ERSITY

UNIVERSITEIT
BRABANT

POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

DENSITY OF THE LEAST SQUARES ESTIMATOR IN THE MULTIVARIATE LINEAR MODEL WITH ARBITRARILY NORMAL VARIABLES

B.B. van der Genugten

R 41

FEW 564

Estimation
Multivariate An

Communicated by Prof.dr. M.H.C. Paardekooper

DENSITY OF THE LEAST SQUARES ESTIMATOR IN THE MULTIVARIATE LINEAR MODEL WITH ARBITRARILY NORMAL VARIABLES

B.B. VAN DER GENUGTEN*

Abstract

An expression is obtained for the density of the least squares estimator $\hat{B} = (X_2'X_2)^{-1}X_2'X_1$ in the multivariate linear model $X_1 = X_2B + E$ for the case that $X = [X_1\ X_2]$ has a normal distribution.

This expression is a multi-dimensional integral with a dimension independent of the number of rows of X.

Two particular cases are worked out in detail.

Keywords
multivariate linear model, least squares estimator, numerical evaluation of probability densities.

AMS classification: 60E05, 62J05.

---

*Department of Econometrics, Tilburg University, Tilburg, The Netherlands.

1 Introduction and statement of the result

Consider the multivariate linear model

$$X_1 = X_2 B + E \tag{1}$$

with dependent variables $X_1 \in \mathbf{R}^{n \times p}$ and explanatory variables $X_2 \in \mathbf{R}^{n \times k}$. The least squares estimator for B is given by

$$\hat{B} = (X_2' X_2)^{-1} X_2' X_1. \tag{2}$$

Assume that $X = [X_1 \ X_2] \in \mathbf{R}^{n \times m}$ (with $m = p + k$) has a non-singular normal distribution:

$$\operatorname{vec}(X) \sim N_{nm}(\mu, \Omega), \quad \Omega > 0. \tag{3}$$

The theorem below gives the density g of $\hat{B}$ as a multi-dimensional integral of the form

$$g(B) = \int_{\mathrm{R}^{k \times p}} \varphi(B, U_{21}) dU_{21}. \tag{4}$$

The dimension does not depend on n.

In section 2 the particular case $\mu = 0$, $\Omega = I_{nm}$ is considered. It is shown that the expression (4) leads to the density of the (central) matrix t-distribution.

In section 3 a detailed expression is given for the two-dimensional case $p = 1$, $k = 2$. This result can be used immediately for numerical calculations.

In the integrand φ of (4) the concept of the determinant- differential operator is used. Such an operator D works on functions $\varphi : \mathbf{R}^{k \times k} \to \mathbf{R}$ and is defined by $D\psi(u) = (\det \partial/\partial U = \{\partial/\partial u_{rs}\}$, $r, s = 1, \ldots, k$. So $D\psi(U)$ is a linear combination of $k!$ partial derivatives of ψ of order k. As usual higher orders are defined by $D^p = D^{p-1}$ ($p = 2, 3, \ldots$). The particular value at $U = U_0$ is denoted by $D^p\psi(U_0)$.

The operator D appears in a natural way in the differentiation of characteristic functions. For $X, U \in \mathbf{R}^{k \times k}$ we have

$$\begin{aligned}\int |X| f(X)\,\mathrm{etr}(iU'X)dx &= \int |X| f(X)\exp\{i\sum_r\sum_s u_{rs}x_{rs}\}dX = \\ &= i^{-k}\sum_{(i_1,\dots,u_k)}\int f(X)(\partial^k/\partial u_{i_k}\dots\partial u_{i_k})\,\mathrm{etr}(iU'X)dX \\ &= i^{-k}D\int f(X)\,\mathrm{etr}(iU'X)dx\end{aligned}$$

and more general for $p = 1, 2, \dots$:

$$\int_{\mathbf{R}^{k\times k}} |X|^p f(X)\,\mathrm{etr}(iU'x)dX = i^{-kp}D^p\int_{\mathbf{R}^{k\times k}} f(X)\,\mathrm{etr}(iU'X)dX \tag{5}$$

Theorem

Under the condition that the integral in (4) is absolutely convergent we have:

$$\varphi(B, U_{21}) = \left(\frac{1}{2\pi i}\right)^{kp} D_{22}^p\psi(U_{21}, -(BU_{21}' + U_{21}B')/2) \tag{6}$$

with determinant-differential operator

$$D_{22} = \det(\partial/\partial U_{22}), \quad U_{22}\in\mathbf{R}^{k\times k} \tag{7}$$

and with

$$\psi(U_{21}, U_{22}) = |\Omega|^{1/2}\exp\left(-\tfrac{1}{2}\mu'\Omega^{-1}\mu\right)\cdot|\Phi|^{-1/2}\exp\left(\tfrac{1}{2}\mu'\Omega^{-1}\Phi^{-1}\Omega^{-1}\mu\right) \tag{8}$$

$$\Phi = \Phi(U_{21}, U_{22}) = \Omega^{-1} - i(V + V') \tag{9}$$

$$V = V(U_{21}, U_{22}) = U\otimes I_n \tag{10}$$

$$U = \begin{bmatrix} 0 & 0 \\ U_{21} & U_{22}\end{bmatrix}\in\mathbf{R}^{m\times m} \tag{11}$$

The theorem shows that for the numerical calculation of the integral (4) an expression for $D_{22}^p\psi$ in (6) is needed. Therefore such a calculation is only straightforward possible with computer packages having the possibility of alpha-numeric evaluation (e.g. Mathematica)

2 The particular case $\mu = 0$, $\Omega = I_{nm}$

Substitution of $\mu = 0$ and $\Omega = I_{nm} = I_m \otimes I_n$ into (9) gives

$$\Phi = \left(\begin{bmatrix} I_p & 0 \\ 0 & I_k \end{bmatrix} - i \begin{bmatrix} 0 & U'_{21} \\ U_{21} & U_{22} + U'_{22} \end{bmatrix} \right) \otimes I_n$$

So, with (8)

$$\psi(U_{21}, U_{22}) = |\Phi|^{-1/2} = |I_k + U_{21}U'_{21} - i(U_{22} + U'_{22})|^{-n/2} \tag{12}$$

It is not easy to calculate $D^p_{22}\psi(U_{21}, U_{22})$ from (12). Using some known results about central Wishart-distributions it can be shown that

$$D^p_{22}\psi(U_{21}, U_{22}) = (2i)^{kp} \frac{\Gamma_k(n/2 + p)}{\Gamma_k(n/2)} \cdot |I_k + U_{21}U'_{21} - i(U_{22} + U'_{22})|^{-n/2-p}, \tag{13}$$

where Γ_k stands for the multivariate gammafunction

$$\Gamma_k(n/2) = \pi^{k(k-1)/4} \prod_{j=1}^{k} \Gamma((n - j + 1)/2). \tag{14}$$

The derivation of (13) from (12) can be found at the end of this section.

Substitution of (13) into (6) leads with (4) to

$$g(B) = \left(\frac{1}{\pi}\right)^{kp} \frac{\Gamma_k(n/2 + p)}{\Gamma_k(n/2)} \int_{\mathbf{R}^{k \times p}} |I_k + U_{21}U'_{21} + i(BU'_{21} + U_{21}B')|^{-n/2-p} dU_{21}$$

The integral can be reduced in the same way as in Phillips (1985), p. 160. Transform with $W = (I_k + BB')^{-1/2}(U_{21} + iB)$. Then

$$I_k + U_{21}U'_{21} + i(BU'_{21} + U_{21}B') = (I_k + BB')^{1/2}(I_k + WW')(I_k + BB')^{1/2}.$$

The transformation has Jacobian $|I_k + BB'|^{p/2}$. The domain of integration can be taken to be $\mathbf{R}^{kl}$ as before using some standard arguments in contour-integration. This leads to

$$g(B) = \left(\frac{1}{\pi}\right)^{kp} \frac{\Gamma_k(n/2 + p)}{\Gamma_k(n/2)} |I_k + BB'|^{-(n+p)/2} \int_{\mathbf{R}^{k \times p}} |I_k + WW'|^{-n/2-p} dW$$

Since (e.g. see Dickey (1967), p. 512)

$$\int_{\mathbf{R}^{k\times p}} |I_k + WW'|^{-n/2-p} dW = \pi^{kp/2} \frac{\Gamma_k((n+p)/2)}{\Gamma_k(n/2+p)}$$

we get finally

$$g(B) = \pi^{-kp/2} \frac{\Gamma_k((n+p)/2)}{\Gamma_k(n/2)} |I_k + BB'|^{-(n+p)/2}. \tag{15}$$

This is the density of the central standardized matrix t distribution $t_{n,k,p}$. By definition $T = J^{-1}C \sim t_{n,k,p}$ if $C \in \mathbf{R}^{k\times p}$ and $J \in \mathbf{R}^{k\times k}$ are independent with C standard normal and $JJ' = \Sigma_1^n u_j u_j' \sim \chi^2_{n,m} = \chi^2_{n,m}(I_m)$, the central standard Wishart distribution with n degrees of freedom and dimension m (i.e. $u_j \sim N_m(0, \Sigma)$ with $\Sigma = I_m$ and independent). For details see Johnson and Kotz (1972), p. 151.

Of course in this particular case there is a more direct way in showing that $\hat{B} \sim t_{n,k,p}$. Since $\mathrm{vec}(\hat{B}) = \mathrm{vec}((X_2'X_2)^{-1}X_2'X_1) = (I_p \otimes (X_2'X_2)^{-1}X_2')\,\mathrm{vec}(X_1)$ we have $\mathcal{L}(\mathrm{vec}(\hat{B})|X_2) = N_{kp}(0, I_p \otimes (X_2'X_2)^{-1})$. So $\mathcal{L}(\mathrm{vec}((X_2'X_2)^{1/2}\hat{B}|X_2) = \mathcal{L}(I_p \otimes (X_2'X_2)^{1/2}\,\mathrm{vec}(\hat{B})|X_2) = N_{kp}(0, I_{kp})$. Hence, $J = (X_2'X_2)^{1/2}$ and $C = (X_2'X_2)^{1/2}\hat{B}$ are independent with C standard normal and $JJ' = X_2'X_2 \sim \chi^2_{n,k}$. Then, by definition, $\hat{B} = J^{-1}C \sim t_{n,k,p}$.

We conclude this section with the derivation of (13). Let $\hat{W} \sim \chi^2_{n,k}(\Sigma)$. Then the characteristic function $\varphi(U) = E\{\mathrm{etr}(iU'\hat{W}\}$, $U \in \mathbf{R}^{k\times k}$ is given by

$$\varphi(U) = \{|\Sigma^{-1}|/|\Sigma^{-1} - i(U + U')|\}^{n/2}$$

and the density $f(W)$ for $W > 0$ by

$$f(W) = K(n, \Sigma^{-1})|W|^{(n-k-1)/2}\,\mathrm{etr}(-\tfrac{1}{2}\Sigma^{-1}W),$$

where

$$K(n, \Sigma^{-1}) = \{2^{nk/2}\Gamma_k(n/2)\}^{-1}|\Sigma|^{n/2}$$

(see Johnson and Kotz (1972), p. 162-163).

It follows from (12) that $|\Sigma|^{n/2}\psi(U_{21}, U_{22})$ is the characteristic function of $\chi^2_{n,k}(\Sigma)$ at U_{22} if $\Sigma = (I_k + U_{21}U'_{21})^{-1}$. Hence,

$$|\Sigma|^{n/2}\psi(U_{21}, U_{22}) = \int K(n, \Sigma^{-1})|W|^{(n-k-1)/2}\,\mathrm{etr}\left(-\tfrac{1}{2}\Sigma^{-1}W + iU'_{22}W\right) dW$$

and so, using (6),

$$\begin{aligned} |\Sigma|^{n/2} D^p_{22}\psi(U_{21}, U_{22}) &= i^{kp}\int K(n, \Sigma^{-1})|W|^{(n-k-1)/2+p}\,\mathrm{etr}\left(-\tfrac{1}{2}\Sigma^{-1}W + iU'_{22}W\right) dW = \\ &= i^{kp}K(n, \Sigma^{-1})/K(n+2p, \Sigma^{-1})|\Sigma|^{n/2+p}|\Sigma^{-1} - i(U_{22}+U'_{22})|^{-n/2-p} \end{aligned}$$

or

$$D^p_{22}\psi(U_{21}, U_{22}) = (2i)^{kp}\frac{\Gamma_k(n/2+p)}{\Gamma_k(n/2)}\left|\Sigma^{-1} - i(U_{22}+U'_{22})\right|^{-n/2-p}$$

This proves (12).

3 The particular case $p = 1, k = 2$

For this most simple non-trivial case explicit expressions for $D^p_{22}\psi = D_{22}\psi$ are manageable. Write $U_{22} = \{u_{rs}\}$, $r, s = 1, 2$. Then

$$D_{22}\psi = \left\{\det\begin{bmatrix} \partial/\partial u_{11} & \partial/\partial u_{12} \\ \partial/\partial u_{21} & \partial/\partial u_{22} \end{bmatrix}\right\}\psi = (\partial^2/\partial u_{11}\partial u_{22} - \partial^2/\partial u_{21}\partial u_{12})\psi.$$

The lemma below gives an explicit expression for $\partial\psi/\partial u_{rs}$ and $\partial^2\psi/\partial u_{rs}\partial u_{pq}$.

Lemma

Let

$$J_{rs} = \tfrac{1}{2}\partial/\partial u_{rs}(V + V'), \tag{16}$$

then

$$\partial\psi/\partial u_{rs} = ia_{rs}\psi \tag{17}$$

with

$$a_{rs} = \mathrm{tr}(\Phi^{-1}I_{rs}) + 2\mu'\Omega^{-1}\Phi^{-1}J_{rs}\Phi^{-1}\Omega^{-1}\mu. \tag{18}$$

Furthermore, let

$$\Phi_{rspq} = J_{rs}\Phi^{-1}J_{pq} + J_{pq}\Phi^{-1}J_{rs}, \tag{19}$$

then

$$\partial^2\psi/\partial u_{rs}\partial u_{pq} = -(a_{rspq} + a_{rs}a_{pq})\psi \tag{20}$$

with

$$a_{rspq} = \mathrm{tr}(\Phi^{-1}\Phi_{rspq}) + 4\mu'\Omega^{-1}\Phi^{-1}\Phi_{rspq}\Phi^{-1}\Omega^{-1}\mu. \tag{21}$$

Proof

We have

$$\begin{aligned}\partial|\Phi|^{-1/2}/\partial u_{rs} &= -\tfrac{1}{2}|\Phi|^{-3/2}\partial|\Phi|/\partial u_{rs} = \\ &= -\tfrac{1}{2}|\Phi|^{-3/2}\cdot|\Phi|\,\mathrm{tr}\{\Phi^{-1}\partial\Phi/\partial u_{rs}\} = i|\Phi|^{-1/2}\,\mathrm{tr}(\Phi^{-1}J_{rs})\end{aligned}$$

$$\partial\Phi^{-1}/\partial u_{rs} = -\Phi^{-1}(\partial\Phi/\partial u_{rs})\Phi^{-1} = 2i\Phi^{-1}J_{rs}\Phi^{-1}.$$

Writing $c = |\Omega|^{1/2}\exp(-\frac{1}{2}\mu'\Omega^{-1}\mu)$ this gives

$$\begin{aligned}\partial\psi/\partial u_{rs} = ci\{&|\Phi|^{-1/2}\,\mathrm{tr}(\Phi^{-1}J_{rs})\cdot\exp(\mu'\Omega^{-1}\Phi^{-1}\Omega^{-1}\mu) + \\ &+ |\Phi|^{-1/2}(2i\mu'\Omega^{-1}J_{rs}\Phi^{-1}\Omega^{-1}\mu)\cdot\exp(\mu'\Omega^{-1}\Phi^{-1}\Omega^{-1}\mu)\} = ia_{rs}\psi,\end{aligned}$$

proving (18). Furthermore,

$$\begin{aligned}\partial\,\mathrm{tr}(\Phi^{-1}J_{rs})/\partial u_{pq} &= \mathrm{tr}\{(\partial\Phi^{-1}/\partial u_{pq})J_{rs}\} = \\ &= 2i\,\mathrm{tr}(\Phi^{-1}J_{pq}\Phi^{-1}J_{rs}) = i\,\mathrm{tr}(\Phi^{-1}\Phi_{rspq})\end{aligned}$$

$$\begin{aligned}\partial\Phi^{-1}J_{rs}\Phi^{-1}/\partial u_{pq} &= 2i\Phi^{-1}J_{pq}\Phi^{-1}J_{rs}\Phi^{-1} + 2i\Phi^{-1}J_{rs}\Phi^{-1}J_{pq}\Phi^{-1} = \\ &= 2i\Phi^{-1}\Phi_{rspq}\Phi^{-1}\end{aligned}$$

Therefore, using (18),

$$\begin{aligned}\partial^2\psi/\partial u_{rs}\partial u_{pq} &= i\{(\partial a_{rs}/\partial u_{pq})\psi + a_{rs}(\partial\psi/\partial u_{pq})\} = \\ &= i\{i\,\mathrm{tr}(\Phi^{-1}\Phi_{rspq}) + 2\mu'\Omega^{-1}(2i\Phi^{-1}\Phi_{rspq}\Phi^{-1})\Omega^{-1}\mu) + ia_{rs}a_{pq}\}\psi = \\ &= -(a_{rspq} + a_{rs}a_{pq})\psi,\end{aligned}$$

proving (21).

4 Proof of the theorem

The proof is given with some preparatory lemma's.

Lemma 1
Let $X \sim N_n(\mu,\Omega)$, $\Omega > 0$. Set $Y_j = X'A_jX$ with symmetric $A_j \in \mathbf{R}^{n\times n}$, $j = 1,\dots,s$ and $Y = (Y_1,\dots,Y_s)'$. Then for $u = (u_1,\dots,u_s)'$:

$$\begin{aligned}\varphi(u) &= E\{\exp(iu'Y)\} = \\ &= |\Omega|^{-1/2}\exp(-\tfrac{1}{2}\mu'\Omega^{-1}\mu)\cdot|\Phi|^{-1/2}\exp(\tfrac{1}{2}\mu'\Omega^{-1}\Phi^{-1}\Omega^{-1}\mu) \qquad (22)\end{aligned}$$

with

$$\Phi = \Phi(u) = \Omega^{-1} - 2i\sum_{j=1}^{s} u_jA_j. \qquad (23)$$

Proof
Follows after a reformulation of Magnus (1986), Lemma 5, p. 102.

Lemma 2

Let $X \in \mathbf{R}^{n\times m}$ with $\mathrm{vec}(X) \sim N_{nm}(\mu, \Omega)$, $\Omega > 0$. Set $Y = X'AX$ with symmetric $A \in \mathbf{R}^{n\times n}$. Then for $U \in \mathbf{R}^{m\times m}$:

$$\begin{aligned} \varphi(U) &= E\{\mathrm{etr}(iU'Y)\} = \\ &= |\Omega|^{-1/2}\exp(-\tfrac{1}{2}\mu'\Omega^{-1}\mu)\cdot|\Phi|^{-1/2}\exp(\tfrac{1}{2}\mu'\Omega^{-1}\Phi^{-1}\Omega^{-1}\mu) \end{aligned} \tag{24}$$

with

$$\Phi = \Phi(U) = \Omega^{-1} - i(U + U')\otimes A \tag{25}$$

Proof

Write $X = [x_1 \ldots x_m]$, $x = \mathrm{vec}(X) = [x_1' \ldots x_m']'$, $Y = (Y_{ij})$. Then $Y_{ij} = (X'AX)_{ij} = x'A_{ij}x$ with $A_{ij} \in \mathbf{R}^{nm\times nm}$ a matrix with A for block (i,j) and zeros elsewhere. Since for $U = (u_{ij})$:

$$\mathrm{tr}(U'Y) = \Sigma\Sigma u_{ij}Y_{ij} = \Sigma\Sigma u_{ij}x'A_{ij}x = \tfrac{1}{2}\Sigma\Sigma u_{ij}x'(A_{ij} + A_{ji})x$$

it follows from (22), (23) with $s = m^2$ that $\varphi(u)$ is given by (24) with

$$\begin{aligned} \Phi = \Phi(U) &= \Omega^{-1} - i\{\Sigma\Sigma u_{ij}(A_{ij} + A_{ji})\} = \\ &= \Omega^{-1} - i\{\Sigma\Sigma(u_{ij} + u_{ji})A_{ij}\} = \Omega^{-1} - i(U + U')\otimes A, \end{aligned}$$

proving (25).

Lemma 3

Let $C_{21} \in \mathbf{R}^{k\times p}$, $C_{22} \in \mathbf{R}^{k\times k}$ with $C_{22} > 0$ a.s. If $E|C_{22}|^p < \infty$ and if $[C_{21}\ C_{22}]$ is absolutely continuous (with respect to C_{22} in the obvious sense) then $\hat{B} = C_{22}^{-1}C_{21}$ has a continuous density g, for $B \in \mathbf{R}^{k\times p}$ given by

$$g(B) = \left(\frac{1}{2\pi i}\right)^{kp}\int_{\mathbf{R}^{k\times p}} D_{22}^{p}\psi(U_{21}, -(BU_{21}' + U_{21}B')/2)dU_{21} \tag{26}$$

provided that the integral is absolutely convergent.

Proof

The Lemma is a slight correction of Phillips (1985), theorem, p. 185.

Proof of the theorem

Follows from Lemma 2 and Lemma 3. Take $A = I_n$ and

$$U = \begin{bmatrix} 0 & 0 \\ U_{21} & U_{22} \end{bmatrix}.$$

Then combination of (24)-(26) leads to (8)-(11).

References

Dickey, J.M. (1967) - Matrixvariate generalizations of the multivariate t-distribution and the inverted multivariate t-distribution - Ann. Math. Stat. 38, 511-518.

Johson, N.L. and Kotz, S. (1972) - Distributions in Statistics: Continuous Multivariate Distributions - Wiley, New York.

Magnus, J.R. (1986) - The exact moments of a ratio of quadratic forms in normal variables - Annales d'Economie et de Statistique 4, 95-109.

Phillips, P.C.B. (1985) - The distribution of Matrix Quotients - J. Multiv. Analysis 16, 157-161.

IN 1991 REEDS VERSCHENEN

466 Prof.Dr. Th.C.M.J. van de Klundert - Prof.Dr. A.B.T.M. van Schaik
Economische groei in Nederland in een internationaal perspectief

467 Dr. Sylvester C.W. Eijffinger
The convergence of monetary policy - Germany and France as an example

468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche

469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication

470 Drs. J. Grazell en Drs. C.H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering

471 P.C. van Batenburg, J. Kriens, W.M. Lammerts van Bueren and R.H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling

472 Marcel Kerkhofs
Identification and Estimation of Household Production Models

473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach

474 Jack P.C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design

475 C.P.M. van Hoesel
An $O(n \log n)$ algorithm for the two-machine flow shop problem with controllable machine speeds

476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance

477 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
Coordinated replenishment systems with discount opportunities

478 A. van den Nouweland, J. Potters, S. Tijs and J. Zarzuelo
Cores and related solution concepts for multi-choice games

479 Drs. C.H. Veld
Warrant pricing: a review of theoretical and empirical research

480 E. Nijssen
De Miles and Snow-typologie: Een exploratieve studie in de meubelbranche

481 Harry G. Barkema
Are managers indeed motivated by their bonuses?

482 Jacob C. Engwerda, André C.M. Ran, Arie L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^{T}X^{-1}A = I$

483 Peter M. Kort
A dynamic model of the firm with uncertain earnings and adjustment costs

484 Raymond H.J.M. Gradus, Peter M. Kort
Optimal taxation on profit and pollution within a macroeconomic framework

485 René van den Brink, Robert P. Gilles
Axiomatizations of the Conjunctive Permission Value for Games with Permission Structures

486 A.E. Brouwer & W.H. Haemers
The Gewirtz graph - an exercise in the theory of graph spectra

487 Pim Adang, Bertrand Melenberg
Intratemporal uncertainty in the multi-good life cycle consumption model: motivation and application

488 J.H.J. Roemen
The long term elasticity of the milk supply with respect to the milk price in the Netherlands in the period 1969-1984

489 Herbert Hamers
The Shapley-Entrance Game

490 Rezaul Kabir and Theo Vermaelen
Insider trading restrictions and the stock market

491 Piet A. Verheyen
The economic explanation of the jump of the co-state variable

492 Drs. F.L.J.W. Manders en Dr. J.A.C. de Haan
De organisatorische aspecten bij systeemontwikkeling
een beschouwing op besturing en verandering

493 Paul C. van Batenburg and J. Kriens
Applications of statistical methods and techniques to auditing and accounting

494 Ruud T. Frambach
The diffusion of innovations: the influence of supply-side factors

495 J.H.J. Roemen
A decision rule for the (des)investments in the dairy cow stock

496 Hans Kremers and Dolf Talman
An SLSPP-algorithm to compute an equilibrium in an economy with linear production technologies

497 L.W.G. Strijbosch and R.M.J. Heuts
Investigating several alternatives for estimating the compound lead time demand in an (s,Q) inventory model

498 Bert Bettonvil and Jack P.C. Kleijnen
Identifying the important factors in simulation models with many factors

499 Drs. H.C.A. Roest, Drs. F.L. Tijssen
Beheersing van het kwaliteitsperceptieproces bij diensten door middel van keurmerken

500 B.B. van der Genugten
Density of the F-statistic in the linear model with arbitrarily normal distributed errors

501 Harry Barkema and Sytse Douma
The direction, mode and location of corporate expansions

502 Gert Nieuwenhuis
Bridging the gap between a stationary point process and its Palm distribution

503 Chris Veld
Motives for the use of equity-warrants by Dutch companies

504 Pieter K. Jagersma
Een etiologie van horizontale internationale ondernemingsexpansie

505 B. Kaper
On M-functions and their application to input-output models

506 A.B.T.M. van Schaik
Produktiviteit en Arbeidsparticipatie

507 Peter Borm, Anne van den Nouweland and Stef Tijs
Cooperation and communication restrictions: a survey

508 Willy Spanjers, Robert P. Gilles, Pieter H.M. Ruys
Hierarchical trade and downstream information

509 Martijn P. Tummers
The Effect of Systematic Misperception of Income on the Subjective Poverty Line

510 A.G. de Kok
Basics of Inventory Management: Part 1
Renewal theoretic background

511 J.P.C. Blanc, F.A. van der Duyn Schouten, B. Pourbabai
Optimizing flow rates in a queueing network with side constraints

512 R. Peeters
On Coloring j-Unit Sphere Graphs

513 Drs. J. Dagevos, Drs. L. Oerlemans, Dr. F. Boekema
Regional economic policy, economic technological innovation and networks

514 Erwin van der Krabben
Het functioneren van stedelijke onroerend-goed-markten in Nederland - een theoretisch kader

515 Drs. E. Schaling
European central bank independence and inflation persistence

516 Peter M. Kort
Optimal abatement policies within a stochastic dynamic model of the firm

517 Pim Adang
Expenditure versus consumption in the multi-good life cycle consumption model

518 Pim Adang
Large, infrequent consumption in the multi-good life cycle consumption model

519 Raymond Gradus, Sjak Smulders
Pollution and Endogenous Growth

520 Raymond Gradus en Hugo Keuzenkamp
Arbeidsongeschiktheid, subjectief ziektegevoel en collectief belang

521 A.G. de Kok
Basics of inventory management: Part 2
The (R,S)-model

522 A.G. de Kok
Basics of inventory management: Part 3
The (b,Q)-model

523 A.G. de Kok
Basics of inventory management: Part 4
The (s,S)-model

524 A.G. de Kok
Basics of inventory management: Part 5
The (R,b,Q)-model

525 A.G. de Kok
Basics of inventory management: Part 6
The (R,s,S)-model

526 Rob de Groof and Martin van Tuijl
Financial integration and fiscal policy in interdependent two-sector economies with real and nominal wage rigidity

527 A.G.M. van Eijs, M.J.G. van Eijs, R.M.J. Heuts
Gecoördineerde bestelsystemen
een management-georiënteerde benadering

528 M.J.G. van Eijs
Multi-item inventory systems with joint ordering and transportation decisions

529 Stephan G. Vanneste
Maintenance optimization of a production system with buffercapacity

530 Michel R.R. van Bremen, Jeroen C.G. Zijlstra
Het stochastische variantie optiewaarderingsmodel

531 Willy Spanjers
Arbitrage and Walrasian Equilibrium in Economies with Limited Information

IN 1992 REEDS VERSCHENEN

532 F.G. van den Heuvel en M.R.M. Turlings
Privatisering van arbeidsongeschiktheidsregelingen
Refereed by Prof.Dr. H. Verbon

533 J.C. Engwerda, L.G. van Willigenburg
LQ-control of sampled continuous-time systems
Refereed by Prof.dr. J.M. Schumacher

534 J.C. Engwerda, A.C.M. Ran & A.L. Rijkeboer
Necessary and sufficient conditions for the existence of a positive definite solution of the matrix equation $X + A^*X^{-1}A = Q$.
Refereed by Prof.dr. J.M. Schumacher

535 Jacob C. Engwerda
The indefinite LQ-problem: the finite planning horizon case
Refereed by Prof.dr. J.M. Schumacher

536 Gert-Jan Otten, Peter Borm, Ton Storcken, Stef Tijs
Effectivity functions and associated claim game correspondences
Refereed by Prof.dr. P.H.M. Ruys

537 Jack P.C. Kleijnen, Gustav A. Alink
Validation of simulation models: mine-hunting case-study
Refereed by Prof.dr.ir. C.A.T. Takkenberg

538 V. Feltkamp and A. van den Nouweland
Controlled Communication Networks
Refereed by Prof.dr. S.H. Tijs

539 A. van Schaik
Productivity, Labour Force Participation and the Solow Growth Model
Refereed by Prof.dr. Th.C.M.J. van de Klundert

540 J.J.G. Lemmen and S.C.W. Eijffinger
The Degree of Financial Integration in the European Community
Refereed by Prof.dr. A.B.T.M. van Schaik

541 J. Bell, P.K. Jagersma
Internationale Joint Ventures
Refereed by Prof.dr. H.G. Barkema

542 Jack P.C. Kleijnen
Verification and validation of simulation models
Refereed by Prof.dr.ir. C.A.T. Takkenberg

543 Gert Nieuwenhuis
Uniform Approximations of the Stationary and Palm Distributions of Marked Point Processes
Refereed by Prof.dr. B.B. van der Genugten

544 R. Heuts, P. Nederstigt, W. Roebroek, W. Selen
Multi-Product Cycling with Packaging in the Process Industry
Refereed by Prof.dr. F.A. van der Duyn Schouten

545 J.C. Engwerda
Calculation of an approximate solution of the infinite time-varying LQ-problem
Refereed by Prof.dr. J.M. Schumacher

546 Raymond H.J.M. Gradus and Peter M. Kort
On time-inconsistency and pollution control: a macroeconomic approach
Refereed by Prof.dr. A.J. de Zeeuw

547 Drs. Dolph Cantrijn en Dr. Rezaul Kabir
De Invloed van de Invoering van Preferente Beschermingsaandelen op Aandelenkoersen van Nederlandse Beursgenoteerde Ondernemingen
Refereed by Prof.dr. P.W. Moerland

548 Sylvester Eijffinger and Eric Schaling
Central bank independence: criteria and indices
Refereed by Prof.dr. J.J. Sijben

549 Drs. A. Schmeits
Geïntegreerde investerings- en financieringsbeslissingen; Implicaties voor Capital Budgeting
Refereed by Prof.dr. P.W. Moerland

550 Peter M. Kort
Standards versus standards: the effects of different pollution restrictions on the firm's dynamic investment policy
Refereed by Prof.dr. F.A. van der Duyn Schouten

551 Niels G. Noorderhaven, Bart Nooteboom and Johannes Berger
Temporal, cognitive and behavioral dimensions of transaction costs; to an understanding of hybrid vertical inter-firm relations
Refereed by Prof.dr. S.W. Douma

552 Ton Storcken and Harrie de Swart
Towards an axiomatization of orderings
Refereed by Prof.dr. P.H.M. Ruys

553 J.H.J. Roemen
The derivation of a long term milk supply model from an optimization model
Refereed by Prof.dr. F.A. van der Duyn Schouten

554 Geert J. Almekinders and Sylvester C.W. Eijffinger
Daily Bundesbank and Federal Reserve Intervention and the Conditional Variance Tale in DM/$-Returns
Refereed by Prof.dr. A.B.T.M. van Schaik

555 Dr. M. Hetebrij, Drs. B.F.L. Jonker, Prof.dr. W.H.J. de Freytas
"Tussen achterstand en voorsprong" de scholings- en personeelsvoorzieningsproblematiek van bedrijven in de procesindustrie
Refereed by Prof.dr. Th.M.M. Verhallen

556 Ton Geerts
Regularity and singularity in linear-quadratic control subject to implicit continuous-time systems
Communicated by Prof.dr. J. Schumacher

557 Ton Geerts
Invariant subspaces and invertibility properties for singular systems: the general case
Communicated by Prof.dr. J. Schumacher

558 Ton Geerts
Solvability conditions, consistency and weak consistency for linear differential-algebraic equations and time-invariant singular systems: the general case
Communicated by Prof.dr. J. Schumacher

559 C. Fricker and M.R. Jaïbi
Monotonicity and stability of periodic polling models
Communicated by Prof.dr.ir. O.J. Boxma

560 Ton Geerts
Free end-point linear-quadratic control subject to implicit continuous-time systems: necessary and sufficient conditions for solvability
Communicated by Prof.dr. J. Schumacher

561 Paul G.H. Mulder and Anton L. Hempenius
Expected Utility of Life Time in the Presence of a Chronic Noncommunicable Disease State
Communicated by Prof.dr. B.B. van der Genugten

562 Jan van der Leeuw
The covariance matrix of ARMA-errors in closed form
Communicated by Dr. H.H. Tigelaar

563 J.P.C. Blanc and R.D. van der Mei
Optimization of polling systems with Bernoulli schedules
Communicated by Prof.dr.ir. O.J. Boxma