TILBURG UNIVERSITY
KATHOLIEKE UNIVERSITEIT BRABANT
POSTBOX 90153
5000 LE TILBURG
THE NETHERLANDS

[redacted]

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

CORES AND RELATED SOLUTION CONCEPTS FOR MULTI-CHOICE GAMES

A. van den Nouweland, J. Potters,
S. Tijs and J. Zarzuelo

FEW 478

CORES AND RELATED SOLUTION CONCEPTS FOR MULTI-CHOICE GAMES

by

A. van den Nouweland[1], J. Potters[2], S. Tijs[1] and J. Zarzuelo[3]

ABSTRACT. A multi-choice game is a generalization of a cooperative game in which each player has several activity levels. Cooperative games form a subclass of the class of multi-choice games.

This paper extends some solution concepts for cooperative games to multi-choice games. In particular, the notions of core, dominance core, Weber set, stable sets and subsolutions are extended. Relations between cores and dominance cores and between cores and Weber sets are extensively studied. A class of flow games is introduced and relations with non-negative games with non-empty cores are investigated.

[1] Department of Econometrics, Tilburg University, P.O. Box 90153, 5000 LE Tilburg, The Netherlands.

[2] Department of Mathematics, University of Nijmegen, Toernooiveld 1, 6525 ED Nijmegen, The Netherlands.

[3] Department of Applied Mathematics, University of Pais Vasco, 48015 Bilbao, Spain.

1. INTRODUCTION

In a cooperative game (N, v), where $v : \{S \mid S \subseteq N\} \to \mathbb{R}$, each player has two choices: to participate or not to participate. If the players in $S \subseteq N$ choose to participate and the players in $N \backslash S$ choose not to participate, then the worth $v(S)$ can be obtained. *Chih-Ru Hsiao* and *Raghavan* (1990) introduced games in which all players have m *activity levels* $(m \geq 2)$ at which they can choose to play. We will still generalize this definition a bit.

A *multi-choice* game is a game in which each player has an arbitrary number of activity levels at which he or she can choose to play. In particular, two players may have different numbers of activity levels. The reward that a group of players can obtain depends on the effort of the cooperating players. This is formalized as follows. Let $N := \{1, \ldots, n\}$ be a set of players $(n \in \mathbb{N})$ and suppose each player $i \in N$ has $m_i + 1 \in \mathbb{N}$ activity levels at which he can play. We set $M_i := \{0, 1, \ldots, m_i\}$ as the action space of player $i \in N$, where the action 0 means not participating. A function $v : \prod_{i \in N} M_i \to \mathbb{R}$ with $v(0) = 0$ gives for each *coalition* $s = (s_1, \ldots, s_n) \in \prod_{i \in N} M_i$ the worth that the players can obtain when each player i plays at level $s_i \in M_i$.

We denote a multi-choice game by a triple (N, m, v), where N is the set of players, $m \in (\mathbb{N} \cup \{0\})^N$ is the vector describing the number of activity levels for all players, and $v : \prod_{i \in N} M_i \to \mathbb{R}$ is the *characteristic function*. If there can be no confusion we will denote a game (N, m, v) by v. We denote the set of all multi-choice games with player set N by MC^N.

An example of a multi-choice game occurs when we consider a large building project with a deadline and a penalty for every day this deadline is exceeded. Obviously, the date of completion depends on the effort of all people involved in the project: the greater their effort the sooner the project will be completed. This situation gives rise to a multi-choice game. The worth of a coalition where each player works at a certain acitivity level is defined as minus the penalty that is to be paid given the date of completion of the project when every player makes the corresponding effort.

In their paper *Chih-Ru Hsiao* and *Raghavan* (1990) introduced extended Shapley values for multi-choice games where all players have the same number of activity levels. They did so by using weights on activity levels, each level having the same weight for all players, and provided axiomatic characterizations of the corresponding Shapley values.

In this paper we extend cores and related solution concepts to multi-choice games. In section 2 we introduce imputations, cores and dominance cores and we investigate relations between those concepts. We introduce a notion of balancedness and prove a theorem in the spirit of the theorem of *Bondareva* (1963) and *Shapley* (1967). Further, in section 3 we introduce Weber sets and we explore the relations between cores and Weber sets, especially for convex games. Also, an extension of the Shapley value is defined.

Based on the notion of dominance, which is introduced in section 2, we introduce stable sets and subsolutions in section 4. Finally, in section 5 we consider a special class of multi-choice games, namely flow games. It is shown that these games can be related to non-negative multi-choice games with non-empty cores.

Notation. Let $N := \{1, \ldots, n\}$ be a set of players and $S \subseteq N$. By e^S we denote the vector in $\mathbb{R}^N$ satisfying $e_i^S = 0$ if $i \notin S$ and $e_i^S = 1$ if $i \in S$.
For two sets A and B in the same vector space we set

$$A + B := \{x + y \mid x \in A \text{ and } y \in B\}$$

and we denote the convex hull of A by $co(A)$. Finally, we define the empty sum to be zero.

2. IMPUTATIONS, CORE AND DOMINANCE CORE

Let $(N, m, v) \in MC^N$. We define $M := \{(i,j) \mid i \in N, j \in M_i\}$. A (level) payoff vector for the game v is a function $x : M \to \mathbb{R}$, where, for all $i \in N$ and $j \in M_i \backslash \{0\}$, x_{ij} denotes the increase in payoff to player i corresponding to a change of activity from level $j-1$ to level j by this player and $x_{i0} = 0$ for all $i \in N$.

A payoff vector is called *efficient* if $\sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} = v(m)$ and it is called *level increase rational* if, for all $i \in N$ and $j \in M_i \backslash \{0\}$, x_{ij} is at least the increase in worth that player i can obtain when he works alone and changes his activity from level $j-1$ to level j, i.e. $x_{ij} \geq v(je^i) - v((j-1)e^i)$.

Definition. A payoff vector is an *imputation* of v if it is efficient and level increase rational.

We denote the set of imputations of the game v by $I(v)$. It is easily seen that

$$I(v) \neq \emptyset \iff \sum_{i \in N} v(m_i e^i) \leq v(m). \tag{1}$$

Now let x be a payoff vector for the game v. If a player i works at his jth level ($j \in M_i$), then he obtains, according to x, the amount $\sum_{k=0}^{j} x_{ik}$. It will often be more natural to look at these accumulated payoffs. For $i \in N$ and $j \in M_i$ we denote $X_{ij} := \sum_{k=0}^{j} x_{ik}$. The members of a coalition $s \in \prod_{i \in N} M_i$ obtain $X(s) := \sum_{i \in N} X_{is_i}$. Using this, we come to the following

Definition. The *core* $C(v)$ of the game v consists of all $x \in I(v)$ that satisfy $X(s) \geq v(s)$ for all $s \in \prod_{i \in N} M_i$.

Now let $s \in \prod_{i \in N} M_i$ and $x, y \in I(v)$. The imputation y dominates the imputation x via coalition s, denoted by $y \operatorname{dom}_s x$, if

$$Y(s) \leq v(s) \text{ and } Y_{is_i} > X_{is_i}$$

for all $i \in C(s)$. Here $C(s) := \{i \in N \mid s_i > 0\}$ is the *carrier* of s, the set of players who participate in s. We say that the imputation y dominates the imputation x if there exists an $s \in \prod_{i \in N} M_i$ such that $y \operatorname{dom}_s x$.

Definition. The *dominance core* $DC(v)$ of the game v consists of all $x \in I(v)$ for which there exists no $y \in I(v)$ such that y dominates x.

In theorems 1, 2 and 3 we deal with the relations between the core and the dominance core.

Theorem 1. For each game v the core $C(v)$ is a subset of the dominance core $DC(v)$.
Proof. Let $x \in C(v)$ and suppose $y \in I(v)$ and $s \in \prod_{i \in N} M_i$, $s \neq 0$, such that $y \operatorname{dom}_s x$. Then

$$v(s) \geq Y(s) = \sum_{i \in N} Y_{is_i} > \sum_{i \in N} X_{is_i} = X(s) \geq v(s),$$

which clearly gives a contradiction. Therefore, x is not dominated. □

To simplify the proof of theorem 2 we introduce zero-normalized games.

Definition. A multi-choice game v is called *zero-normalized* if the players cannot gain anything by working alone, i.e. $v(je^i) = 0$ for all $i \in N$ and $j \in M_i$.
A multi-choice game a is called *additive* if the worth of each coalition s equals the sum of the worths of the players when they all work alone at the level they work at in s, or, in formula,

$$a(s) = \sum_{i \in N} a(s_i e^i)$$

for all $s \in \prod_{i \in N} M_i$.
For an arbitrary multi-choice game v the *zero-normalization of* v is the game v_0 that is obtained by subtracting from v the additive game a with

$$a(je^i) := v(je^i)$$

for all $i \in N$ and $j \in M_i \backslash \{0\}$.

Let v be a zero-normalized game and x a payoff vector for v. Then the condition of level increase rationality boils down to the condition $x \geq 0$. For an additive game a we have $C(a) = DC(a) = I(a) = \{x\}$, where $x : M \to \mathbb{R}$ is the payoff vector where

$$x_{ij} := a(je^i) - a((j-1)e^i)$$

for all $i \in N$ and $j \in M_i \backslash \{0\}$. Now we have the following

Proposition 1. Let v be an arbitrary game and v_0 its zero-normalization. Let x be a payoff vector for this game. Define $y : M \to \mathbb{R}$ by $y_{ij} := x_{ij} - v(je^i) + v((j-1)e^i)$ for all $i \in N$ and $j \in M_i \backslash \{0\}$. Then we have

(i) $x \in I(v) \iff y \in I(v_0)$

(ii) $x \in C(v) \iff y \in C(v_0)$

(iii) $x \in DC(v) \iff y \in DC(v_0)$.

We leave the proof of this proposition as an exercise to the reader.

Theorem 2. Let v be a multi-choice game with a non-empty dominance core. Then the core $C(v)$ equals the dominance core $DC(v)$ if and only if the zero-normalization v_0 of v satisfies $v_0(s) \leq v_0(m)$ for all coalitions s.

Proof. Because of proposition 1 it suffices to prove this theorem for zero-normalized games. So suppose v is zero-normalized.

Suppose $C(v) = DC(v)$ and let $x \in C(v)$. Then

$$v(m) = X(m) = \sum_{i \in N} \sum_{j=1}^{s_i} x_{ij} + \sum_{i \in N} \sum_{j=s_i+1}^{m_i} x_{ij} \geq v(s)$$

for all $s \in \prod_{i \in N} M_i$.

Now suppose $v(s) \leq v(m)$ for all $s \in \prod_{i \in N} M_i$. Since $C(v) \subseteq DC(v)$ (theorem 1), it suffices to prove that $x \notin DC(v)$ for all $x \in I(v) \backslash C(v)$. Let $x \in I(v) \backslash C(v)$ and $s \in \prod_{i \in N} M_i$ such that $X(s) < v(s)$. Define $y : M \to \mathbb{R}$ as follows

$$y_{ij} := \begin{cases} x_{ij} + \frac{v(s) - X(s)}{\sum_{k \in N} s_k} & \text{if } i \in N, j \in \{1, \ldots, s_i\} \\ \frac{v(m) - v(s)}{\sum_{k \in N} (m_k - s_k)} & \text{if } i \in N, j \in \{s_i + 1, \ldots, m_i\}. \end{cases}$$

It follows readily from the definition of y that y is efficient. Since $x \geq 0$, $v(s) > X(s)$ and $v(m) \geq v(s)$, it follows that $y \geq 0$. Hence, y is also level increase rational and we conclude that $y \in I(v)$.

For $i \in N$ and $j \in \{1, \ldots, s_i\}$ we have that $y_{ij} > x_{ij}$. Hence, $Y_{is_i} > X_{is_i}$ for all $i \in N$. This and the fact that

$$Y(s) = X(s) + \sum_{i \in N} \sum_{j=1}^{s_i} \frac{v(s) - X(s)}{\sum_{k \in N} s_k} = v(s)$$

imply that $y \,\mathrm{dom}_s x$. Hence, $x \notin DC(v)$. □

Theorem 2 was inspired by a similar theorem for cooperative games by *Derks* (1986). Using theorems 1 and 2 we can easily prove

Theorem 3. Let v be a multi-choice game with a non-empty core. Then the core $C(v)$ equals the dominance core $DC(v)$.

Proof. It suffices to prove the theorem for zero-normalized games (see proposition 1). So suppose v is zero-normalized. From the first part of the proof of theorem 2 we see that the fact that $C(v) \neq \emptyset$ implies that $v(s) \leq v(m)$ for all $s \in \prod_{i \in N} M_i$. Because $C(v) \subseteq DC(v)$ (cf. theorem 1), we know that $DC(v) \neq \emptyset$. Now theorem 2 immediately implies $C(v) = DC(v)$. □

Considering theorem 3 one might ask oneself if there actually exist games where the core is not equal to the dominance core. The answer to this question is given in example 1, where we provide a multi-choice game with an empty core and a non-empty dominance core. To simplify the notations in examples we represent a payoff vector $x : M \to \mathbb{R}$ by a deficient matrix $[a_{ij}]_{i=1,j=1}^{n\ \ \max\{m_1,\ldots,m_n\}}$, where $a_{ij} := x_{ij}$ if $i \in N$ and $j \in M_i \backslash \{0\}$ and a_{ij} is left out $(*)$ if $i \in N$ and $j > m_i$.

Example 1. Let (N, m, v) be the multi-choice game where $N = \{1, 2\}$, $m = (2, 1)$ and $v((1,0)) = v((0,1)) = 0$, $v((2,0)) = 1/4$ and $v((1,1)) = v((2,1)) = 1$. An imputation x should satisfy the following (in)equalities:

$$\begin{aligned} x_{1,1} + x_{1,2} + x_{2,1} &= 1 \\ x_{1,1} \geq 0, x_{2,1} &\geq 0 \\ x_{1,2} &\geq 1/4. \end{aligned}$$

Hence, we obtain

$$I(v) = co\left\{\begin{bmatrix} 0 & 1/4 \\ 3/4 & * \end{bmatrix}, \begin{bmatrix} 3/4 & 1/4 \\ 0 & * \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & * \end{bmatrix}\right\}.$$

Note that for this game an imputation can only dominate another imputation via the coalition (1,1) and, since $x_{1,1} + x_{2,1} \leq 3/4$ for all $x \in I(v)$, this gives us

$$DC(v) = co\left\{\begin{bmatrix} 0 & 1/4 \\ 3/4 & * \end{bmatrix}, \begin{bmatrix} 3/4 & 1/4 \\ 0 & * \end{bmatrix}\right\}.$$

Finally, for none of the elements x of the dominance core $x_{1,1} + x_{2,1} \geq v((1,1))$. Since $C(v) \subseteq DC(v)$ this gives us $C(v) = \emptyset$. Note that for the zero-normalization v_0 of v it holds that $v_0((1,1)) = 1 > 3/4 = v_0((2,1))$.

For the game in example 1 both the core and the dominance core are convex sets. This is generally true, as is stated in

Theorem 4. Let v be a multi-choice game. Then the following two assertions hold:
(i) $C(v)$ is convex
(ii) $DC(v)$ is convex.

Proof. We omit the proof of part (i), because this is a simple exercise. We now prove part (ii). It suffices to prove that $DC(v)$ is convex if v is zero-normalized. So, suppose v is zero-normalized. Obviously, if $DC(v) = \emptyset$, then it is convex. Now suppose $DC(v) \neq \emptyset$. We define a game (N, m, w) and we show that $DC(v) = DC(w) = C(w)$. For all $s \in \prod_{i \in N} M_i$

$$w(s) := \min\{v(s), v(m)\}.$$

It is easily seen that

$$w(m) = v(m). \tag{2}$$

Since $DC(v) \neq \emptyset$, we know that $I(v) \neq \emptyset$. Since v is zero-normalized, this implies $v(m) \geq 0$ (cf. (1)) and

$$w(je^i) = \min\{v(je^i), v(m)\} = 0 \tag{3}$$

for all $i \in N$ and $j \in M_i$.
Using (2) and (3) we see that

$$I(w) = I(v).$$

Now let $s \in \prod_{i \in N} M_i$ and let $x, y \in I(v) = I(w)$. Since $w(s) \leq v(s)$ we see that if $x \,\mathrm{dom}_s y$ in w, then $x \,\mathrm{dom}_s y$ in v. On the other hand, if $x \,\mathrm{dom}_s y$ in v, then

$$X(s) \leq v(s)$$

and

$$X(s) = \sum_{i \in N} \sum_{j=1}^{m_i} x_{ij} - \sum_{i \in N} \sum_{j=s_i+1}^{m_i} x_{ij} \leq v(m)$$

and therefore $X(s) \leq w(s)$ and $x \,\mathrm{dom}_s y$ in w.
We conclude that

$$DC(w) = DC(v). \tag{4}$$

This implies that $DC(w) \neq \emptyset$. Since w is zero-normalized (cf. (3)) and

$$w(s) = \min\{v(s), v(m)\} \leq v(m) = w(m),$$

theorem 2 shows that

$$C(w) = DC(w). \tag{5}$$

Now (4), (5) and part (i) of this theorem immediately imply that $DC(v)$ is convex. $\square$

The next theorem is an extension of the theorem of *Bondareva* (1963) and *Shapley* (1967) to multi-choice games and gives a necessary and sufficient condition for the non-emptiness of the core of a game.

Definition. A multi-choice game v is called *balanced* if for all maps $\lambda : \prod_{i\in N} M_i \to \mathbb{R}_+$ satisfying

$$\sum_{s\in\prod_{i\in N} M_i} \lambda(s)e^{C(s)} = e^N$$

it holds that

$$\sum_{s\in\prod_{i\in N} M_i} \lambda(s)v_0(s) \leq v_0(m),$$

where v_0 is the zero-normalization of v.

Note that this definition coincides with the familiar definition of balancedness for cooperative games $(N, (1, \ldots, 1), v) \in MC^N$.

Theorem 5. Let v be a multi-choice game. Then the core $C(v)$ of v is non-empty if and only if v is balanced.

Proof. It suffices to prove the theorem for zero-normalized games. So suppose v is zero-normalized.

Suppose $C(v) \neq \emptyset$ and $x \in C(v)$. Then we define a payoff vector $y : M \to \mathbb{R}$ by

$$y_{ij} := \begin{cases} 0 & \text{if } i \in N \text{ and } j \in \{2, \ldots, m_i\} \\ \sum_{j=1}^{m_i} x_{ij} & \text{if } i \in N \text{ and } j = 1. \end{cases}$$

Then, obviously, $y \in C(v)$. Further, we can identify y with the vector $(y_{1,1}, \ldots, y_{n,1})$. This proves that $C(v) \neq \emptyset$ if and only if there exist $z_1, \ldots, z_n \in \mathbb{R}_+$ such that

$$\sum_{i\in N} z_i = v(m) \tag{6}$$

and

$$\sum_{i\in C(s)} z_i \geq v(s) \tag{7}$$

for all $s \in \prod_{i\in N} M_i$.

Obviously, there exist $z_1, \ldots, z_n \in \mathbb{R}_+$ satisfying (6) and (7) if and only if

$$v(m) = \min\{\sum_{i\in N} z_i \mid z_i \in \mathbb{R} \text{ for all } i \in N \text{ and } \sum_{i\in C(s)} z_i \geq v(s) \text{ for all } s \in \prod_{i\in N} M_i\}. \tag{8}$$

From the duality theorem of linear programming theory we know that (8) is equivalent to

$$v(m) = \max\{ \sum_{s \in \prod_{i\in N} M_i} \lambda(s)v(s) \mid \sum_{s \in \prod_{i\in N} M_i} \lambda(s)e^{C(s)} = e^N,$$
$$\lambda(s) \geq 0 \text{ for all } s \in \prod_{i \in N} M_i\}. \tag{9}$$

Having in mind the map $\lambda : \prod_{i\in N} M_i \to \mathbb{R}$ with $\lambda(m) = 1$ and $\lambda(s) = 0$ for all $s \neq m$, we see that (9) is equivalent to v being balanced. □

3. THE WEBER SET

Weber (1988) considered for each cooperative game (N, v) the convex hull of all $n!$ marginal vectors corresponding to v and he showed that the core of a game is always a subset of this so-called Weber set. *Shapley* (1971) showed that for convex games the core coincides with the Weber set and *Ichiishi* (1983) proved the converse, i.e. a game for which the core coincides with the Weber set is convex.

In this section we will extend the definition of the Weber set to multi-choice games and investigate the relations between the core and the Weber set of a multi-choice game. First we define the marginal vectors of a multi-choice game. Let $(N, m, v) \in MC^N$. Suppose the coalition m forms step by step, starting from the coalition $(0, \ldots, 0)$ and where in each step the level of one of the players is increased by 1. So, in particular, there are $\sum_{i\in N} m_i$ steps in this procedure. Now assign for every player to each level the marginal value that is created when that player reaches that particular level from the level directly below. This is formalized as follows:
Define $M^+ := \{(i, j) \mid i \in N,\ j \in M_i \backslash \{0\}\}$. An *admissible permutation* (for v) is a bijection $\sigma : M^+ \to \{1, \ldots, \sum_{i\in N} m_i\}$ satisfying

$$\sigma((i,j)) < \sigma((i,j+1))$$

for all $i \in N$ and $j \in \{1, \ldots, m_i - 1\}$. The number of admissible permutations for v is

$$\frac{(\sum_{i\in N} m_i)!}{\prod_{i\in N}(m_i!)}.$$

Now let σ be an admissible permutation and let $(i, j) \in M^+$. The *predecessing coalition of* (i, j) *with respect to* σ, denoted by $p(\sigma, (i, j))$, is given by

$$p_k(\sigma, (i,j)) = \max\{\ell \in M_k \mid \sigma((k,\ell)) < \sigma((i,j))\} \cup \{0\}$$

for all $k \in N$, and the *marginal vector* $w^\sigma : M \to \mathbb{R}$ corresponding to σ is defined by

$$w^\sigma_{i,j} := v(p(\sigma,(i,j)) + e^i) - v(p(\sigma,(i,j)))$$

for all $i \in N$ and $j \in M_i \backslash \{0\}$.
In general the marginal vectors of a multi-choice game are not necessarily imputations, but for zero-monotonic games they are.

Definition. A multi-choice game v is called *zero-monotonic* if its zero-normalization is monotonic, i.e.

$$v_0(s) \le v_0(t)$$

for all $s, t \in \prod_{i \in N} M_i$ with $s \le t$.

We leave the proof of the following theorem to the reader.

Theorem 6. Let v be a zero-monotonic multi-choice game. Then for every admissible permutation σ the marginal vector corresponding to σ is an imputation of v.

Definition. The *Weber set* $W(v)$ of a multi-choice game v is the convex hull of the marginal vectors of v, or, in formula,

$$W(v) := co\{w^\sigma \mid \sigma \text{ is an admissible permutation for } v\}.$$

Let $(N, m, v) \in MC^N$ and let $x : M \to \mathbb{R}$ and $y : M \to \mathbb{R}$ be two payoff vectors for the game v. We say *x is weakly smaller than y* if

$$X(s) \le Y(s)$$

for all $s \in \prod_{i \in N} M_i$. Note that this does not imply that $x_{ij} \le y_{ij}$ for all $i \in N$ and $j \in M_i$. Let us consider an example.

Example 2. Let (N, m, v) be the multi-choice game where $N = \{1, 2\}$, $m = \{2, 1\}$ and $v((1,0)) = v((0,1)) = 1$, $v((2,0)) = 2$, $v((1,1)) = 3$ and $v((2,1)) = 5$. Now consider the two core-elements x and y, defined by

$$x = \begin{bmatrix} 1 & 2 \\ 2 & * \end{bmatrix}, \; y = \begin{bmatrix} 2 & 1 \\ 2 & * \end{bmatrix}.$$

Then x is weakly smaller than y, since $X((1,0)) < Y((1,0))$, $X((1,1)) < Y((1,1))$ and $X(s) = Y(s)$ for all other s.
What is causing this is the fact that, although according to both payoff vectors player 1 gets 3 for playing at his second level, according to y he gets 2 for playing at his first level and according to x he gets only 1 at the first level.

Now we are ready to formulate

Theorem 7. For each multi-choice game v and each core-element x of v there is a vector y in the Weber set of v that is weakly smaller than x.

Proof. We will actually prove that for each multi-choice game v and each $x \in \widetilde{C}(v)$ there is a vector $y \in W(v)$ such that y is weakly smaller than x, where

$$\widetilde{C}(v) := \{x : M \to \mathbb{R} \mid X(m) = v(m),\ X(s) \geq v(s) \text{ for all } s \in \prod_{i \in N} M_i$$
$$\text{and } x_{i0} = 0 \text{ for all } i \in N\}$$

is a core-catcher (i.e. $C(v) \subseteq \widetilde{C}(v)$). We will do so by induction to the number of levels involved in the game v. We distinguish two basic steps.

• Let (N, m, v) be a multi-choice game where $|N| = 1$ and $m_1 \in \mathbb{N}$ is arbitrary. Then there is only one marginal vector y, which satisfies

$$y_{1j} = v(je^1) - v((j-1)e^1)$$

for all $j \in \{1, \ldots, m_1\}$. Suppose $x \in \widetilde{C}(v)$. Then

$$X(m_1e^1) = v(m_1e^1) = Y(m_1e^1) \text{ and}$$
$$X(je^1) \geq v(je^1) = Y(je^1) \text{ for all } j \in \{1, \ldots, m_1\}.$$

Hence, y is weakly smaller than x.

• Let (N, m, v) be a multi-choice game where $|N| = 2$ and $m = (1,1)$. Then there are two marginal vectors,

$$y^1 = \begin{bmatrix} v(e^1) \\ v(e^1 + e^2) - v(e^1) \end{bmatrix} \text{ and } y^2 = \begin{bmatrix} v(e^1 + e^2) - v(e^2) \\ v(e^2) \end{bmatrix}.$$

Suppose $x \in \widetilde{C}(v)$. Then

$$x_{1,1} \geq v(e^1),\ x_{2,1} \geq v(e^2) \text{ and } x_{1,1} + x_{2,1} = v(e^1 + e^2).$$

Hence, x is a convex combination of y^1 and y^2. We conclude that $x \in W(v)$.

• Now let (N, m, v) be a multi-choice game such that $|\{i \in N \mid m_i > 0\}| \geq 2$ and $\sum_{i \in N} m_i > 2$. Suppose we already proved the statement for all multi-choice games $(\overline{N}, \overline{m}, \overline{v})$ with $\sum_{i \in \overline{N}} \overline{m}_i < \sum_{i \in N} m_i$. Since, obviously, $\widetilde{C}(v)$ and $W(v)$ are both convex sets, it suffices to prove that for all extreme points x of $\widetilde{C}(v)$ we can find a $y \in W(v)$ such that y is weakly smaller than x. So, let x be an extreme point of $\widetilde{C}(v)$. Then let

$t \in \prod_{i \in N} M_i$ be such that $1 \leq \sum_{i \in N} t_i \leq \sum_{i \in N} m_i - 1$ and $X(t) = v(t)$. We split up (N, m, v) into two games, (N, t, u) and $(N, m - t, w)$, defined by

$$u(s) := v(s) \text{ for all } s \in \prod_{i \in N} m_i \text{ with } s \leq t$$

and

$$w(s) := v(s + t) - v(t) \text{ for all } s \in \prod_{i \in N} M_i \text{ with } s \leq m - t.$$

We also split up x into two parts, $x^u : \{(i, j) \mid i \in N,\ j \in \{0, \ldots, t_i\}\} \to \mathbb{R}$ and $x^w : \{(i, j) \mid i \in N,\ j \in \{0, \ldots, m_i - t_i\}\} \to \mathbb{R}$ defined by

$$x^u_{i,j} := x_{ij} \text{ for all } i \in N \text{ and } j \in \{0, \ldots, t_i\}$$

and

$$x^w_{ij} := \begin{cases} x_{i,j+t_i} & \text{if } i \in N \text{ and } j \in \{1, \ldots, m_i - t_i\} \\ 0 & \text{if } i \in N \text{ and } j = 0. \end{cases}$$

Now $x^u \in \widetilde{C}(u)$, because $X^u(t) = X(t) = v(t) = u(t)$ and $X^u(s) = X(s) \geq v(s) = u(s)$ for all $s \in \prod_{i \in N} M_i$ with $s \leq t$. Further, $x^w \in \widetilde{C}(w)$, because

$$X^w(m - t) = \sum_{i \in N} \sum_{j=1}^{m_i - t_i} x_{i,j+t_i} = X(m) - X(t) = v(m) - v(t) = w(m - t)$$

and

$$X^w(s) = \sum_{i \in N} \sum_{j=1}^{s_i} x_{i,j+t_i} = X(s + t) - X(t) \geq v(s + t) - v(t) = w(s)$$

for all $s \in \prod_{i \in N} M_i$ with $s \leq m - t$.

Hence, using the induction hypothesis, we can find $y^u \in W(u)$ such that y^u is weakly smaller than x^u and $y^w \in W(w)$ such that y^w is weakly smaller than x^w.

For the payoff vector $z^1 : \{(i, j) \mid i \in N, j \in \{0, \ldots, t_i\}\} \to \mathbb{R}$ for u and the payoff vector $z^2 : \{(i, j) \mid i \in N, j \in \{0, \ldots, m_i - t_i\}\} \to \mathbb{R}$ for w we define the payoff vector $(z^1, z^2) : M \to \mathbb{R}$ for v as follows:

$$(z^1, z^2)_{ij} := \begin{cases} z^1_{ij} & \text{if } i \in N \text{ and } j \in \{0, \ldots, t_i\} \\ z^2_{i,j-t_i} & \text{if } i \in N \text{ and } j \in \{t_i + 1, \ldots, m_i\}. \end{cases}$$

Then $y := (y^u, y^w)$ is weakly smaller than $x = (x^u, x^w)$. Hence, the only thing to prove yet is that $y \in W(v)$.

We prove that $(W(u), W(w)) := \{(z^1, z^2) \mid z^1 \in W(u),\ z^2 \in W(w)\}$ is a subset of $W(v)$. Note that $(W(u), W(w))$ and $W(v)$ are convex sets. Hence, it suffices to prove that the extreme points of $(W(u), W(w))$ are elements of $W(v)$. Suppose (z^1, z^2) is an extreme point of $(W(u), W(w))$. Then, obviously, z^1 is a marginal vector of u and z^2 is a marginal vector of w. Let σ be an admissible permutation for u and ρ an admissible permutation for w such that z^1 is the marginal vector of u corresponding to σ and z^2 is the marginal vector of w corresponding to ρ. Then (z^1, z^2) is the marginal vector of v corresponding to the admissible permutation τ for v defined by

$$\tau((i,j)) := \begin{cases} \sigma((i,j)) & \text{if } i \in N \text{ and } j \in \{1, \ldots, t_i\} \\ \rho((i, j-t_i)) + \sum_{i \in N} t_i & \text{if } i \in N \text{ and } j \in \{t_i + 1, \ldots, m_i\}. \end{cases}$$

Hence, $(z^1, z^2) \in W(v)$ and this completes the proof. □

For the class of convex multi-choice games we can say more about the relation between the core and the Weber set.

Definition. A multi-choice game v is called *convex* if

$$v(s \wedge t) + v(s \vee t) \geq v(s) + v(t) \tag{10}$$

for all $s, t \in \prod_{i \in N} M_i$. Here

$$(s \wedge t)_i := \min\{s_i, t_i\} \text{ and } (s \vee t)_i := \max\{s_i, t_i\}$$

for all $i \in N$.

For a convex game v it holds that

$$v(s + t) - v(s) \geq v(\overline{s} + t) - v(\overline{s}) \tag{11}$$

for all $s, \overline{s}, t \in \prod_{i \in N} M_i$ satisfying $\overline{s} \leq s$, $\overline{s}_i = s_i$ for all $i \in C(t)$ and $s + t \in \prod_{i \in N} M_i$. This is seen by putting s and $\overline{s} + t$ in the roles of s and t in expression (10). In fact, every game satisfying expression (11) is convex, but we do not need this fact.

Theorem 8. Let v be a convex multi-choice game. Then the Weber set $W(v)$ is a subset of the core $C(v)$.

Proof. Let σ be an admissible bijection for v. Note that it suffices to prove $w^\sigma \in C(v)$. Efficiency of w^σ follows immediately from its definition. That w^σ is level increase rational follows straightforwardly when we use expression (11). Now let $s \in \prod_{i \in N} M_i$. The bijection σ induces an admissible bijection $\sigma' : \{(i,j) \mid i \in N,\ j \in \{1, \ldots, s_i\}\} \to \{1, \ldots, \sum_{i \in N} s_i\}$ in an obvious way. Since $p(\sigma', (i,j)) \leq p(\sigma, (i,j))$ for all $i \in N$ and $j \in \{1, \ldots, s_i\}$, convexity of v implies

$$w_{ij}^{\sigma'} \leq w_{ij}^{\sigma}$$

for all $i \in N$ and $j \in \{1, \ldots, s_i\}$. Hence,

$$\sum_{i \in N} \sum_{j=0}^{s_i} w_{ij}^{\sigma} \geq \sum_{i \in N} \sum_{j=0}^{s_i} w_{ij}^{\sigma'} = v(s).$$

We conclude that $w^{\sigma} \in C(v)$. □

The converse of theorem 8 is not true. We provide an example of a convex game v with $W(v) \subsetneq C(v)$.

Example 3. Let (N, m, v) be the multi-choice game where $N = \{1, 2\}$, $m = (2, 1)$ and v is the convex game defined by $v((1,0)) = v((2,0)) = v((0,1)) = 0$, $v((1,1)) = 2$ and $v((2,1)) = 3$. There are three marginal vectors,

$$w_1 = \begin{bmatrix} 0 & 0 \\ 3 & * \end{bmatrix},\ w_2 = \begin{bmatrix} 0 & 1 \\ 2 & * \end{bmatrix} \text{ and } w_3 = \begin{bmatrix} 2 & 1 \\ 0 & * \end{bmatrix}.$$

Some calculation shows that $C(v) = co\{w_1, w_2, w_3, x\}$, where

$$x = \begin{bmatrix} 3 & 0 \\ 0 & * \end{bmatrix}$$

We see that $x \notin co\{w_1, w_2, w_3\} = W(v)$.

The core element x in example 3 seems to be too large: note that w_3 is weakly smaller than x and w_3 is still in the core $C(v)$. This inspires to the following

Definition. For a multi-choice game v the set $C_{\min}(v)$ of *minimal core elements* is defined as follows

$$C_{\min}(v) := \{x \in C(v) \mid \text{ there is no } y \in C(v) \text{ such that } y \neq x \text{ and } y \text{ is weakly smaller than } x\}.$$

Now we can formulate

Theorem 9. Let v be a convex multi-choice game. Then the Weber set $W(v)$ is the convex hull of the set $C_{\min}(v)$ of minimal core elements.

Proof. We start by proving that all marginal vectors are minimal core elements. Let σ be an admissible bijection for v. Then $w^{\sigma} \in C(v)$ (cf. theorem 8). Suppose $y \in C(v)$ is such that $y \neq w^{\sigma}$ and y is weakly smaller than w^{σ}. Let $i \in N$ and $j \in M_i \backslash \{0\}$ be such that $Y(je^i) < \sum_{\ell=1}^{j} w_{i\ell}^{\sigma}$ and consider $t := p(\sigma, (i,j)) + e^i$. Then

$$Y(t) = \sum_{k \in N} Y(t_k e^k) < \sum_{k \in N} \sum_{\ell=0}^{t_k} w_{k\ell}^{\sigma} = v(t), \tag{12}$$

where the inequality follows from the fact that $t_i = j$ and the last equality follows from the definitions of t and w^σ. Now (12) implies that $y \notin C(v)$. Hence, we see that $w^\sigma \in C_{\min}(v)$. This immediately implies that

$$W(v) \subseteq co(C_{\min}(v)). \tag{13}$$

Now let x be a minimal core element. We prove that $x \in W(v)$. According to theorem 7 we can find a payoff vector $y \in W(v)$ that is weakly smaller than x. Using (13) we see that $y \in co(C_{\min}(v)) \subseteq C(v)$. Since x is minimal we may conclude that $x = y \in W(v)$. Hence, $W(v) = co(C_{\min}(v))$. □

Note that theorem 9 implies that for a convex cooperative game (N, v) the core $C(v)$ equals the Weber set $W(v)$. Instead of concentrating on the convex hull of the marginal vectors of a multi-choice game, we can also consider the average of the marginal vectors of a game. For cooperative games this will give us the Shapley value.

Definition. Let (N, m, v) be a multi-choice game. Then the *extended Shapley value* $\Phi(N, m, v)$ is the average of all marginal vectors of v, or, in formula

$$\Phi(N, m, v) := \frac{\prod_{i \in N}(m_i!)}{(\sum_{i \in N} m_i)!} \cdot \sum_\sigma w^\sigma,$$

where the sum is taken over all admissible permutations for v. An interesting question now is if this value does have anything to do with the values that *Chih-Ru Hsiao* and *Raghavan* (1990) defined, starting with axioms that are analogues of axioms that characterize the Shapley value for cooperative games. We found an example of a multi-choice game for which the extended Shapley value does not equal any of the values of *Chih-Ru Hsiao* and *Raghavan* (1990). For this example we refer to a future paper.

4. STABLE SETS AND SUBSOLUTIONS

In section 2 we introduced the notion of dominance between imputations. The dominance core was defined using this notion: $DC(v)$ is the set of undominated imputations. In this section we introduce some other sets of payoff vectors for multi-choice games which are based on the notion of domination, stable sets and subsolutions.

Let $(N, m, v) \in MC^N$ and let $2^{I(v)} := \{A \mid A \subseteq I(v)\}$. We introduce two maps, $D : 2^{I(v)} \to 2^{I(v)}$ and $U : 2^{I(v)} \to 2^{I(v)}$, where

$$D(A) := \{x \in I(v) \mid \text{ there exists an } a \in A \text{ that dominates } x\}$$

and

$$U(A) := I(v) \backslash D(A)$$

for all $A \subseteq I(v)$. The set $D(A)$ consists of all imputations that are dominated by some element of A. The set $U(A)$ consists of all imputations that are undominated by elements of A. Hence,

$$DC(v) = U(I(v)).$$

A set $A \subseteq I(v)$ is *internally stable* if elements of A do not dominate each other, i.e. $A \cap D(A) = \emptyset$, and it is *externally stable* if all imputations not in A are dominated by an imputation in A, i.e. $I(v) \backslash A \subseteq D(A)$.

Definition (cf. *von Neumann and Morgenstern* (1944)). A set $A \subseteq I(v)$ is called a *stable set* if it is both internally and externally stable. It is easily seen that a set $A \subseteq I(v)$ is stable if and only if A is a fixed point of U, i.e. $U(A) = A$.

The extension of the following theorem from cooperative games towards multi-choice games is straightforward and therefore we will omit the proof.

Theorem 10. Let v be a multi-choice game. Then the following two assertions hold:

(i) Every stable set contains the dominance core as a subset.

(ii) If the dominance core is a stable set, then there are no other stable sets.

Lucas (1966) showed that there exist cooperative games without a stable set. Therefore, since all our definitions are consistent with the corresponding definitions for cooperative games, we may conclude that multi-choice games do not always have a stable set.

Now let $A \subseteq I(v)$ and $x \in I(v)$. Then *x is protected by A* if each imputation that dominates x is on its turn dominated by an element of A. The set A is *self-protecting* if each $a \in A$ is protected by A. It is not hard to see that for all $A \subseteq I(v)$

$$\{x \in I(v) \mid x \text{ is protected by } A\} = U^2(A).$$

Definition (cf. *Roth* (1976)). A set $A \subseteq I(v)$ is called a *subsolution* if it is internally stable, self-protecting and does not protect any element of $I(v) \backslash A$. It is easily seen that a set A is a subsolution if and only if it is internally stable and a fixed point of U^2.

The following theorem is a straightforward extension of a corresponding theorem of *Roth* (1976) for cooperative games.

Theorem 11. Let v be a multi-choice game. Then the following assertions hold:

(i) Each stable set is a subsolution.

(ii) A subsolution cannot contain a stable set as a proper subset.

(iii) Every subsolution contains the dominance core as a subset.

(iv) Every multi-choice game has at least one subsolution.

Let N be a set of players and let $m \in (\mathbb{N} \cup \{0\})^N$. A flow situation consists of a directed network with one source and one sink and for each arc its capacity and a simple multi-choice game, the control game, that describes which coalitions are allowed to use the arc.

Definition. A multi-choice game v is called *simple* if $v(s) \in \{0,1\}$ for all $s \in \prod_{i \in N} M_i$ and $v(m) = 1$.

If ℓ is an arc in the network and w is the (simple) control game for arc ℓ, then a coalition s is allowed to use arc ℓ if and only if $w(s) = 1$. The capacity of an arc ℓ in the network is denoted by $c_\ell \in (0, \infty)$. The flow game corresponding to a flow situation assigns to a coalition s the maximal flow that coalition s can send through the network from the source to the sink.

For cooperative games it was shown by *Curiel, Derks* and *Tijs* (1989) that a non-negative game is balanced if and only if it is a flow game corresponding to a flow situation in which all control games are balanced. Furthermore, *Kalai* and *Zemel* (1982) proved that a non-negative cooperative game is totally balanced if and only if it is a flow game corresponding to a flow situation in which all arcs are controlled by a single player. Here, a cooperative game is called totally balanced if all its subgames are balanced. Example 4 shows that we cannot generalize the theorems mentioned above to multi-choice games.

Definition. A simple multi-choice game v is called *dictatorial* if there exist $i \in N$ and $j \in M_i \backslash \{0\}$ such that $v(s) = 1$ if and only if $s_i \geq j$ for all $s \in \prod_{i \in N} M_i$.
A multi-choice game v is called *totally balanced* if for every $s \in \prod_{i \in N} M_i$ the subgame $(N, s, v|_s)$ is balanced, where $v|_s(t) := v(t)$ for all $t \in \prod_{i \in N} M_i$ with $t \leq s$.

Example 4. Let (N, m, v) be the multi-choice game where $N = \{1, 2\}$, $m = (2, 1)$ and v is the flow game corresponding to the flow situation with the directed network as represented in figure 1, the capacities $c_{\ell_1} = 2$, $c_{\ell_2} = c_{\ell_3} = c_{\ell_4} = 1$ and the control games $w_{\ell_1}, \ldots, w_{\ell_4}$ defined by

$$w_{\ell_1}(s) = w_{\ell_2}(s) := \begin{cases} 1 & \text{if } s_1 \geq 1 \\ 0 & \text{else,} \end{cases}$$

$$w_{\ell_3}(s) := \begin{cases} 1 & \text{if } s_1 \geq 2 \\ 0 & \text{else} \end{cases}$$

and

$$w_{\ell_4}(s) := \begin{cases} 1 & \text{if } s_2 \geq 1 \\ 0 & \text{else.} \end{cases}$$

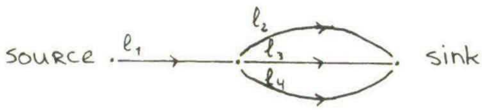

Figure 1

The games $w_{\ell_1}, \ldots, w_{\ell_4}$ are dictatorial. We calculate $v((0,1)) = 0$, $v((1,0)) = 1$, and $v((2,0)) = v((1,1)) = v((2,1)) = 2$. It is easily seen that v is not even balanced.

In order to avoid the difficulties we have in example 4, we restrict ourselves to zero-normalized games. Then we have the following

Theorem 12. Consider a flow situation in which all control games are zero-normalized and balanced. Then the corresponding flow game is non-negative, zero-normalized and balanced.

Proof. It is obvious that v is zero-normalized and non-negative. Now let $L = \{\ell_1, \ldots, \ell_p\}$ be a set of arcs with capacities $c_1, \ldots, c_p$ and control games $w_1, \ldots, w_p$ such that every directed path from the source to the sink contains an arc in L and the capacity of L, $\sum_{r=1}^{p} c_r$, is minimal. From a theorem of ***Ford*** and ***Fulkerson*** (1956) we find that $v(m) = \sum_{r=1}^{p} c_r$ and $v(s) \leq \sum_{r=1}^{p} c_r w_r(s)$ for all $s \in \prod_{i \in N} M_i$. Now let $x^r \in C(w_r)$ for all $r \in \{1, \ldots, p\}$. Define $y := \sum_{r=1}^{p} c_r x^r$. Then

$$Y(m) = \sum_{r=1}^{p} c_r X^r(m) = \sum_{r=1}^{p} c_r w_r(m) = v(m) \tag{14}$$

and

$$Y(s) = \sum_{r=1}^{p} c_r X^r(s) \geq \sum_{r=1}^{p} c_r w_r(s) \geq v(s), \tag{15}$$

for all $s \in \prod_{i \in N} M_i$.

Now let $i \in N$ and $j \in M_i \backslash \{0\}$. Since $c_r \geq 0$ and $x_{ij}^r \geq 0$ for all $r \in \{1, \ldots, p\}$ it easily follows that

$$y_{ij} = \sum_{r=1}^{p} c_r x_{ij}^r \geq 0. \tag{16}$$

Now (14), (15) and (16) imply that $y \in C(v)$. Hence, v is balanced. □

We can prove the converse of theorem 12 using

Theorem 13. Each non-negative zero-normalized balanced multi-choice game is a non-negative linear combination of zero-normalized balanced simple games.

Proof. Let v be a non-negative zero-normalized balanced game. We provide an algorithm to write v as a non-negative linear combination of zero-normalized balanced simple games.

Suppose $v \neq 0$ and let $x \in C(v)$. Let $k \in N$ be the smallest integer in

$\{i \in N \mid \text{there exists a } j \in M_i \backslash \{0\} \text{ such that } x_{ij} > 0\}$

and let ℓ be the smallest integer in $\{j \in M_k \backslash \{0\} \mid x_{kj} > 0\}$. Further,

$$\beta := \min\{\ x_{k\ell},\ \min\{\ v(s) \mid s \in \prod_{i \in N} M_i, s_k \geq \ell,\ v(s) > 0\}\}$$

and

$$w(s) := \begin{cases} 1 & \text{if } s_k \geq \ell \text{ and } v(s) > 0 \\ 0 & \text{else.} \end{cases}$$

Let $\overline{v} := v - \beta w$ and let $\overline{x} : M \to \mathbb{R}$ be defined by

$$\overline{x}_{ij} := \begin{cases} x_{k\ell} - \beta & \text{if } i = k \text{ and } j = \ell \\ x_{ij} & \text{else.} \end{cases}$$

Then w is a zero-normalized balanced simple game and $\beta > 0$. Furthermore, $\overline{v}$ is a non-negative zero-normalized game and $\overline{x} \in C(\overline{v})$. Note that $v = \overline{v} + \beta w$. Further,

$|\{(i,j) \in M \mid \overline{x}_{ij} > 0\}| < |\{(i,j) \in M \mid x_{ij} > 0\}|$ or

$|\{s \in \prod_{i \in N} M_i \mid \overline{v}(s) > 0\}| < |\{s \in \prod_{i \in N} M_i \mid v(s) > 0\}|$.

If $\overline{v} \neq 0$ we can follow the same procedure with $\overline{v}$ in the role of v and $\overline{x}$ in the role of x. It is easily seen that if we keep on repeating this procedure, then after only finitely many steps we will obtain the zero game. Suppose this happens after q steps. Then we have found $\beta_1, \ldots, \beta_q > 0$ and zero-normalized balanced simple games $w_1, \ldots, w_q$ such that

$$v = \sum_{r=1}^{q} \beta_r w_r. \qquad \square$$

The algorithm we described in the proof of theorem 13 is a generalization of an algorithm by *Derks* (1987) for traditional cooperative games.

Theorem 14. Let v be a non-negative zero-normalized balanced game. Then v is a flow game corresponding to a flow situation in which all control games are zero-normalized and balanced.

Proof. According to theorem 13 we can find $k \in \mathbb{N}$, $\beta_1, \ldots, \beta_k > 0$ and zero-normalized balanced simple games $w_1, \ldots, w_k$ such that

$$v = \sum_{r=1}^{k} \beta_r w_r.$$

Consider the flow situation with k arcs as represented in figure 2,

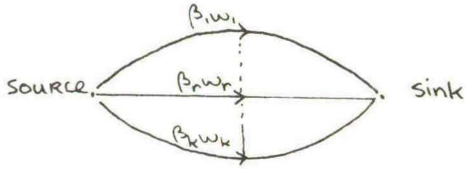

Figure 2

where for each $r \in \{1, \ldots, k\}$ the capacity restriction of arc ℓ_r is given by β_r and the control game of ℓ_r is w_r. It is easily seen that the flow game corresponding to the flow situation described is the game v. □

Combining theorems 12 and 14 we obtain

Corollary 1. Let v be a non-negative zero-normalized multi-choice game. Then v is balanced if and only if v is a flow game corresponding to a flow situation in which all control games are zero-normalized and balanced.

REFERENCES

Bondareva ON (1963) Certain applications of the methods of linear programming to the theory of cooperative games (In Russian). Problemy Kibernetiki 10, 119-139.

Chih-Ru Hsiao, Raghavan TES (1990) Shapley value for multi-choice cooperative games (I). Discussion paper of the University of Illinois at Chicago, Chicago.

Curiel I, Derks J, Tijs S (1989) On balanced games and games with committee control. OR Spektrum 11, 83-88.

Derks JJM (1986) Personal communication.

Derks JJM (1987) Decomposition of games with non-empty cores into veto-controlled simple games. OR Spektrum 9, 81-85.

Ford LR, Fulkerson DR (1956) Maximal flow through a network. Canad J of Math 8, 399-404.

Ichiishi T (1983) Game theory for economic analysis. Academic press, New York.

Kalai E, Zemel E (1982) Totally balanced games and games of flow. Math of Oper Res 7, 476-479.

Lucas WF (1966) n-Person games with only 1, $n-1$ and n-person coalitions. Z Wahr ver Geb 6, 287-292.

von Neumann J, Morgenstern O (1944) Theory of games and economic behavior. Princeton University Press, Princeton New Jersey.

Roth A (1976) Subsolutions and the supercore of cooperative games. Math Oper Res 1, 43-49.

Shapley LS (1967) On balanced sets and cores. Nav Res Log Quart 14, 453-460.

Shapley LS (1971) Cores of convex games. Int J of Game Theory 1, 11-26.

Weber RJ (1988) Probabilistic values for games. In: The Shapley value, Roth AE (Ed.), Cambridge University Press, Cambridge, 101-119.

IN 1990 REEDS VERSCHENEN

419 Bertrand Melenberg, Rob Alessie
A method to construct moments in the multi-good life cycle consumption model

420 J. Kriens
On the differentiability of the set of efficient (μ,σ^2) combinations in the Markowitz portfolio selection method

421 Steffen Jørgensen, Peter M. Kort
Optimal dynamic investment policies under concave-convex adjustment costs

422 J.P.C. Blanc
Cyclic polling systems: limited service versus Bernoulli schedules

423 M.H.C. Paardekooper
Parallel normreducing transformations for the algebraic eigenvalue problem

424 Hans Gremmen
On the political (ir)relevance of classical customs union theory

425 Ed Nijssen
Marketingstrategie in Machtsperspectief

426 Jack P.C. Kleijnen
Regression Metamodels for Simulation with Common Random Numbers: Comparison of Techniques

427 Harry H. Tigelaar
The correlation structure of stationary bilinear processes

428 Drs. C.H. Veld en Drs. A.H.F. Verboven
De waardering van aandelenwarrants en langlopende call-opties

429 Theo van de Klundert en Anton B. van Schaik
Liquidity Constraints and the Keynesian Corridor

430 Gert Nieuwenhuis
Central limit theorems for sequences with m(n)-dependent main part

431 Hans J. Gremmen
Macro-Economic Implications of Profit Optimizing Investment Behaviour

432 J.M. Schumacher
System-Theoretic Trends in Econometrics

433 Peter M. Kort, Paul M.J.J. van Loon, Mikulás Luptacik
Optimal Dynamic Environmental Policies of a Profit Maximizing Firm

434 Raymond Gradus
Optimal Dynamic Profit Taxation: The Derivation of Feedback Stackelberg Equilibria

435 Jack P.C. Kleijnen
Statistics and Deterministic Simulation Models: Why Not?

436 M.J.G. van Eijs, R.J.M. Heuts, J.P.C. Kleijnen
Analysis and comparison of two strategies for multi-item inventory systems with joint replenishment costs

437 Jan A. Weststrate
Waiting times in a two-queue model with exhaustive and Bernoulli service

438 Alfons Daems
Typologie van non-profit organisaties

439 Drs. C.H. Veld en Drs. J. Grazell
Motieven voor de uitgifte van converteerbare obligatieleningen en warrantobligatieleningen

440 Jack P.C. Kleijnen
Sensitivity analysis of simulation experiments: regression analysis and statistical design

441 C.H. Veld en A.H.F. Verboven
De waardering van conversierechten van Nederlandse converteerbare obligaties

442 Drs. C.H. Veld en Drs. P.J.W. Duffhues
Verslaggevingsaspecten van aandelenwarrants

443 Jack P.C. Kleijnen and Ben Annink
Vector computers, Monte Carlo simulation, and regression analysis: an introduction

444 Alfons Daems
"Non-market failures": Imperfecties in de budgetsector

445 J.P.C. Blanc
The power-series algorithm applied to cyclic polling systems

446 L.W.G. Strijbosch and R.M.J. Heuts
Modelling (s,Q) inventory systems: parametric versus non-parametric approximations for the lead time demand distribution

447 Jack P.C. Kleijnen
Supercomputers for Monte Carlo simulation: cross-validation versus Rao's test in multivariate regression

448 Jack P.C. Kleijnen, Greet van Ham and Jan Rotmans
Techniques for sensitivity analysis of simulation models: a case study of the CO_2 greenhouse effect

449 Harrie A.A. Verbon and Marijn J.M. Verhoeven
Decision-making on pension schemes: expectation-formation under demographic change

IN 1991 REEDS VERSCHENEN

466 Prof.Dr. Th.C.M.J. van de Klundert - Prof.Dr. A.B.T.M. van Schaik
Economische groei in Nederland in een internationaal perspectief

467 Dr. Sylvester C.W. Eijffinger
The convergence of monetary policy - Germany and France as an example

468 E. Nijssen
Strategisch gedrag, planning en prestatie. Een inductieve studie binnen de computerbranche

469 Anne van den Nouweland, Peter Borm, Guillermo Owen and Stef Tijs
Cost allocation and communication

470 Drs. J. Grazell en Drs. C.H. Veld
Motieven voor de uitgifte van converteerbare obligatieleningen en warrant-obligatieleningen: een agency-theoretische benadering

471 P.C. van Batenburg, J. Kriens, W.M. Lammerts van Bueren and R.H. Veenstra
Audit Assurance Model and Bayesian Discovery Sampling

472 Marcel Kerkhofs
Identification and Estimation of Household Production Models

473 Robert P. Gilles, Guillermo Owen, René van den Brink
Games with Permission Structures: The Conjunctive Approach

474 Jack P.C. Kleijnen
Sensitivity Analysis of Simulation Experiments: Tutorial on Regression Analysis and Statistical Design

475 An $O(nlogn)$ algorithm for the two-machine flow shop problem with controllable machine speeds
C.P.M. van Hoesel

476 Stephan G. Vanneste
A Markov Model for Opportunity Maintenance

477 F.A. van der Duyn Schouten, M.J.G. van Eijs, R.M.J. Heuts
Coordinated replenishment systems with discount opportunities