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RESEARCH MEMORANDUM

EXTENSIONS OF THE τ-VALUE TO NTU-GAMES

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EXTENSIONS OF THE τ-VALUE TO NTU-GAMES

Saskia Oortwijn[1)] Peter Borm[2)] Hans Keiding[3)] and Stef Tijs[1)]

Abstract. This paper introduces two extensions of the τ-value for TU-games towards NTU-games: the compromise value and the NTU τ-value.

The compromise value is based upon upper and lower bounds for the core and coincides with the Kalai-Smorodinsky solution for two-person bargaining games. The NTU τ-value is defined in analogy to the NTU (Shapley)-value and coincides with the Nash solution for two-person bargaining games.

Both values are computed for the Roth-Schafer examples. For special classes of NTU-games existence is shown and an axiomatic characterization of the compromise value is provided.

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I. INTRODUCTION

The Shapley value of TU (=Transferable Utility)-games, introduced by *Shapley* (1953), has been generalized towards NTU (=Non Transferable Utility)-games in various ways. *Shapley* (1969) defined the NTU-value, *Harsanyi* (1959, 1963), *Owen* (1971) and *Imai* (1983) considered other possible extensions. For the NTU-value an axiomatic characterization has been provided by *Aumann* (1985a) and *Kern* (1985), for the Harsanyi solution by *Hart* (1985a) and for monotonic solutions by *Kalai* and *Samet* (1985).

This paper introduces two extensions of the τ-value of *Tijs* (1981) for quasi-balanced TU-games towards special subclasses of NTU-games: the compromise value and the NTU τ-value. The compromise value is defined in section 3 for the class of compromise admissible NTU-games. It is a one-point solution concept that is based upon the upper and lower bounds for the core of an NTU-game that are given in section 2. An axiomatic characterization is provided and it is shown that the compromise value coincides with the solution of *Kalai* and *Smorodinsky* (1975) for the special case of two-person bargaining games.

Section 4 introduces the NTU τ-value as a multi-valued solution concept that is defined by means of the τ-value of corresponding (quasi-balanced) λ-transfer games. The procedure is analogous to the definition of the NTU-value with respect to the Shapley value. Also similar to the NTU-value, we show existence of the NTU τ-value for a class of compactly generated NTU-games.

Roth (1980) and *Shafer* (1980) introduced two special classes of games for which, in their opinion, the NTU-value leads to a strange and counterintuitive outcome. This led to an interesting discussion of the NTU-value in the papers of *Aumann* (1985b, 1986), *Roth* (1986) and *Hart* (1985b). In the present paper the Roth-Shafer examples are discussed in detail and it is seen that the alternative outcomes of Roth and Shafer come forward if one applies the compromise value or the NTU τ-value.

Notations. For $x, y \in \mathbf{R}^n$, let $x \geq (>)\ y$ if and only if $x_i \geq (>)\ y_i$ for all $i \in \{1, \ldots, n\}$, $xy := \sum_{i=1}^{n} x_i y_i \in \mathbf{R}$ and $x * y := (x_1 y_1, \ldots, x_n y_n) \in \mathbf{R}^n$. $\mathbf{R}^n_+ := \{x \in \mathbf{R}^n | x \geq 0\}$.
For $C, D \subset \mathbf{R}^n$ the algebraic sum $C + D \subset \mathbf{R}^n$ is given by

$$C + D := \{c + d \in \mathbf{R}^n | c \in C, d \in D\},$$

and $\text{Aff}(C)$ ($\text{Conv}(C)$) denotes the affine (convex) hull of C.
Finally, with $N := \{1, \ldots, n\}$, $S \subset N$ and $x \in \mathbf{R}^n$, $x_S := (x_i)_{i \in S} \in \mathbf{R}^S$ and x is identified with $(x_S, x_{N \setminus S})$.

2. NTU-GAMES: BOUNDS FOR THE CORE

An *NTU-game* is a pair (N, V) where $N := \{1, 2, \ldots, n\}$ is the set of players and V is a set-valued function that assigns to each *coalition* $S \in 2^N \backslash \{\varnothing\}$ a non-empty set $V(S) \subset \mathbf{R}^S$ of *attainable* payoff vectors. For each player $i \in N$ we assume there is a *individual rational* payoff $v(i) \in \mathbf{R}$ such that $V(\{i\}) = \{a \in \mathbf{R} \mid a \leq v(i)\}$ while, for each $S \in 2^N \backslash \{\varnothing\}$,

(i) $V(S)$ is closed and comprehensive (i.e. if $a \in V(S)$ and $b \in \mathbf{R}^S$ is such that $b \leq a$, then $b \in V(S)$).

(ii) $V(S) \cap \{a \in \mathbf{R}^S \mid a \geq \left(v(j)\right)_{j \in S}\}$ is bounded.

An NTU-game (N, V) will be often identified with V. The *core* $C(V)$ consists of those attainable payoff vectors for the grand coalition N which are stable with respect to (strict) domination. More specifically, with

$$\text{Dom}\,(V(S)) := \{a \in \mathbf{R}^S \mid \exists_{b \in V(S)} : b > a\} \tag{1}$$

representing the set of dominated payoff vectors for a coalition $S \in 2^N \backslash \{\varnothing\}$,

$$C(V) := \{a \in V(N) \mid \neg \exists_{S \in 2^N \backslash \{\varnothing\}} : a_S \in \text{Dom}\,(V(S))\}. \tag{2}$$

Let $i \in N$. Assuming that the coalition $N \backslash \{i\}$ will never agree with a payoff vector $a \in \mathbf{R}^{N \backslash \{i\}}$ with $a \in \text{Dom}\,(V(N \backslash \{i\}))$ or $a_j < v(j)$ for some $j \in N \backslash \{i\}$, the highest possible marginal contribution of player i by joining the coalition $N \backslash \{i\}$ is given by

$$\begin{aligned} K_i(V) := \sup\{t \in \mathbf{R} \mid \exists_{a \in \mathbf{R}^{N \backslash \{i\}}} :&(a, t) \in V(N), a \notin \text{Dom}\,(V(N \backslash \{i\})) \\ &\text{and } a \geq (v(j))_{j \in N \backslash \{i\}}\}. \end{aligned} \tag{3}$$

$K_i(V)$ is called the ***utopia payoff*** of player i. By assumption (ii) in the definition of an NTU-game we have that $K_i(V) < \infty$. However, $K_i(V) = -\infty$ might occur.

Assume $K_j(V) \in \mathbf{R}$ for all $j \in N$ and consider a coalition S to which player i belongs. The formation of such a coalition is attractive for a player $j \in S \backslash \{i\}$ if he gets (slightly) more that the utopia payoff $K_j(V)$. Thus, player i can lay a rightful claim on the *remainder* $\rho_i^S(V)$ which is given by

$$\rho_i^S(V) := \sup\{t \in \mathbf{R} \mid \exists_{a \in \mathbf{R}^{S \backslash \{i\}}} : (t, a) \in V(S) \text{ and } a > K_{S \backslash \{i\}}(V)\}. \tag{4}$$

Among the 2^{n-1} possible coalitions with $i \in S$, player i can choose one where this remainder is maximal. Let

$$k_i(V) := \max_{S: i \in S} \rho_i^S(V) \tag{5}$$

denote the *minimal right* of player i. Clearly $k_i(V) \geq v(i)$, but it might occur that $k_i(V) = \infty$. In this paper we concentrate on NTU-games for which all utopia payoffs and minimal rights for the various players are real numbers. In particular, this is the case for NTU-games with a non-empty core: theorem 1 shows that $K(V) = (K_j(V))_{j\in N}$ and $k(V) = (k_j(V))_{j\in N}$ establish an upper and lower bound for the core, respectively.

THEOREM 1. Let (N, V) be an NTU-game with $x \in C(V)$. Then

$$k(V) \leq x \leq K(V).$$

Proof. Obviously, (2) and (3) imply

$$K_j(V) \geq \sup\{t \in \mathbf{R} \mid \exists_{a\in\mathbf{R}^{N\setminus\{j\}}} : (a,t) \in C(V)\} \geq x_j$$

for all $j \in N$. Hence, $x \leq K(V)$.
Let $i \in N$ and choose a coalition $T \ni i$ such that $k_i(V) = \rho_i^T(V) = \max_{S:i\in S} \rho_i^S(V)$.
Suppose $k_i(V) > x_i$. Then we can choose $\varepsilon > 0$ such that $k_i(V) > x_i + \varepsilon$. Further, by (4), there exists a vector $a \in \mathbf{R}^{T\setminus\{i\}}$ such that $(x_i + \varepsilon, a) \in V(T)$ and $a > K_{T\setminus\{i\}}(V)$.
However, this would imply that

$$x_T \leq (x_i, K_{T\setminus\{i\}}(V)) < (x_i + \varepsilon, a) \in V(T),$$

which contradicts the fact that $x \in C(V)$. Hence, $k(V) \leq x$. □

The vectors $k(V)$ and $K(V)$ induce familar bounds for TU-games and two-person bargaining games.
(a) **TU-games.** A *TU-game* is a pair (N, v) where v is a function that assigns to each coalition S a real number $v(S)$ with $v(\varnothing) = 0$. The core $C(v)$ is defined by
$C(v) := \{a \in \mathbf{R}^N \mid \sum_{i\in N} a_i = v(N), \sum_{i\in S} a_i \geq v(S) \text{ for all } S \subset N\}$.
For a TU-game (N, v), *Tijs* (1981) introduced a utopia vector $M(v) \in \mathbf{R}^N$ and a minimal right vector $m(v) \in \mathbf{R}^N$ as follows. For $i \in N$,

$$M_i(v) := v(N) - v(N\setminus\{i\}) \text{ and } m_i(V) := \max_{S:i\in S}(v(S) - \sum_{j\in S\setminus\{i\}} M_j(v)). \tag{6}$$

For $x \in C(v)$, it was shown that $m(v) \leq x \leq M(v)$.
Associating an NTU-game (N, V) to a TU-game (N, v) by defining

$$V(S) := \{a \in \mathbf{R}^S \mid \sum_{i\in S} a_i \leq v(S)\} \tag{7}$$

for all $S \in 2^N \backslash \{\emptyset\}$, it is straighforward to verify that $C(v) = C(V)$, and that $M(v) = K(V)$ and $m(v) = k(V)$ if v is such that $v(N\backslash\{i\}) \geq \sum_{j \in N\backslash\{i\}} v(j)$ for all $i \in N$.

(b) **Bargaining games.** In a two-person person *bargaining game* (C,d) the non-empty set $C \subset \mathbf{R}^2$ represents the set of feasible outcomes and $d \in C$ is the disagreement point. Moreover, we assume that the following properties are satisfied:

(i) C is closed, convex and comprehensive.

(ii) There is an $x^0 \in C$ with $x^0 > d$.

(iii) $C_d := \{x \in C \mid x \geq d\}$ is bounded.

For each bargaining game (C,d), *Kalai* and *Smorodinsky* (1975) introduced the utopia point $u(C,d) \in \mathbf{R}^2$ by defining $u_1(C,d) := \max\{a \in \mathbf{R} \mid \exists_{b \in \mathbf{R}} : (a,b) \in C_d\}$; $u_2(C,d)$ is defined analogously. Associating an NTU-game $(\{1,2\},V)$ to a bargaining game (C,d) by setting $V(\{i\}) := \{a \in \mathbf{R} \mid a \leq d_i\}$ for $i \in \{1,2\}$ and $V(\{1,2\}) := C$, one obtains that $u(C,d) = K(V)$ and $d = k(V)$.

The class of games we consider in the following example was first introduced by *Roth* (1980).

EXAMPLE 1. Let $N = \{1,2,3\}$. For a parameter p with $0 \leq p \leq \frac{1}{2}$, the NTU-game (N, V_p) is defined by (the subscripts denote players):

$V_p(\{i\}) = \{a_i \in \mathbf{R} \mid a_i \leq 0\} \qquad (i \in N)$

$V_p(\{1,2\}) = \{(a_1,a_2) \in \mathbf{R}^2 \mid (a_1,a_2) \leq (\frac{1}{2},\frac{1}{2})\}$

$V_p(\{1,3\}) = \{(a_1,a_3) \in \mathbf{R}^2 \mid (a_1,a_3) \leq (p, 1-p)\}$

$V_p(\{2,3\}) = \{(a_2,a_3) \in \mathbf{R}^2 \mid (a_2,a_3) \leq (p, 1-p)\}$

$V_p(\{1,2,3\}) = \{a=(a_1,a_2,a_3) \in \mathbf{R}^3 \mid a \leq b \text{ for some } b \in \mathrm{Conv}\{(\frac{1}{2},\frac{1}{2},0),(p,0,1-p),(0,p,1-p)\}\}$

If $0 \leq p < \frac{1}{2}$, then

$$C(V_p) = \mathrm{Conv}\{(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},p,0)\} \cup \mathrm{Conv}\{(\frac{1}{2},\frac{1}{2},0),(p,\frac{1}{2},0)\},$$

$$\{K(V_p)\} = \{(\frac{1}{2},\frac{1}{2},0)\} \text{ and } k(V_p) = (p,p,0).$$

Further, $(N, V_{\frac{1}{2}})$ is a symmetric game with

$$C(V_{\frac{1}{2}}) = \{(\frac{1}{2},\frac{1}{2},0),(\frac{1}{2},0,\frac{1}{2}),(0,\frac{1}{2},\frac{1}{2})\}, K(V_{\frac{1}{2}}) = (\frac{1}{2},\frac{1}{2},\frac{1}{2}) \text{ and } k(V_{\frac{1}{2}}) = (0,0,0).$$

The class of games in example 1 provoked an interesting discussion between A. Roth and R. Aumann about the credibility and interpretation of the NTU-value as introduced by *Shapley* (1953, 1969). *Roth* (1980 , 1986) argued that the (unique) NTU-value $(\frac{1}{3},\frac{1}{3},\frac{1}{3})$ for all games V_p with $0 < p \leq \frac{1}{2}$ seems rather counterintuitive and that the only reasonable outcome for these games is the core-element $(\frac{1}{2},\frac{1}{2},0)$. Roth's arguments were countered by *Aumann* (1985b, 1986).

One of Aumann's arguments involves the role of safety (of accepting an offer from player 3 by player 1 or 2) as opposed to coordination (between player 1 and 2).
For a detailed discussion we refer to the papers of Aumann and Roth and also to *Hart* (1985a).

3. THE COMPROMISE VALUE

In this section the compromise value is introduced as an extension of the τ-value for quasi-balanced TU-games (cf. *Tijs* (1981)) towards compromise admissible NTU-games.
Here, an NTU-game (N, V) is called *compromise admissible* if the utopia vector $K(V)$ and the minimal right vector $k(V)$ of section 2 satisfy the following two properties:

(i) $k(V) \leq K(V)$.

(ii) $k(V) \in V(N)$, $K(V) \notin \text{Dom}(V(N))$.

By $\mathcal{C}^N$ we denote the does of all compromise admissible NTU-games with player set N. Clearly, we have

LEMMA 2. Every NTU-game with a non-empty core is compromise admissible.

Proof. Let (N, V) be an NTU-game with $x \in C(V)$. Then, using theorem 1, $k(V) \leq x \leq K(V)$. In particular, since $x \in V(N)$, comprehensiveness implies that $k(V) \in V(N)$.
Suppose $K(V) \in \text{Dom}(V(N))$. Then there is an $y \in V(N)$ such that $y > K(V) \geq x$. However, this contradicts the fact that $x \in C(V)$. We may conclude that the conditions (i) and (ii) are satisfied. □

For $V \in \mathcal{C}^N$ the *compromise value* $T(V) \in \mathbf{R}^N$ is defined as the unique vector on the line segment between $k(V)$ and $K(V)$ which lies in $V(N)$ and is closest to the utopia vector $K(V)$. More specifically,

$$T(V) := \lambda_V K(V) + (1 - \lambda_V) k(V), \tag{8}$$

where

$$\lambda_V := \max\{\lambda \in [0,1] \mid \lambda K(V) + (1-\lambda) k(V) \in V(N)\}. \tag{9}$$

Note that λ_V is well-defined because $k(V) \in V(N)$ and $V(N)$ is closed and comprehensive.

EXAMPLE 2. For the games V_p of example 1 with $0 \leq p < \frac{1}{2}$ it follows that the compromise value $T(V_p)$ equals the core element $K(V) = (\frac{1}{2}, \frac{1}{2}, 0)$. Further, one finds that $\lambda_{V_{\frac{1}{2}}} = \frac{2}{3}$. So, for $p = \frac{1}{2}$, the compromise value equals the (unique) NTU-value $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$.

Using the notations of section 2, a TU-game (N, v) is called *quasi-balanced* if $m(v) \leq M(v)$ and $\sum_{i \in N} m_i(v) \leq v(N) \leq \sum_{i \in N} M_i(v)$. For a quasi-balanced TU-game (N, v) the *τ-value* $\tau(v) \in \mathbf{R}^N$

is defined as the unique vector lying on the line segment between $m(v)$ and $M(v)$ which is efficient, i.e. $\sum_{i\in N} \tau_i(v) = v(N)$.

Assume $v(N\backslash\{i\}) \geq \sum_{j\in N\backslash\{i\}} v(j)$ for all $i \in N$ and let V be the NTU-game corresponding to v (cf. (7)). One easily verifies that v is quasi-balanced if and only if V is compromise admissible, and that the τ-value of v coincides with the compromise value of V.

For a two-person bargaining game (C,d) one finds that the compromise value of the corresponding NTU-game V is the unique undominated feasible outcome lying on the line segment between the disagreement point d and the utopia point $u(C,d)$. By definition, this outcome corresponds to the Kalai-Smorodinsky solution for the bargaining game (C,d).

We now provide an axiomatic characterization fo the (compromise) rule $T : \mathcal{C}^N \to \mathbf{R}^N$ à la *Tijs* (1987).

Let $F : \mathcal{C}^N \to \mathbf{R}^N$. The rule F is said to have the *minimal right property* if

$$F(V) = k(V) + F(V - k(V)) \quad \text{for all } V \in \mathcal{C}^N, \tag{10}$$

where the (minimal right reduced) game $(N, V - k(V))$ is defined by

$$(V - k(V))(S) := \{a - k_S(V) \mid a \in V(S)\} \quad \text{for all} \quad S \in 2^N\backslash\{\emptyset\} \tag{11}$$

One easily checks that $V \in \mathcal{C}^N$ implies that $V - k(V) \in \mathcal{C}^N$, while $K(V - k(V)) = K(V) - k(V)$ and $k(V - k(V)) = 0$.

Further, the rule F is called *efficient* if

$$F(V) \in V(N)\backslash\text{Dom}(V(N)) \quad \text{for all } V \in \mathcal{C}^N \tag{12}$$

and F has the *restricted proportionality property* if $F(V)$ is a multiple of the utopia vector $K(V)$ for all $V \in \mathcal{C}^N$ with $k(V) = 0$. These three properties characterize the compromise value.

THEOREM 3. The compromise value T is the unique rule on $\mathcal{C}^N$ that satisfies the minimal right property, efficiency and the restricted proportionality property.

Proof. (a) We first show that the compromise value $T : \mathcal{C}^N \to \mathbf{R}^N$ satisfies the three properties. The minimal right property and the restricted proportionality property are obvious. Let $V \in \mathcal{C}^N$. We show efficiency by proving that $T(V) \in V(N)\backslash\text{Dom}(V(N))$. By definition $T(V) \in V(N)$. Suppose $T(V) \in \text{Dom}(V(N))$. Then there is an $y \in V(N)$ such that $y > T(V)$. Comprehensiveness and the definition of $T(V)$ imply that $T(V) = K(V)$. Choose $i \in N$ with $y_i > K_i(V)$. By comprehensiveness, $(K_{N\backslash\{i\}}(V), y_i) \in V(N)$. Since $K_{N\backslash\{i\}}(V) \geq k_{N\backslash\{i\}}(V) \geq (v(j))_{j\in N\backslash\{i\}}$ and $y_i > K_i(V)$, the definition of $K_i(V)$ (cf. (3)) implies that $K_{N\backslash\{i\}}(V) \in \text{Dom}\ (V(N\backslash\{i\}))$. Let $z \in V(N\backslash\{i\})$ be such that $z > K_{N\backslash\{i\}}(V)$. Then, however,

$$k_j(V) \geq \rho_j^{N\backslash\{i\}}(V) \geq z_j > K_j(V)$$

for all $j \in N\backslash\{i\}$, which contradicts the fact that V is compromise admissible.
(b) Let $F : \mathcal{C}^N \to \mathbf{R}^N$ satisfy the three properties stated in the theorem. Let $V \in \mathcal{C}^N$. We prove that $F(V) = T(V)$.
Using the minimal right property we have that $F(V) = k(V) + F(V - k(V))$.
Since $k(V - k(V)) = 0$, the restricted propertionality property implies there is a $\lambda \in \mathbf{R}$ such that $F(V) = k(V) + \lambda K(V - k(V)) = \lambda K(V) + (1 - \lambda)k(V)$.
By efficiency of F, it is clear that $\lambda = \lambda_V$ with λ_V as in (9). Hence, $F(V) = T(V)$. □

Some further properties of the compromise value T are summarized below.
Clearly, $T : \mathcal{C}^N \to \mathbf{R}^N$ is *individual rational* because $T(V) \geq k(V) \geq (v(j))_{j \in N}$ for all $V \in \mathcal{C}^N$.
Two players $i, j \in N$ are called *symmetric* in the game $V \in \mathcal{C}^N$ if the following two assertions hold:

(i) for all $S \subset N\backslash\{i,j\}$, $a \in \mathbf{R}^S$ and $t \in \mathbf{R}$ we have that $(a,t) \in V(S \cup \{i\})$ if and only if $(a,t) \in V(S \cup \{j\})$

(ii) if $a \in V(N)$ and $b \in \mathbf{R}^N$ is given by $b_k = a_k$ for $k \in N\backslash\{i,j\}$, $b_i = a_j$ and $b_j = a_i$, then $b \in V(N)$.

Obviously, if the players i and j are symmetric in V, then $K_i(V) = K_j(V)$, $k_i(V) = k_j(V)$ and, consequently, $T_i(V) = T_j(V)$.
Finally, according the compromise value, each *dummy player* gets his individual rational payoff. More specifically, if $V \in \mathcal{C}^N$ and $i \in N$ are such that $V(S \cup \{i\}) = V(S) \times V(\{i\})$ for all $S \in 2^N\backslash\{\emptyset\}$ with $i \notin S$, we have that $T_i(V) = v(i)$.

4. THE NTU τ-VALUE

In *Shapley* (1969) each NTU-game is associated to so-called λ-transfer TU-games and the NTU (Shapley)-value is obtained from the Shapley value of these games. Analogously, this section introduces the NTU τ-value by means of the τ-value of quasi-balanced λ-transfer games.

Let (N,V) be an NTU-game. Define $\Delta_N := \{\lambda \in \mathbf{R}^N \mid \lambda \geq 0, \sum_{i \in N} \lambda_i = 1\}$. A vector $\lambda \in \Delta_N$ is called *V-feasible* if $\sup\{\sum_{i \in S} \lambda_i a_i \mid a \in V(S)\} < \infty$ for all $S \in 2^N\backslash\{\emptyset\}$. For each V-feasible vector $\lambda \in \Delta_N$ the NTU-game (N, v_λ) with

$$v_\lambda(\emptyset) := 0, \quad v_\lambda(S) := \sup\{\sum_{i \in S} \lambda_i a_i \mid a \in V(S)\} \quad \text{for } S \in 2^N\backslash\{\emptyset\}\} \tag{13}$$

is called a *λ-transfer game* corresponding to V.
If for all V-feasible λ the corresponding λ-transfer games are quasi-balanced, then the game V is called *τ-admissible*. By $\mathcal{A}^N$ we denote the class of all τ-admissible NTU-games with player

set N. For $V \in \mathcal{A}^N$ the *NTU τ-value* $\tau(V) \in \mathbf{R}^N$ is defined by

$$\tau(V) := \{x \in V(N) \mid \text{there is a } V\text{-feasible } \lambda \in \Delta_N \text{ such that } \tau(v_\lambda) = \lambda * x\} \tag{14}$$

A special class of τ-admissible NTU-games is given by the class Q^N of quasi-balanced NTU-games. Here, an NTU-game (N,V) is called *quasi-balanced* if the following two properties hold:

(i) $V^0(S) + \sum_{i \in S} V^0(N\backslash\{i\}) \subset |S| V^0(N)$ for all $S \in 2^N \backslash \{\varnothing\}$.

(ii) $\sum_{i \in N} [V^0(S_i) + \sum_{j \in S_i \backslash \{i\}} V^0(N\backslash\{j\})] \subset (1 - n + \sum_{i \in N} |S_i|) V^0(N)$ for all $(S_1, S_2, \ldots, S_n)$ with $S_i \in 2^N$ and $i \in S_i$ $(i \in N)$.

where $V^0(S) := \{x \in \mathbf{R}^N \mid x_S \in V(S) \text{ and } x_{N\backslash S} = 0\}$, $S \in 2^N \backslash \{\varnothing\}$, corresponds to the *zero-representation* of the game V.

This notion of a quasi-balanced NTU-game was suggested in a private correspondence by *Kern* (1983). It may be noted that the properties (i) and (ii) are direct extensions of the conditions provided in the characterization of quasi-balanced TU-games given in *Tijs* (1981). Moreover, (i) and (ii) are special balancedness conditions. For this, recall that an NTU-game is *cardinally balanced* if for each balanced map $\gamma : 2^N \backslash \{\varnothing\} \to \mathbf{R}_+$ (i.e. such that $\sum_S \gamma(S) e^S = e^N$) it holds that $\sum_{S \in 2^N \backslash \{\varnothing\}} \gamma(S) V^0(S) \subset V^0(N)$.

Summarizing, it is straightforward to prove

LEMMA 4. (i) Every cardinally balanced NTU-game is quasi-balanced.
(ii) Every quasi-balanced NTU-game is τ-admissible.

As an illustration we consider a modified version of the example of *Shafer* (1980) due to *Hart* and *Kurz* (1983).

EXAMPLE 3. Consider an exchange market with three traders and two commodities, where the initial endowment $\omega_i \in \mathbf{R}^2_+$ and the utility function $u_i : \mathbf{R}^2_+ \to \mathbf{R}$ of trader $i \in \{1,2,3\}$ are given by

$$\omega_1 = (1-\varepsilon, 0), \quad \omega_2 = (0, 1-\varepsilon), \quad \omega_3 = (\varepsilon, \varepsilon)$$

$$u_1(c_1,c_2) = u_2(c_1,c_2) = \min\{c_1,c_2\} \text{ and } u_3(c_1,c_2) = \frac{1}{2}(c_1+c_2) \text{ for all } (c_1,c_2) \in \mathbf{R}^2_+.$$

for some $0 < \varepsilon < \frac{1}{6}$.
This exchange market corresponds to an NTU-game (N,V) with $N = \{1,2,3\}$ and

$$V(S) := \{a \in \mathbf{R}^S \mid \exists_{f:S \to \mathbf{R}^2} \forall_{i \in S} : u_i(f(i)) \geq a_i, \sum_{j \in S} f(j) = \sum_{j \in S} \omega_j\} \text{ for all } S \in 2^N \backslash \{\varnothing\}.$$

So, in particular, with subscripts representing players,

$$V(\{1\}) = \{a_1 \in \mathbf{R} \mid a_1 \leq 0\},\ V(\{2\}) = \{a_2 \in \mathbf{R} \mid a_2 \leq 0\},\ V(\{3\}) = \{a_3 \in \mathbf{R} \mid a_3 \leq \varepsilon\},$$
$$V(\{1,2\}) = \{(a_1,a_2) \in \mathbf{R}^2 \mid a_1 + a_2 \leq 1 - \varepsilon,\ a_1 \leq 1 - \varepsilon,\ a_2 \leq 1 - \varepsilon\},$$
$$V(\{1,3\}) = \{(a_1,a_3) \in \mathbf{R}^2 \mid a_1 + a_3 \leq \frac{1}{2} + \frac{1}{2}\varepsilon,\ a_1 \leq \varepsilon,\ a_3 \leq \frac{1}{2} + \frac{1}{2}\varepsilon\},$$
$$V(\{2,3\}) = \{(a_2,a_3) \in \mathbf{R}^2 \mid a_2 + a_3 \leq \frac{1}{2} + \frac{1}{2}\varepsilon,\ a_2 \leq \varepsilon,\ a_3 \leq \frac{1}{2} + \frac{1}{2}\varepsilon\},$$
$$V(\{1,2,3\}) = \{(a_1,a_2,a_3) \in \mathbf{R}^3 \mid a_1 + a_2 + a_3 \leq 1,\ a_1 \leq 1,\ a_2 \leq 1,\ a_3 \leq 1\}.$$

One easily checks that each $\lambda \in \Delta_N$ is V-feasible and that V is cardinally balanced. Further,

$$C(V) = \{(a_1,a_2,a_3) \in \mathbf{R}^3 \mid a_1 + a_2 + a_3 = 1,\ a_1 \geq \varepsilon, a_2 \geq \varepsilon, a_3 = \varepsilon\}.$$

Let $\lambda \in \Delta_N$ and $x \in V(N)$ be such that $\lambda * x = \tau(v_\lambda)$.
Suppose there exists a player $i \in N$ such that $\lambda_i < \max_{j \in N} \lambda_j$. Since $v_\lambda(N) = \max_{j \in N} \lambda_j$ and $\sum_{j \in N} \tau_j(v_\lambda) = v(N)$, one finds that $x_i = 0$ and, consequently $\tau_i(v_\lambda) = 0$. Distinguishing cases, some calculation shows that $M_i(v_\lambda) > 0$, $m_i(v_\lambda) \geq 0$ and $\sum_{j \in N} m_j(v_\lambda) \neq v_\lambda(N)$. However, since this should imply that $\tau_i(v_\lambda) > 0$, we arrive at a contradiction.
We may conclude that $\lambda = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. Then v_λ is given by

$$v_\lambda(\emptyset) = 0,\ v_\lambda(\{1\}) = v_\lambda(\{2\}) = 0,\ v_\lambda(\{3\}) = \frac{1}{3}\varepsilon,\ v_\lambda(\{1,2\}) = \frac{1}{3} - \frac{1}{3}\varepsilon,$$
$$v_\lambda(\{1,3\}) = v_\lambda(\{2,3\}) = \frac{1}{6} + \frac{1}{6}\varepsilon \text{ and } v_\lambda(\{1,2,3\}) = \frac{1}{3}.$$

Hence, $\tau(v_\lambda) = M(v_\lambda) = m(v_\lambda) = (\frac{1}{6} - \frac{1}{6}\varepsilon, \frac{1}{6} - \frac{1}{6}\varepsilon, \frac{1}{3}\varepsilon)$ and $\tau(V) = \{(\frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} - \frac{1}{2}\varepsilon, \varepsilon)\}$.
It may be noted that, since the utopia vector $K(V)$ also equals $(\frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} - \frac{1}{2}\varepsilon, \varepsilon)$, the compromise value and the (unique) NTU τ-value of V coincide.

The (unique) NTU-value of the game in example 3 is given by $(\frac{5}{12} - \frac{5}{12}\varepsilon, \frac{5}{12} - \frac{5}{12}\varepsilon, \frac{1}{6} + \frac{5}{6}\varepsilon)$. So, according to the NTU-value, player 3 who enters the market with an initial endowment of $(\varepsilon, \varepsilon)$, should end up with a utility of at least $\frac{1}{6}$, no matter how small ε is. *Shafer* (1980) assails this outcome and argues in favour of the symmetric core element $(\frac{1}{2} - \frac{1}{2}\varepsilon, \frac{1}{2} - \frac{1}{2}\varepsilon, \varepsilon)$: this outcome is prefered to the NTU-value by both player 1 and player 2 and can be accomplished without the help of player 3.

For TU-games the NTU τ-value coincides with the τ-value. Consider an NTU-game (N, V) that arises from a quasi-balanced TU-game (N, v). Obviously, $\lambda = (\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n})$ is the unique V-feasible vector in Δ_N. Further, since $v_\lambda(S) = \frac{1}{n}v(S)$ for all $S \in 2^N$, $\tau(v_\lambda) = \frac{1}{n}\tau(v) = \lambda * \tau(v)$. Hence, $\tau(V) = \{\tau(v)\}$.

For two-person bargaining games, the NTU τ-value and the Nash bargaining solution (cf. *Nash* (1950)) coincide. Let $(\{1,2\},V)$ correspond to a bargaining game (C,d). Obviously, V is τ-admissible. Since for each (quasi-balanced) two-person TU-game, the Shapley value and the τ-value coincide, it follows that the NTU-value and the NTU τ-value are the same for each (two-person) bargaining game. Moreover, since the NTU-value coincides with the Nash bargaining solution (cf. *Shapley* (1969)), this also holds for the NTU τ-value.

We now show existence of the NTU τ value for the class of (τ-admissible) zero-adjusted compactly generated NTU-games.
An NTU-game (N,V) is called *zero-adjusted* if $v(i) \geq 0$ for all $i \in N$ and *compactly generated* if, for each $S \in 2^N \backslash \{\emptyset\}$ there exists a compact set $C(S) \subset \mathbf{R}^S$ such that

$$V(S) = \{a \in \mathbf{R}^S \mid \exists_{c \in C(S)} : a \leq c\}. \tag{15}$$

The Shafer game of example 3 satisfies these two properties.

The proof of theorem 5 follows the same line of argument as the existence proof of the NTU-value given by *Shapley* (1969).

THEOREM 5. Let the NTU-game (N,V) be τ-admissible, zero-adjusted and compactly generated. Then $\tau(V) \neq \emptyset$.

Proof. Clearly, since V is compactly generated, each $\lambda \in \Delta_N$ is V-feasible. So it suffices to prove that there is a $\lambda \in \Delta_N$ and an $x \in V(N)$ such that $\lambda * x = \tau(v_\lambda)$.
For each $S \in 2^N \backslash \{\emptyset\}$ and $\lambda \in \Delta_N$, let $C(S)$ be as in (15) and define

$$C(S,\lambda) := \{y \in C(S) \mid \sum_{i \in S} \lambda_i y_i = v_\lambda(S)\}$$

Obviously, $C(S,\lambda)$ is non-empty, convex and compact. For fixed $S \in 2^N \backslash \{\emptyset\}$, using a maximum theorem (cf. theorem 9.2.1. in *Klein* and *Thompson* (1984)), it follows that the multifunction $\lambda \mapsto C(S,\lambda)$ is upper semicontinuous and that the function $\lambda \mapsto v_\lambda(S)$ is continuous. Consider the multifunction $H : \Delta_N \twoheadrightarrow \text{Aff}(\Delta_N)$ defined by

$$H(\lambda) := \{\lambda + \tau(v_\lambda)\} - \lambda * C(N,\lambda))$$

where $\lambda * C := \{\lambda * c \mid c \in C\}$ for $C \subset \mathbf{R}^N$.
Using the corresponding properties of the set $C(N,\lambda)$ it follows that $H(\lambda)$ is non empty, convex and compact for all $\lambda \in \Delta_N$. Further, since $\lambda \mapsto C(N,\lambda)$ is upper semicontinuous and $\lambda \mapsto \tau(v_\lambda)$ is continuous, we have that H is upper semicontinuous. If we can prove that H has a fixed point $\hat{\lambda} \in \Delta_N$ such that $\hat{\lambda} \in H(\hat{\lambda})$, we are finished because this implies there is an $\hat{x} \in C(N)$ such that $\hat{\lambda} * \hat{x} = \tau(v_{\hat{\lambda}})$.

Since H is upper semicontinuous and compact-valued, we have that $H(\Delta_N)$ is compact. Consequently, we can choose a simplex $\overline{\Delta}_N$ such that $H(\Delta_N) \subset \overline{\Delta}_N \subset \text{Aff}(\Delta_N)$. One can extend the multifunction H to $\overline{\Delta}_N$ by defining

$$\overline{H}(\mu) := H(f(\mu)) \text{ for all } \mu \in \overline{\Delta}_N,$$

where the continuous mapping $f : \overline{\Delta}_N \to \Delta_N$ is defined by

$$f(\mu)_i := \frac{\max\{\mu_i, 0\}}{\sum\limits_{j \in N} \max\{\mu_j, 0\}} \quad \text{for all } i \in N.$$

The multifunction $\overline{H}$ satisfies all conditions of Kakutani's fixed point theorem, so there exists a $\hat{\mu} \in \overline{\Delta}_N$ such that $\hat{\mu} \in \overline{H}(\hat{\mu})$. Let $\hat{\lambda} := f(\hat{\mu}) \in \Delta_N$. Suppose $\hat{\lambda} \neq \hat{\mu}$. Then there is a player $i \in N$ such that $\hat{\mu}_i < 0$ and $\hat{\lambda}_i = 0$. Since $\hat{\mu} \in H(\hat{\lambda})$, there is a $z \in \mathbf{R}^N$ such that $\hat{\mu} = \hat{\lambda} + z$ and $z \in \{\tau(v_{\hat{\lambda}})\} - \hat{\lambda} * C(N, \hat{\lambda})$. However, since V is zero-adjusted and τ-value is individual rational, this would imply that

$$0 > \hat{\mu}_i = z_i = \tau_i(v_{\hat{\lambda}}) \geq v_{\hat{\lambda}}(\{i\}) \geq 0.$$

Hence $\hat{\lambda} = \hat{\mu} \in \overline{H}(\hat{\mu}) = H(\hat{\lambda})$ and $\hat{\lambda}$ is a fixed point of H. □

Let (N, V) be an NTU-game. So far the NTU τ-value is defined only if for *all* V-feasible $\lambda \in \Delta_N$ the corresponding λ-transfer games are quasi-balanced. However, the definition can be extended to a larger class of games by requiring that only *some* feasible $\lambda \in \Delta_N$ give rise to quasi-balanced λ-transfer games. More specifically, we introduce

$$\tau^*(V) := \{x \in V(N) \mid \text{there is a } V\text{-feasible } \lambda \in \Delta_N \text{ such that } v_\lambda \text{ is quasi-balanced and } \lambda * x = \tau(v_\lambda)\} \tag{16}$$

Obviously, if V is τ-admissible, then $\tau^*(V) = \tau(V)$. Using this extended definition, the NTU τ-value can be calculated for the Roth games V_p of example 1.

EXAMPLE 4. For $0 \leq p \leq \frac{1}{2}$, let (N, V_p) be as in example 1. Since V_p is compactly generated, each $\lambda \in \Delta_N$ is V_p-feasible. The corresponding λ-transfer games $v_{p,\lambda}$ are given by $v_{p,\lambda}(\{i\}) = 0$ for all $i \in N$,

$$v_{p,\lambda}(\{1,2\}) = \frac{1}{2}(\lambda_1+\lambda_2), v_{p,\lambda}(\{1,3\}) = p\lambda_1+(1-p)\lambda_3, v_{p,\lambda}(\{2,3\}) = p\lambda_2+(1-p)\lambda_3 \text{ and}$$
$$v_{p,\lambda}(N) = \max\{\frac{1}{2}(\lambda_1 + \lambda_2), p\lambda_1 + (1-p)\lambda_3,\ p\lambda_2 + (1-p)\lambda_3\}.$$

Note that V_p is not τ-admissible because for $\overline{\lambda} = (\frac{1}{10}, \frac{1}{10}, \frac{8}{10})$ we have that

$$M_1(v_{p,\overline{\lambda}}) = 0 < \frac{1}{10} = m_1(v_{p,\overline{\lambda}}),$$

which implies that $v_{p,\bar{\lambda}}$ is not quasi-balanced.

Define $\theta(\lambda, V_p) \subset V_p(N)$ by

$$\theta(\lambda, V_p) := \begin{cases} \{x \in V_p(N) \mid \lambda * x = \tau(v_{p,\lambda})\} & \text{if } v_{p,\lambda} \text{ is quasi-balanced} \\ \varnothing & \text{otherwise} \end{cases} \tag{17}$$

For calculating $\tau^*(V_p) = \bigcup_{\lambda \in \Delta_N} \theta(\lambda, V_p)$ we distinguish between two cases.

(a) Let $0 \leq p < \frac{1}{2}$. We show that

$$\tau^*(V) = \{(\frac{1}{2}, \frac{1}{2}, 0)\} \cup \{x \in \mathbf{R}^N \mid \sum_{j \in N} x_j \leq 1,\ x_1 \leq p,\ x_2 \leq p,\ x_3 = 1-p\}$$

It is straightforward to verify that $v_{p,\lambda}$ is (quasi)-balanced if and only if $\lambda = (0,0,1)$ or it holds that

$$\lambda_1 + \lambda_2 \geq \frac{2-2p}{2\frac{1}{2} - 3p} \text{ and } \frac{1}{2}(\lambda_1 + \lambda_2) > \max\{p\lambda_1 + (1-p)\lambda_3, p\lambda_2 + (1-p)\lambda_3\}. \tag{18}$$

If $\lambda = (0,0,1)$, then $\tau(v_{p,\lambda}) = (0,0,1-p)$. Hence, $\{x \in V_p(N) \mid x_3 = 1-p\} \subset \tau^*(V)$.

Let $\lambda \in \Delta_N$ satisfy (18). We prove that $\theta(\lambda, V_p) \subset \{(\frac{1}{2}, \frac{1}{2}, 0)\}$. First note that by choosing $\hat{\lambda} := (\frac{1}{2}\alpha(p), \frac{1}{2}\alpha(p), 1-\alpha(p))$ with $\alpha(p) := \frac{2-2p}{2\frac{1}{2}-3p}$, we have that $\tau(v_{p,\hat{\lambda}}) = (\frac{1}{4}\alpha(p), \frac{1}{4}\alpha(p), 0)$ and, consequently, $\theta(\hat{\lambda}, V_p) = \{(\frac{1}{2}, \frac{1}{2}, 0)\}$.

- Assume $\lambda_1 = \lambda_3 = 0$. Then $\tau_1(v_{p,\lambda}) = \frac{1}{4} - \frac{1}{2}p > 0$ and $0 = \lambda_1 x_1 \neq \tau_1(v_{p,\lambda})$ for all $x \in V_p(N)$. Hence, $\theta(\lambda, V_p) = \varnothing$. Similarly, $\theta(\lambda, V_p) = \varnothing$ if $\lambda_2 = \lambda_3 = 0$.
- Assume $\lambda_3 = 0$, $\lambda_1 > 0$ and $\lambda_2 > 0$. Then $v_{p,\lambda}(N) = \frac{1}{2}$ and $\{x \in V_p(N) \mid \lambda x = v_{p,\lambda}(N)\} = \{(\frac{1}{2}, \frac{1}{2}, 0)\}$. Hence, $\theta(\lambda, V_p) \subset \{(\frac{1}{2}, \frac{1}{2}, 0)\}$.
- Assume $\lambda_3 > 0$ and $\lambda_1 = 0$. Using (18), $M_3(v_{p,\lambda}) = 0$ and

  $$\tau_1(v_{p,\lambda}) \geq m_1(v_{p,\lambda}) \geq v_{p,\lambda}(\{1,3\}) - M_3(v_{p,\lambda}) = (1-p)\lambda_3 > 0.$$

  Hence, $0 = \lambda_1 x_1 \neq \tau_1(v_{p,\lambda})$ for all $x \in V_p(N)$ and $\theta(\lambda, V_p) = \varnothing$. Analogously, $\theta(\lambda, V_p) = \varnothing$ if $\lambda_3 > 0$ and $\lambda_2 = 0$.
- Assume $\lambda_1 > 0$, $\lambda_2 > 0$ and $\lambda_3 > 0$. Using (18), $\tau_3(v_{p,\lambda}) = M_3(v_{p,\lambda}) = m_3(v_{p,\lambda}) = 0$. Then, since $\lambda_3 > 0$, $x \in \theta(\lambda, V_p)$ implies that $x_3 = 0$. Consequently, using the fact that $\lambda_1 > 0$, $\lambda_2 > 0$ and $v_{p,\lambda}(N) = \frac{1}{2}\lambda_1 + \frac{1}{2}\lambda_2$ it follows that

  $$\theta(\lambda, V_p) \subset \{x \in V_p(N) \mid \lambda x = v_{p,\lambda}(N),\ x_3 = 0\} \subset \{(\frac{1}{2}, \frac{1}{2}, 0)\}.$$

(b) Let $p = \frac{1}{2}$. We show that

$$\tau^*(V_p) = \{x \in \mathbf{R}^N \mid x_1 = \frac{1}{2},\ x_2 + x_3 \leq \frac{1}{2}\} \cup \{x \in \mathbf{R}^N \mid x_2 = \frac{1}{2}, x_1 + x_3 \leq \frac{1}{2}\} \cup \{x \in \mathbf{R}^N \mid x_3 = \frac{1}{2},\ x_1 + x_2 \leq \frac{1}{2}\}.$$

One can check that $v_{p,\lambda}$ is quasi-balanced if and only if $\lambda_i = 0$ for some $i \in N$. Using symmetry considerations, it suffices to prove that

$$\bigcup_{\lambda \in \Delta_N : \lambda_3 = 0} \theta(\lambda, v_p) = \{x \in V_p(N) \mid x_1 = \frac{1}{2} \text{ or } x_2 = \frac{1}{2}\}$$

Let $\lambda \in \Delta_N$ be such that $\lambda_3 = 0$.

- Assume $\lambda_2 = 0$. Then $\tau(v_{p,\lambda}) = (\frac{1}{2}, 0, 0)$ and $\theta(\lambda, V_p) = \{x \in V_p(N) \mid x_1 = \frac{1}{2}\}$ Analogously, if $\lambda_1 = 0$, then $\theta(\lambda, V_p) = \{x \in V_p(N) \mid x_2 = \frac{1}{2}\}$.
- Assume $\lambda_1 > 0$ and $\lambda_2 > 0$. Then $v_{p,\lambda}(N) = 1$, $\{x \in V_p(N) \mid \lambda x = v_{p,\lambda}(N)\} = \{(\frac{1}{2}, \frac{1}{2}, 0)\}$ and $\theta(\lambda, V_p) \subset \{(\frac{1}{2}, \frac{1}{2}, 0)\}$.

Remark. Should one restrict attention to *positive* V-feasible vectors λ only, there does not exist an NTU τ-value in case $p = \frac{1}{2}$ and, for $0 \le p < \frac{1}{2}$, there is a unique NTU τ-value $(\frac{1}{2}, \frac{1}{2}, 0)$.

Finally, we discuss some properties of the (extended) NTU τ-value. Let (N, V) be an NTU-game. It is straightforward to verify that the NTU τ-value is *efficient*, i.e. if $x \in \tau^*(V)$, then $x \in V(N)\backslash \text{Dom}(V(N))$, and *individual rational* in the sense that for each $x \in \tau^*(V)$ which corresponds to a positive V-feasible $\lambda \in \Delta_N$ we have that $x \ge (v(i))_{i \in N}$. Further, the NTU τ-value is *symmetric*: if $i, j \in N$ are symmetric in V, $x \in \tau^*(V)$ and $y \in \mathbf{R}^N$ is such that $y_i = x_j$, $y_j = x_i$ and $y_k = x_k$ for all $k \in N \backslash \{i, j\}$, then $y \in \tau^*(V)$.

5. CONCLUDING REMARKS

(i) It would be interesting to provide an axiomatic characterization of the NTU τ-value. Probably, as in the characterization of *Aumann* (1985a) for the NTU-value, one should restrict attention to a special subclass of NTU-games.

(ii) The definition of the compromise value given in section 3 is based upon bounds for the core. Analogously, one can introduce a *strong compromise value* by means of bounds for the strong core, and obtain similar results.

Schematically, this proceeds as follows. Let (N, V) be an NTU-game. The strong core $SC(V)$ is defined by

$$SC(V) := \{x \in V(N) \mid \neg\exists_{S \in 2^N \backslash \{\emptyset\}} : x_S \in \overline{\text{Dom}}(V(S))\},$$

where $\overline{\text{Dom}}(V(S)) := \{a \in \mathbf{R}^S \mid \exists_{b \in V(S)} : b \ge a,\ b \ne a\}$ for all $S \in 2^N \backslash \{\emptyset\}$.
Note that $\text{Dom}(V(S)) \subset \overline{\text{Dom}}(V(S))$ and $SC(V) \subset C(V)$. For $i \in N$, define

$$\overline{K}_i(V) := \sup\{t \in \mathbf{R} \mid \exists_{a \in \mathbf{R}^{N \backslash \{i\}}} : (a, t) \in V(N), a \notin \overline{\text{Dom}}(V(N) \backslash \{i\})), a \ge (v(j))_{j \in N \backslash \{i\}}\}$$

and

$$\overline{k}_i(V) := \max_{S:i\in S} \overline{\rho}_i^S(V),$$

where

$$\overline{\rho}_i^S(V) := \sup\{t \in \mathbf{R} \mid (t, \overline{K}_{S\setminus\{i\}}(V)) \in V(S)\}.$$

Straightforwardly it follows that $\overline{K}(V) \leq K(V)$, $k(V) \leq \overline{k}(V)$ and $x \in SC(V)$ implies that $\overline{k}(V) \leq x \leq \overline{K}(V)$. Further, if V arises from a TU-game, we have that $\overline{K}(V) = K(V)$ and $\overline{k}(V) = k(V)$. For (two-person) bargaining games the two upper bounds again coincide, but the lower bounds need not.

Defining an NTU-game V to be *strongly compromise admissible* if $\overline{k}(V) \leq \overline{K}(V)$, $\overline{k}(V) \in V(N)$ and $\overline{K}(V) \notin \overline{\mathrm{Dom}}(V(N))$, the definition of the strong compromise value proceeds analogously to (8). Moreover, modifying the characterizing properties given in theorem 3 in the obvious way (introducing among others strong efficiency) the same kind of characterization carries through.

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