of w, u and v:

$$g(w,u,v;A) := \|W^{-1}AW\|_F^2\;. \qquad (3.13)$$

With simple but cumbersome calculations one proves the following lemma.

LEMMA 3.4. Let be $(c'_{i,j}) = (A')^*A' - A'(A')^*$, where $A' = W^{-1}AW$ with W as defined in (3.6) and (3.7). Then

$$\sum_{\ell=1}^{k} \begin{bmatrix} c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ c'_{2\ell-1,2\ell} \\ c'_{2\ell,2\ell-1} \end{bmatrix} = \tfrac{1}{2} \begin{bmatrix} |p|^2+|s|^2-|q|^2-|r|^2 & \bar{p}r-\bar{q}s & p\bar{r}-q\bar{s} \\ \bar{p}q - \bar{r}s & \bar{p}s & q\bar{r} \\ p\bar{q} - r\bar{s} & \bar{q}r & p\bar{s} \end{bmatrix} \begin{bmatrix} g_w \\ g_u+ig_v \\ g_u-ig_v \end{bmatrix} \qquad (3.14)$$

where g as defined in (3.11). □

THEOREM 3.5. The function g is stationary in $(w,u,v) \in \mathbb{R}^3$ iff

$$\sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = 0 \text{ and } \sum_{\ell=1}^{k} c'_{2\ell-1,2\ell} = 0.$$

PROOF. The determinant of the coefficient matrix in (3.12) equals $\tfrac{1}{2}|ps - qr|^2(|p|^2+|q|^2+|r|^2+|s|^2) = t > 0$. This proofs the theorem. □

Theorem 3.3 implies that the determination of the optimal normreducing diagonal of shears requires the minimization of a quadratic function subject to $xy - u^2 - v^2 = 1$. Neither the function f in (3.10) nor $h(x,y,z) = xy - |z|^2$ are analytic, in contrast with the corresponding real functions (2.10) and (2.9) resp.

Let be $\mathbf{d} = (d_1,d_2,d_{3},_4) = (x,y,u,v)$,

$$H = \begin{bmatrix} 0 & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{3.15}$$

and

$$\mathcal{H} = \{\mathbf{d} \in \mathbf{R}^4 | \mathbf{d}^T H\mathbf{d} = 1,\ d_1 > 0\} \tag{3.16}$$

Further $M = \sum_{\ell,m=1}^{k} P_{\ell,m}$, where $P_{\ell,m} \in \mathbf{R}^{4\times4}$ defined in theorem 3.3 and let be $h(\mathbf{d}) := \mathbf{d}^T M\mathbf{d},\ d \in \mathbf{R}^4$.

With these notations the minimization problem becomes: solve

$$\min\{h(\mathbf{d})|\mathbf{d} \in \mathcal{H}\}\ . \tag{3.17}$$

Let be λ_j, $j = 1,\ldots,n$ the eigenvalues of A. Then

$$\sum_{j=1}^{n} |\lambda_j|^2 \leqq \|W^{-1}AW\|_F^2 = h(\mathbf{d}) + \sum_{\ell,m=1}^{k} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2)$$

for each $\mathbf{d} \in \mathcal{H}$. So h is bounded below on $\mathcal{H}$. Let be

$$\eta = \inf\{h(\mathbf{d})|\mathbf{d} \in \mathcal{H}\}\ . \tag{3.18}$$

THEOREM 3.6. Let be η as defined in (3.18).

(i) If infimum η is assumed in $\mathbf{d} \in \mathcal{H}$ then $M\mathbf{d} = \eta H\mathbf{d}$.

(ii) If $h(\mathbf{x}) > \eta$ for each $\mathbf{x} \in \mathcal{H}$ then there exist a $\mathbf{d} = (d_1,d_2,d_3,d_4)^T$ such that $M\mathbf{d} = \eta H\mathbf{d}$, $\|\mathbf{d}\| = 1$, $d_1 > 0$ and $\mathbf{d}^T H\mathbf{d} = 0$.

PROOF.

(i) The Lagrange multiplier method gives $\mathbf{Md} = \lambda\mathbf{Hd}$ and $\eta = \mathbf{d}^T\mathbf{Md} = \lambda\mathbf{d}^T\mathbf{Hd} = \lambda$. So $\mathbf{Md} = \eta\mathbf{Hd}$.

(ii) Let be $\mathcal{K} = \{\mathbf{d} \in \mathrm{R}^4 | \mathbf{d}^T\mathbf{Hd} \geqq 0\}$ and $S = \{\mathbf{x} \in \mathcal{K} | \|\mathbf{x}\| = 1\}$. Without loss of generality we may assume $\eta = 0$. There exists a sequence $\{\mathbf{x}_h\}$ in $\mathcal{H}$ and a corresponding sequence $\{\hat{\mathbf{x}}_n\}$ in S, with $\hat{\mathbf{x}}_n = \mathbf{x}_n/\|\mathbf{x}_n\|$, such that

$$\mathbf{x}_n^T\mathbf{M}\mathbf{x}_n = \hat{\mathbf{x}}_n^T\mathbf{M}\mathbf{x}_n/\hat{\mathbf{x}}_n^T\mathbf{H}\hat{\mathbf{x}}_n \downarrow \eta \ (,\ n \to \infty) \ . \tag{3.19}$$

Some subsequence $\{\hat{\mathbf{x}}_{n_k}\}$ of $\{\hat{\mathbf{x}}_n\}$ is convergent; let be $\mathbf{d}$ its limit; $\mathbf{d} \in \partial\mathcal{K}$ for otherwise the infimum η would be assumed on $\mathcal{H}$. So $\mathbf{d}^T\mathbf{Hd} = 0$. Hence also $\mathbf{d}^T\mathbf{Md} = 0$. It is clear that $\mathbf{x}^T\mathbf{Mx} \geqq 0$ for each $\mathbf{x} \in \partial\mathcal{K}$. So $h|\partial K$ assumed its minimum in $\mathbf{d}$. Application of Lagranges multiplier method for that minimum gives: there exists a $\lambda \in \mathbf{R}$ such that

$$\mathbf{Md} = \lambda\mathbf{Hd} \ . \tag{3.20}$$

Let be $\mathbf{x} = \mathbf{d} + \mathbf{h} \in S$. Since $\mathbf{d}^T\mathbf{Md} = \mathbf{d}^T\mathbf{Hd}$ we get with (3.20)

$$\frac{\mathbf{x}^T\mathbf{Mx}}{\mathbf{x}^T\mathbf{Hx}} = \frac{2\lambda(\mathbf{Bd})^T\mathbf{h} + \mathbf{h}^T\mathbf{Mh}}{2(\mathbf{Bd})^T\mathbf{h} + \mathbf{h}^T\mathbf{Hh}} \ . \tag{3.21}$$

Now $(\mathbf{Bd})^T\mathbf{h} > 0$ when $\mathbf{h} \in \mathrm{int}(\mathcal{K})$. Hence

$$\mathbf{x}^T\mathbf{Mx}/\mathbf{x}^T\mathbf{Hx} \to \lambda \quad (,\ \mathbf{x} \in S,\ \mathbf{x} \to \mathbf{d}) \ .$$

So $\lambda = \eta$. □

COROLLARY. If $\mathbf{x}^T\mathbf{Mx} > \eta$ for each $x \in \mathcal{H}$ then the intersection of $\mathcal{H}$ and the subspace $\{\tau_1(1,1,0,0)^T + \tau_2\mathbf{d} | \tau_1, \tau_2 > 0\}$ is a curve

$$\Gamma : \mathbf{x}(t) = t(d_1+d_2)^{-1}\mathbf{d} + t^{-1}(d_1+d_2)^{-1}(d_2, d_1, -d_3, -d_4)^T$$

along which $\mathbf{x}(t)^T\mathbf{Mx}(t)$ tends to η for $t \to \infty$. □

EXAMPLE 3.1.

Let be

$$A = \begin{bmatrix} 0 & -i & 1 & 2 \\ -i & 2 & 2 & 1+4i \\ 4+i & 1-3i & 5+i & 2+i \\ 1-3i & 10+3i & 2+i & 3+5i \end{bmatrix}$$

Then

$$M = \begin{bmatrix} 20 & 0 & 0 & 40 \\ 0 & 20 & 0 & 40 \\ 0 & 0 & 40 & 0 \\ 40 & 40 & 0 & 120 \end{bmatrix}, \quad H^{-1}M = 40\begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & -1 & 0 \\ -1 & -1 & 0 & -3 \end{bmatrix}.$$

$\lambda = -40$ is the fourfold eigenvalue of the pair (M,H). The eigenspace is spanned by the three vectors $(-1,1,0,0)^{\top}$, $(-2,0,0,1)^{\top}$ and $(0,0,1,0)^{\top}$. None linear combination of these vectors is in $\mathcal{H}$; $\mathbf{d} = (\sqrt{3})^{-1}(1,1,0,-1)^{\top}$ satisfies the equality $d_1d_2 - d_3^2 - d_4^2 = 0$. The line $\mathbf{x}(\tau) = \tau(1,1,0,-1)^{\top}$ is the asymptote of the curve $\Gamma : \mathbf{x}(\tau) = (2\tau)^{-1}(\tau^2+1,\ \tau^2+1,\ 0,\ \tau^2-1)^{\top}$ in $\mathcal{H}$. Γ is the intersection of the plane $\{\tau_1(1,1,0,0)^{\top} + \tau_2\mathbf{d} \mid \tau_1,\tau_2 > 0\}$ and $\mathcal{H}$. Along Γ holds $h(\mathbf{x}(\tau)) \to \eta = -40$. Remark that $-40 + \sum_{\ell,m=1}^{2} (|\alpha_{\ell,m}|^2 + |\beta_{\ell,m}|^2) = 168$, being the sum of the squares of the moduli of the eigenvalues of the four non-diagonalizable matrices $A_{\ell,m}$ of A. □

4. PRECONDITIONED PARALLEL NORMREDUCTION

This section describes a preconditioning transformation by unitary matrices, and the following suboptimal normreduction. This combination brings about convergence to normality. In the first step that twofold action results in

$$A' = W^{-1}U^*AUW = W^{-1}(U^*AU)W \tag{4.1}$$

where

$$U = \mathrm{diag}(U_1,\ldots,U_k),\ W = \mathrm{diag}(S,\ldots,S) \tag{4.2}$$

with

$$U_\ell = \begin{bmatrix} \cos\varphi_\ell & -e^{-i\vartheta_\ell}\sin\varphi_\ell \\ e^{i\vartheta_\ell}\sin\varphi_\ell & \cos\varphi_\ell \end{bmatrix}, \quad \ell = 1,\ldots,k \tag{4.3}$$

and

$$S = \begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}, \quad ps = 1$$

a 2×2 submatrix in the diagonal of shears W.

As in Sameh's algorithm [5] the *rectifier* U is chosen such that grad $g(0,0,0;W^{-1}AW)$ (see (2.11) and (3.17)) has maximal length. For complex A we use theorem 3.4.

THEOREM 4.1. Let be $U = \mathrm{drag}(U_1,\ldots,U_k)$, with U_ℓ, $\ell = 1,\ldots,k$, as given in (4.3) and let be

$$v_\ell(A) = v_\ell = \begin{bmatrix} c_{2\ell-1,2\ell-1} - c_{2\ell,2\ell} \\ 2c_{2\ell-1,2\ell} \end{bmatrix}, \quad \ell = 1,\ldots,k \tag{4.4}$$

where $(c_{i,j}) = C(A) = A^*A - AA^*$. Then the maximum length of grad $g(0,0,0;U^*AU)$ with respect to U equals $\sum_{\ell=1}^{k} |\mathbf{v}_\ell|$ and is assumed for

$$\begin{bmatrix} \cos 2\varphi_\ell \\ \\ \sin 2\varphi_\ell \end{bmatrix} = \begin{cases} (1,0)^T & , \text{ if } \mathbf{v}_\ell = \underline{0}, \\ & \\ (c_{2\ell-1,2\ell-1} - c_{2\ell,2\ell},\ 2(c_{2\ell-1,2\ell}))|\mathbf{v}_\ell|^{-1}, & \text{if } \mathbf{v}_\ell \neq 0 \end{cases} \tag{4.5}$$

with $|\varphi_\ell| \leqq \frac{\pi}{2}$ and

$$e^{i\vartheta} = \begin{cases} 1 & , \text{ if } c_{2\ell-1,2\ell} = 0 \\ & \\ (c_{2\ell,2\ell-1}/|c_{2\ell-1,2\ell}|, & \text{if } c_{2\ell-1,2\ell} \neq 0 \ . \end{cases} \tag{4.6}$$

Then $g_w(0,0,0;U^*AU) \geqq 0$, $g_u(0,0,0;U^*AU) = g_v(0,0,0;U^*AU) = 0$.

PROOF. It follows from the corollary of theorem 3.4 that grad $g(0,0,0;U^*AU) = \sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell},\ 2\mathrm{Re}(c_{2\ell-1,2\ell}),\ 2\mathrm{Im}(c_{2\ell-1,2\ell}))$ where $(c'_{i,j}) = C(U^*AU)$. Since $C(U^*AU) = U^*C(A)U$ we find

$$\begin{bmatrix} c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ e^{i\vartheta} c'_{2\ell-1,2\ell} \end{bmatrix} = \begin{bmatrix} \cos 2\varphi & \sin 2\varphi & \sin 2\varphi \\ -\frac{1}{2}\sin 2\varphi & \cos^2\varphi & -\sin^2\varphi \end{bmatrix} \begin{bmatrix} c_{2\ell-1,2\ell-1} - c_{2\ell,2\ell} \\ e^{i\vartheta}\ c_{2\ell-1,2\ell} \\ e^{-i\vartheta}\ \bar{c}_{2\ell-1,2\ell} \end{bmatrix}$$

With $\cos\varphi$, $\sin\varphi$ and $e^{i\vartheta}$ as given in (4.6) and (4.5)

$$\begin{bmatrix} c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell} \\ \\ 2c_{2\ell-1,2\ell} \end{bmatrix} = \mathbf{v}_\ell(U^*AU) = \begin{bmatrix} |\mathbf{v}_\ell(A)| \\ \\ 0 \end{bmatrix} . \tag{4.7}$$

With these transformations $U_1,\ldots,U_k$ the k vectors $\mathbf{v}_\ell(U^*AU)$ have the same direction, the vectors $\mathbf{v}_\ell(A)$ are *rectified*. Thus

$$|g(0,0,0;U^*AU)| = |\sum_{\ell=1}^{k} \mathbf{v}_\ell(U^*AU)| = \sum_{\ell=1}^{k} |\mathbf{v}_\ell(U^*AU)| = \sum_{\ell=1}^{k} \mathbf{v}_\ell(A)|.$$

Now $g_u(0,0,0;U^*AU) = g_v(0,0,0;U^*AU) = 0$ and $g_w(0,0,0;U^*AU) \geqq 0$. □

THEOREM 4.2. Let be U a unitary matrix as defined in (4.2), (4.3), (4.5) and (4.6). Then there exists a diagonal matrix W such that

$$\|A\|_F^2 - \|W^{-1}U^*AUW\|_F^2 \geqq \frac{1}{8}\|A\|_F^2 \sum_{\ell=1}^{k} \|v_\ell\|^2 \tag{4.8}$$

where v_ℓ as defined in (4.4).

PROOF. Let be $A' = U^*AU$ and W a diagonal of identical diagonal shears $\begin{bmatrix} p & 0 \\ 0 & s \end{bmatrix}$. According to (3.12)

$$\|W^{-1}A'W\|_F^2 = \sum_{\ell,m=1}^{k} (|\alpha'_{\ell,m}|^2 + |\beta'_{\ell,m}|^2 + |\mu'_{\ell,m}|^2 x^{-2} + |\sigma'_{\ell,m}|^2 x^2) ,$$

for the Euclidean parameter z of a diagonal shear equals zero. Let be $\sum^{k}|\sigma'_{\ell,m}|^2 = c_1$, $\sum_{\ell,m}^{k} |\mu'_{\ell,m}|^2 = c_2$.

1. Let be $c_1c_2 \neq 0$. Then $c_1x^2 + c_2x^{-2}$ is minimal for $x = (c_2/c_1)^{\frac{1}{4}}$. With the Euclidean parameters $((c_2,c_1)^{\frac{1}{4}},(c_1/c_2)^{\frac{1}{4}},0)$ the decrease of the Euclidean norm equals

$$\|A\|_F^2 - \|W^{-1}U^*AUW\|_F^2 = (\sqrt{c}_1 - \sqrt{c}_2)^2 .$$

Since $c_1 - c_2 = \frac{1}{2}\, g_w(0,0,0;A')$

$$\sqrt{c}_1 - \sqrt{c}_2 = (\sqrt{c}_1 + \sqrt{c}_2)^{-1}(c_1 - c_2) = \tfrac{1}{2}(\sqrt{c}_1 + \sqrt{c}_2)^{-1} g_w(0,0,0;A^1) .$$

Now $\sqrt{c}_1 + \sqrt{c}_2 \leqq \sqrt{2}(c_1+c_2)^{\frac{1}{2}} \leqq \sqrt{2}\|A'\|_F = \sqrt{2}\|A\|_F$ and

$$g_w(0,0,0;A') = \sum_{\ell=1}^{k} (c'_{2\ell-1,2\ell-1} - c'_{2\ell,2\ell}) = \sum_{\ell=1}^{k} |v_\ell(U^*AU)| \geqq \sum_{\ell=1}^{k} |v_\ell(A)|$$

as can be seen from (4.7). Consequently

$$\|A\|_f^2 - \|W^{-1}U^*AUW\|_F^2 \geqq \frac{1}{8}\|A\|_F^{-2}\left(\sum_{\ell=1}^{k}\|v_\ell(A)\|\right)^2 \geqq \frac{1}{8}\|A\|_F^{-2}\sum_{\ell=1}^{k}\|v_\ell(A)\|^2.$$

2. In case $c_1 \neq 0$ and $c_2 = 0$, choose x so small that $x^2 \leq 1 - \|A\|_F^{-2}c_1/8$. Then

$$\|A\|_F^2 - \|W^{-1}U^*AUW\|_F^2 = c_1(1-x^2) \geqq \|A\|_F^{-2}c_1^2/8 \geqq \frac{1}{8}\|A\|_F^{-2}\sum_{\ell=1}^{k}\|v_\ell(A)\|^2$$

for $c_1 = \sum_{\ell=1}^{k}(c'_{2\ell-1,2\ell-1}-c'_{2\ell,2\ell}) = \sum_{\ell=1}^{k}\|v_\ell(U^*AU)\| = \sum_{\ell=1}^{k}\|v_\ell(A)\|$.

3. In case $c_2 = 0$ and $c_1 \neq 0$ choose $x^2 \geq (1 - \|A\|_F^{-2}c_2/8)^{-1}$. □

The pivot strategy guaranteeing that the Euclidean norm decreases in sufficient degree for convergence to normality [2,3] will be derived from lower bound (4.8). Therefore we need

THEOREM 4.3. There exists a set of k distinct index pairs (ℓ_j,m_j), with $\ell_j < m_j$, $j = 1,\dots,k$, such that

$$\sum_{j=1}^{k}(c_{\ell_j,\ell_j}-c_{m_j,m_j}) + 4|c_{\ell_j,m_j}|^2) \geqq \frac{4}{n-1}\|C(A)\|_F^2\,. \tag{4.9}$$

PROOF. We have $\sum_{\ell\neq m}(c_{\ell,\ell}-c_{m,m})^2 = 2(n-1)\sum_{\ell=1}^{n}c_{\ell,\ell}^2 - 2\sum_{\ell\neq m}^{m}c_{\ell,\ell}c_{m,m}$. But since $\sum_{\ell=1}^{n}c_{\ell,\ell} = 0$, $(\sum_{\ell=1}^{n}c_{\ell,\ell})^2 = \sum_{\ell=1}^{n}c_{\ell,\ell}^2 + \sum_{\ell\neq m}c_{\ell,\ell}c_{m,m} = 0$. Hence for $n \geqq 2$: $\sum_{\ell\neq m}(c_{\ell,\ell}-c_{m,m})^2 = 2n\sum_{\ell=1}^{n}c_{\ell,\ell}^2 \geqq 4\sum_{\ell=1}^{n}c_{\ell,\ell}^2$. Consequently

$$\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2) \geqq 4\|C(A)\|_E^2 \tag{4.10}$$

Let be Ω the collection of all sets ω of k distinct index pairs (ℓ_j,m_j). The number of sets w in Ω is $n!/(k!2^k)$, and each pair (ℓ,m), with $\ell \neq m$, occurs in $(n-2)!/((k-1)!2^{k-1})$ sets of Ω. Thus

$$\sum_{\omega\in\Omega}\ \sum_{(\ell,m)\in\omega}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2) = \sum_{\ell\neq m}\ \sum_{\substack{\omega\in\Omega\\(\ell,m)\in\omega}}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2)$$

$$= \frac{(n-2)!}{(k-1)!2^{k-1}}\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2) .$$

Hence the mean of $\Sigma_{(\ell,m)\in\omega}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2)$ overall $\omega \in \Omega$ equals

$$\left[\frac{n!}{k!2^k}\right]^{-1}\frac{(n-2)!}{(k-1)!2^{k-1}}\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2 =$$

$$(n-1)^{-1}\sum_{\ell\neq m}((c_{\ell,\ell}-c_{m,m})^2 + 4|c_{\ell,m}|^2) .$$

This result, together with (4.10), proves the theorem. □

THEOREM 4.4. Let a sequence $\{A^{(j)}\}$, starting with $A^{(0)} = A$, be constructed by

$$A^{(j+1)} = (P^{(j)}U^{(j)}W^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)}W^{(j)} \quad , \quad j = 1,2,\ldots \qquad (4.11)$$

where in each step k disjunct index pairs $(\ell_{1,j},m_{1,j}),\ldots,(\ell_{k,j},m_{k,j})$ are selected according to rule (4.9). $P^{(j)}$ is a permutation with $P(\ell_{1,j},m_{1,j},\ldots,\ell_{k,j},m_{k,j}) = (1,2,\ldots,n-1,n)$. $U^{(j)}$ is a preconditioning unitary block diagonal matrix a described in (4.2), (4.3), (4.5) and (4.6) and $\tilde{W}^{(j)} = \mathrm{diag}(x_j^{\frac{1}{2}},x_j^{-\frac{1}{2}},\ldots,x_j^{\frac{1}{2}},x_j^{-\frac{1}{2}})$ that reduces the Frobenius norm of $(P^{(j)}U^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)}$ as described in theorem 4.2. Then $\{A^{(j)}\}$ converges to normality.

PROOF. $\{A_F^{(j)}\}$ decreases monotonically and is bounded below. Therefore $\delta_j := \|A^{(j)}\|_F^2 - \|A^{(j+1)}\|_F^2 \downarrow 0$, $(,j \to \infty)$. Since by theorem 4.2 and theorem 4.3

$$\begin{aligned} \delta_j &\geqq \frac{1}{8}\, \|A\|_F^{-2} \sum_{\ell=1}^{k} \|v_\ell((P^{(j)}U^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)})\|^2 \\ &\geqq \frac{1}{2(n-1)}\, \|A\|_F^{-2}\|C((P^{(j)}U^{(j)}A^{(j)}P^{(j)}U^{(j)}\|_F^2 \\ &= \frac{1}{2(n-1)}\, \|A\|_F^{-2}\|C(A^{(j)})\|_F^2 \,, \end{aligned}$$

we have $C(A^{(j)}) \to 0$ $(,\ j \to \infty)$. □

THEOREM 4.5. Let $\{A^{(j)}\}$ be constructed recursively by

$$A^{(j+1)} = (P^{(j)}U^{(j)}W^{(j)})^{-1}A^{(j)}P^{(j)}U^{(j)}W^{(j)} \,, \qquad j = 1,2,\ldots \tag{4.12}$$

with $P^{(j)}$ and $U^{(j)}$ as in theorem 4.4 but $W^{(j)}$ an optimal parallel norm-reducing shear as described in section 3. Then $\{A^{(j)}\}$ converges to normality.

PROOF. $\delta_j = \|A^{(j)}\|_F^2 - \|A^{(j+1)}\|_F^2 \downarrow 0$ $(,\ j \to \infty)$ for now $W^{(j)}$ is even optimal. As in the proceding theorem $C(A^{(j)}) \to 0$ $(,\ j \to \infty)$. □

REFERENCES

1. P.J. Eberlein (1962). A Jacobi-like Method for the Automatic Computation of Eigenvalues and Eigenvectors of an Arbitrary Matrix, J. Soc. Indust. Appl. Math., 10, pp. 74-88.

2. L. Elsner, M.H.C. Paardekooper (1987). On Measures of Nonnormality of Matrices, Lin. Algebra Appl., 92, 107-124.

3. M.H.C. Paardekooper (1969). An Eigenvalue Algorithm Based on Normreducing Transformations, Technological University Eindhoven, 144 pp.

4. M.H.C. Paardekooper (1987). Sameh's Parallel Eigenvalue Algorithm Revisited, in Algorithm and Applications of Vector and Parallel Computers, (H.J.J. te Riele, T.J. Dekker and H.A. van der Vorst, Eds.), Elseviers Science Publishers, pp. 351-371.

5. A.H. Sameh (1971). On Jacobi and Jacobi-like Algorithms for a Parallel Computer, Math. Comp., 25, pp. 579-590.

6. K. Veselič (1976). A Convergence Jacobi Method for Solving the Eigenproblem of Arbitrary Real Matrices, Num. Math., 25, pp. 178-184.

IN 1989 REEDS VERSCHENEN

368 Ed Nijssen, Will Reijnders
"Macht als strategisch en tactisch marketinginstrument binnen de distributieketen"

369 Raymond Gradus
Optimal dynamic taxation with respect to firms

370 Theo Nijman
The optimal choice of controls and pre-experimental observations

371 Robert P. Gilles, Pieter H.M. Ruys
Relational constraints in coalition formation

372 F.A. van der Duyn Schouten, S.G. Vanneste
Analysis and computation of (n,N)-strategies for maintenance of a two-component system

373 Drs. R. Hamers, Drs. P. Verstappen
Het company ranking model: a means for evaluating the competition

374 Rommert J. Casimir
Infogame Final Report

375 Christian B. Mulder
Efficient and inefficient institutional arrangements between governments and trade unions; an explanation of high unemployment, corporatism and union bashing

376 Marno Verbeek
On the estimation of a fixed effects model with selective nonresponse

377 J. Engwerda
Admissible target paths in economic models

378 Jack P.C. Kleijnen and Nabil Adams
Pseudorandom number generation on supercomputers

379 J.P.C. Blanc
The power-series algorithm applied to the shortest-queue model

380 Prof. Dr. Robert Bannink
Management's information needs and the definition of costs, with special regard to the cost of interest

381 Bert Bettonvil
Sequential bifurcation: the design of a factor screening method

382 Bert Bettonvil
Sequential bifurcation for observations with random errors

383 Harold Houba and Hans Kremers
Correction of the material balance equation in dynamic input-output models

384 T.M. Doup, A.H. van den Elzen, A.J.J. Talman
Homotopy interpretation of price adjustment processes

385 Drs. R.T. Frambach, Prof. Dr. W.H.J. de Freytas
Technologische ontwikkeling en marketing. Een oriënterende beschouwing

386 A.L.P.M. Hendrikx, R.M.J. Heuts, L.G. Hoving
Comparison of automatic monitoring systems in automatic forecasting

387 Drs. J.G.L.M. Willems
Enkele opmerkingen over het inversificerend gedrag van multinationale ondernemingen

388 Jack P.C. Kleijnen and Ben Annink
Pseudorandom number generators revisited

389 Dr. G.W.J. Hendrikse
Speltheorie en strategisch management

390 Dr. A.W.A. Boot en Dr. M.F.C.M. Wijn
Liquiditeit, insolventie en vermogensstructuur

391 Antoon van den Elzen, Gerard van der Laan
Price adjustment in a two-country model

392 Martin F.C.M. Wijn, Emanuel J. Bijnen
Prediction of failure in industry
An analysis of income statements

393 Dr. S.C.W. Eijffinger and Drs. A.P.D. Gruijters
On the short term objectives of daily intervention by the Deutsche Bundesbank and the Federal Reserve System in the U.S. Dollar - Deutsche Mark exchange market

394 Dr. S.C.W. Eijffinger and Drs. A.P.D. Gruijters
On the effectiveness of daily interventions by the Deutsche Bundesbank and the Federal Reserve System in the U.S. Dollar - Deutsche Mark exchange market

395 A.E.M. Meijer and J.W.A. Vingerhoets
Structural adjustment and diversification in mineral exporting developing countries

396 R. Gradus
About Tobin's marginal and average q
A Note

397 Jacob C. Engwerda
On the existence of a positive definite solution of the matrix equation $X + A^T X^{-1} A = I$

398 Paul C. van Batenburg and J. Kriens
Bayesian discovery sampling: a simple model of Bayesian inference in auditing

399 Hans Kremers and Dolf Talman
Solving the nonlinear complementarity problem

400 Raymond Gradus
Optimal dynamic taxation, savings and investment

401 W.H. Haemers
Regular two-graphs and extensions of partial geometries

402 Jack P.C. Kleijnen, Ben Annink
Supercomputers, Monte Carlo simulation and regression analysis

403 Ruud T. Frambach, Ed J. Nijssen, William H.J. Freytas
Technologie, Strategisch management en marketing

404 Theo Nijman
A natural approach to optimal forecasting in case of preliminary observations

405 Harry Barkema
An empirical test of Holmström's principal-agent model that tax and signally hypotheses explicitly into account

406 Drs. W.J. van Braband
De begrotingsvoorbereiding bij het Rijk

407 Marco Wilke
Societal bargaining and stability

408 Willem van Groenendaal and Aart de Zeeuw
Control, coordination and conflict on international commodity markets

409 Prof. Dr. W. de Freytas, Drs. L. Arts
Tourism to Curacao: a new deal based on visitors' experiences

410 Drs. C.H. Veld
The use of the implied standard deviation as a predictor of future stock price variability: a review of empirical tests

411 Drs. J.C. Caanen en Dr. E.N. Kertzman
Inflatieneutrale belastingheffing van ondernemingen

412 Prof. Dr. B.B. van der Genugten
A weak law of large numbers for m-dependent random variables with unbounded m

413 R.M.J. Heuts, H.P. Seidel, W.J. Selen
A comparison of two lot sizing-sequencing heuristics for the process industry

414 C.B. Mulder en A.B.T.M. van Schaik
Een nieuwe kijk op structuurwerkloosheid

415 Drs. Ch. Caanen
De hefboomwerking en de vermogens- en voorraadaftrek

416 Guido W. Imbens
Duration models with time-varying coefficients

417 Guido W. Imbens
Efficient estimation of choice-based sample models with the method of moments

418 Harry H. Tigelaar
On monotone linear operators on linear spaces of square matrices

IN 1990 REEDS VERSCHENEN

419 Bertrand Melenberg, Rob Alessie
A method to construct moments in the multi-good life cycle consumption model

420 J. Kriens
On the differentiability of the set of efficient (μ,σ^2) combinations in the Markowitz portfolio selection method

421 Steffen Jørgensen, Peter M. Kort
Optimal dynamic investment policies under concave-convex adjustment costs

422 J.P.C. Blanc
Cyclic polling systems: limited service versus Bernoulli schedules