subfaculteit der econometrie

RESEARCH MEMORANDUM

[redacted]

TILBURG UNIVERSITY
DEPARTMENT OF ECONOMICS
Postbus 90153 - 5000 LE Tilburg
Netherlands

[redacted]

HORIZONTAL MIXED DECOMPOSITION

Bert R. Meijboom

March 1985

Contents

HORIZONTAL MIXED DECOMPOSITION

B.R. Meijboom[1)]

Subfaculty of Econometrics, Tilburg University, The Netherlands

Abstract

In this contribution a new decomposition approach for block-angular programming problems is presented. Historically, decomposition methods are either price or resource directive. The present paper integrates the two in the sense that part of the common constraints are coordinated by prices, while, at the same time, the other part of the common constraints are coordinated by direct allocations. We find increasing lower and decreasing upper bounds for the optimal value and globally feasible solutions with improving value can easily be obtained.
The resulting algorithm has an appealing economic interpretation in terms of mixed price-budget oriented planning in a two-level organization.

1) The author is grateful to Prof. Dr. J.F. Benders for many fruitful discussions and useful suggestions. The idea for the convergence proof is mainly due to J.P. Boly and C.P.M. van Hoesel.
The research is supported by a grant from the Common Research Pool of the Tilburg University and the Technical University Eindhoven (Samenwerkingsorgaan KHT-THE) in the Netherlands.

Horizontal mixed decomposition

1. Introduction

Block-angular LP problems can be decomposed in various ways. Both problem and solution method often have an economic interpretation in terms of a multilevel organization. Therefore classifications of decomposition algorithms are frequently based on the distinction "price-directive" versus "resource-directive". This paper contains a hybrid algorithm in the sense that the pricing operations and resource allocation operations occur simultaneously. Following Obel (1978), who has introduced this mixed approach, we will speak of "horizontal mixed decomposition". The present method is essentially an extension to the theory because we explicitly analyse the master problem. Similar ideas, with emphasis on the economic implications, have been found in a working paper by Atkins (1979).

After the introduction of the problem (section 2), it is reformulated (section 3) and then replaced by two approximating formulations (section 4) so that an iterative two-level solution strategy can be applied (section 5). Realization of near-optimal solution values by globally feasible solutions is then discussed (section 6) followed by the economic interpretation of the algorithm (section 7).

2. Problem formulation

The class of problems to be analysed is of the form

$$
\begin{aligned}
\text{Maximize } & c_1x_1 + \ldots + c_nx_n \\
\text{s.t.} \quad & A_1x_1 + \ldots + A_nx_n \leq a \qquad (*) \\
& B_1x_1 + \ldots + B_nx_n \leq b \qquad (**) \\
& D_1x_1 \leq d_1 \\
& \qquad \vdots \\
& D_nx_n \leq d_n \\
& x_1, \ldots, x_n \geq 0
\end{aligned}
\tag{2.1}
$$

Here: a, b, c_j, d_j ($j=1,\ldots,n$) are known vectors,
A_j, B_j, D_j ($j=1,\ldots,n$) are known matrices,
x_j ($j=1,\ldots,n$) are variable vectors,
all of appropriate dimensions.

It is well known that problem (2.1) can be viewed as a model of a divisionalized organization (Dirickx and Jennergren (1979, chapter 6). The common constraints (*) en (**) express the common use of certain resources by all divisions. In this paper we will develop a two-level solution algorithm in which the common resources (*) are coordinated by prices while the common resources (**) are coordinated by allocations. This treatment of the common constraints (*) and (**) will be referred to as horizontal mixed decomposition.

Before we present our algorithm, a few assumptions and definitions are in order.
We presume the existence of a feasible solution of problem (2.1). The

divisional feasible regions X_j, which are defined as $X_j :=$ $\{x_j \mid D_j x_j < d_j,\ x_j > 0\}$ are assumed to be bounded. So (2.1) has a finite optimum.

For $j = 1,\dots,n$, the set Y_j is defined as follows:

$$y_j \in Y_j \ :\leftrightarrow \{x_j \mid x_j \in X_j,\ B_j x_j < y_j\} \neq \phi \qquad (2.2)$$

Set Y_j will be called the set of feasible allocations for division j concerning common resources (**).

For notational convenience, the set Y is introduced:

$$Y := \{(y_1,\dots,y_n) \mid \sum_{j=1}^{n} y_j < b;\ y_j \in Y_j,\ j = 1,\dots,n\} \qquad (2.3)$$

Set Y will be called the set of globally feasible allocations concerning common resources (**).

3. Reformulation of the original problem

We define the Lagrangian function relative to the common constraints (*):

$$L(x,\pi) := \sum_{j=1}^{n} c_j x_j + \pi(a - \sum_{j=1}^{n} A_j x_j)$$

where x replaces $x_1,\dots,x_n$. Now the (Lagrangian) dual of (2.1) is:

$$\min_{\pi \geq 0} \max_{x} L(x,\pi) \tag{3.1}$$

$$\text{s.t. } x_j \in X_j,\ j = 1,\ldots,n,\ \sum_{j=1}^{n} B_j x_j \leq b$$

Optimal solutions $(\tilde{x}, \tilde{\pi})$ to the minmax problem (3.1) are saddle points of $L(x, \pi)$. The next theorem provides for necessary and sufficient conditions for the existence of a saddle point.

Theorem 1:

A vector $\tilde{x} = (\tilde{x},\ldots, \tilde{x}_n)$ is a solution to the LP problem (2.1) if and only if there exists a vector $\tilde{\pi} \geq 0$ such that $(\tilde{x}, \tilde{\pi})$ is a saddle point of $L(x,\pi)$.

Proof: see Appendix A.

Now the usefulness of a saddle point is evident: if $(\tilde{x}, \tilde{\pi})$ is a saddle point of $L(x,\pi)$, then $\tilde{x}$ solves the original problem (2.1).

If we use the set Y (see formula (2.3)) and introduce the functions $\Phi_j(y_j,\pi)$ defined as

$$\Phi_j(y_j,\pi) := \text{Max}\{(c_j - \pi A_j)x_j \mid x_j \in X_j,\ B_j x_j \leq y_j\},$$

$j = 1,\ldots,n$, problem (3.1) can be rewritten as

$$\min_{\pi \geq 0} \max_{y \in Y} \sum_{j=1}^{n} \Phi_j(y_j,\pi) + \pi a \tag{3.2}$$

Problem (3.1) and (3.2) are equivalent because, for every $\pi \geq 0$, the inner maximization problems are equivalent (see Lasdon (1970, p. 462)).

The functions $\Phi_j(y_j,\pi)$ are concave in y_j and convex in π. Moreover, they are even piecewise linear in both y_j and π.

Due the the presumed existence of a finite optimum to (2.1), the original Lagrangian $L(x,\pi)$ has a saddle point. So the function $\sum_{j=1}^{n} \Phi_j(y_j,\pi) + \pi a$ has a saddle point, too. Hence, it is allowed to reverse the order of maximization and minimization in (3.2) (see Zangwill (1969, p. 45-46), thereby obtaining:

$$\underset{y \in Y}{\text{Max}} \ \underset{\pi \geq 0}{\text{Min}} \ \sum_{j=1}^{n} \Phi_j(y_j,\pi) + \pi a \tag{3.3}$$

In the sequel, (3.2) will be referred to as (D), the dual problem, and (3.3) as (P), the primal problem. The problem (P) and (D) have the same optimal solutions. Equivalent formulations of (P) and (D) are

$$\underset{y \in Y}{\text{Max}} \ w \quad \text{s.t. } w \leq \sum_{j=1}^{n} \Phi_j(y_j,\pi) + \pi a, \ \pi \geq 0 \tag{3.4}$$

and

$$\underset{\pi \geq 0}{\text{Min}} \ v \quad \text{s.t. } v \geq \sum_{j=1}^{n} \Phi_j(y_j,\pi) + \pi a, \ y \in Y \tag{3.5}$$

respectively.

4. Relaxation of both primal and dual problem

In this section, tangential approximation is applied to derive appropriate relaxed versions of (P) and (D).

Suppose we have at hand r tuples $(\pi^k, y_1^k, \ldots, y_n^k)$, $k = 1,\ldots,r$ of trial prices and allocations with respect to the common resources (*) and (**), respectively. Let $\pi^k \geq 0$, $y_j^k \in Y_j$, $j = 1,\ldots, n$, and $\sum_{j=1}^{n} y_j^k \leq b$ for all k.

From $w \leq \sum_{j=1}^{n} \Phi_j(y_j,\pi) + \pi a$, $\pi \geq 0$, it follows that

$$w \leq \sum_{j=1}^{n} \Phi_j(y_j,\pi^k) + \pi^k a, \quad k = 1,\ldots,r \tag{4.1}$$

Similarly

$$v \geq \sum_{j=1}^{n} \Phi_j(y_j^k,\pi) + \pi a, \quad k = 1,\ldots,r \tag{4.2}$$

The r right-hand sides of (4.1) are not easily handled so we apply a further relaxation. Tangential approximations are readily obtained by solving the subproblems

$$\begin{aligned} \text{Maximize } & (c_j - \pi^k A_j)x_j \\ \text{s.t. } & B_j x_j \leq y_j^k \\ & x_j \in X_j \end{aligned}$$

For let $\hat{x}_j^k$ be an optimal solution, let $\hat{u}_j^k$ be an optimal dual solution associated to the constraint $B_j x_j \leq y_j^k$, while the optimal solution value is $\Phi_j(y_j^k, \pi^k)$, by definition. Then it is a trivial task to show that

$$\Phi_j(y_j,\pi^k) \leq \Phi_j(y_j^k,\pi^k) + \hat{u}_j^k(y_j - y_j^k), \quad y_j \in Y_j \tag{4.3}$$

(See Geoffrion (1970, p. 381) or Dirickx and Jennergren (1979, p. 69).

The right-hand sides of (4.2) can also be approximated

$$\pi a + \sum_{j=1}^{n} \Phi_j(y_j^k,\pi) \geq f^k + (\pi-\pi^k)\, \Delta_a^k \tag{4.4}$$

where $(k = 1,\ldots, r)$:

$$\begin{aligned} f^k &:= \sum_{j=1}^{n} \Phi_j(y_j^k,\pi^k) + \pi^k a = \sum_{j=1}^{n} (c_j-\pi^k A_j)\hat{x}_j^k + \pi^k a \\ \Delta_a^k &:= a - \sum_{j=1}^{n} A_j\, \hat{x}_j^k \end{aligned} \tag{4.5}$$

(See Appendix B.)

Combining (3.6), (4.1) and (4.3) leads to the following relaxed primal problem:

$$\begin{aligned} &\text{Max} \quad w \\ &\text{s.t. } w \leq f^k + \sum_{j=1}^{n} \hat{u}_j^k(y_j-y_j^k) \quad , k = 1,\ldots,r \\ &\qquad (y_1,\ldots,y_n) \in Y \end{aligned} \tag{4.6}$$

From (3.7), (4.2) and (4.4), we derive the following relaxed dual problem:

$$\begin{aligned} &\text{Min} \quad v \\ &\text{s.t. } v \geq f^k + (\pi-\pi^k)\, \Delta_a^k \qquad , k = 1,\ldots, r \\ &\qquad \pi \geq 0 \end{aligned} \tag{4.7}$$

We conclude this section with a useful statement concerning the optimal solution values of (4.6) and (4.7).
Let w_p be the optimal solution value of (3.4), let v_D be the optimal solution value of (3.5). We know that $w_D = v_P$. Now, if v_r and w_r are the optimal solution values of (4.5) and (4.6), respectively, it holds that

$$v_r \leq v_D = w_P \leq w_r \tag{4.8}$$

So the optimal solution value of the original problem lies between v_r and w_r.

5. Solution strategy

Our algorithm approximates the solution of (P) and (D) by solving relaxed versions of (P) and (D), and adding new constraints to the relaxated problems when necessary.

Let $\varepsilon > 0$ be the desired accuracy. Suppose one has arrived at a relaxed primal and a relaxed dual problem of the form (4.5) and (4.6), respectively, each with r constraints. We call these problems (P_r) and (D_r). Solve (P_r) and denote the optimal solution and objective function value by $(y_1^{r+1},\ldots,y_n^{r+1})$ and w_r. Similarly, π^{r+1} and v_r are the optimal solution and objective function value to (D_r).
If $w_r - v_r < \varepsilon$, we may terminate. Now we can generate a globally feasible solution with value $\geq v_r$ (see section 6).
Otherwise, if $w_r - v_r \geq \varepsilon$, solve the following subproblems:

$$\begin{array}{lll} \text{Max} & (c_j - \pi^{r+1} A_j) x_j & \\ \text{s.t.} & B_j \, x_j \leq y_j^{r+1} & \\ & x_j \in X_j & \end{array} \tag{5.1}$$

This yields $\hat{x}_j^{r+1}$, $\hat{u}_j^{r+1}$, $\Phi_j(y_j^{r+1}, \pi^{r+1})$, $j = 1,\ldots,n$, from which appropriate constraints to be added to (P_r) and (D_r), can be deduced. The augmented problems are called (P_{r+1}) and (D_{r+1}).
If we solve (P_{r+1}) and (D_{r+1}), w_{r+1} and v_{r+1} will come out. These are, possibly better, upper/lower bounds for the optimal objective function value of the original problem as

$$v_r \leq v_{r+1} \leq v_D = w_P \leq w_{r+1} \leq w_r$$

Now it is clear that, by successively solving (P_r) and (D_r) and adding new constraints to them, we expect to find shrinking intervals $[v_r, w_r]$, $r = 1,2,\ldots$, which contain the optimal objective function value. Indeed, the difference between v_r and w_r converges to zero. A convergence proof is given in Appendix C.
The algorithm can be summarized as follows:

Summary of algorithm:

Step 0. Choose $\pi^1 \geq 0$, and $y_j^1 \in Y_j$, $j=1,\ldots,n$, such that $\sum_{j=1}^{n} y_j^1 \leq b$.

Set $r := 0$.

Step 1. For $j = 1,\ldots,n$, solve (5.1), which yields $\hat{x}_j^{r+1}$, $\hat{u}_j^{r+1}$ and $\Phi_j(y_j^{r+1}, \pi^{r+1})$.

Step 2. Compute f^{r+1} and Δ_a^{r+1}, and add appropriate constraints to (P_r) and (D_r) thereby obtaining (P_{r+1}) and (D_{r+1}), respectively.

Step 3. Set $r := r + 1$ and solve (P_r) and (D_r) which yields w_r, v_r, $y_1^{r+1},\ldots,y_n^{r+1}$, π^{r+1}.

Step 4. Optimality test:

if $w_r - v_r < \varepsilon$ then terminate, otherwise return to step 1.

Up till now, we have treated the set Y as if it is completely known. In practical applications it is usually impossible to obtain the set Y (or the sets $Y_1,\ldots,Y_n$) in explicit form. The literature, in particular Geoffrion (1970, section 3.1), offers several useful methods to generate the sets $Y_1,\ldots,Y_n$ during the iterations of the just described algorithm. Especially in the present, linear case, each Y_j can be specified without approximation by a finite collection of linear equalities. Each of these inequalities can be added to (4.6) "when needed".
This aspect even more stresses the importance of the algorithm as an information collecting procedure.

6. Generation of globally feasible solutions

So far, we have described a procedure which simultaneously generates a decreasing sequence of upper bounds as well as an increasing sequence of lower bounds for the optimal solution value. Moreover, both sequences, i.e. $(v_r)_1^\infty$, and $(w_r)_1^\infty$, converge to this value. In this section, we will show that, without much extra effort, a globally feasible solution can be computed.

The relaxed dual D_r is simply an LP problem and can be written as

$$\begin{aligned} &\text{Minimize } v \\ &\text{s.t.} \quad v - \pi \Delta_a^k \geq f^k - \pi^k \Delta_a^k, \; k = 1,\ldots,r \\ &\qquad\quad \pi \geq 0 \end{aligned}$$

Dualization yields

$$\begin{aligned} &\text{Maximize } \sum_{k=1}^{r} \lambda^k (f^k - \pi^k \Delta_a^k) \\ &\text{s.t.} \quad \sum_{k=1}^{r} \lambda^k = 1 \\ &\qquad - \sum_{k=1}^{r} \lambda^k \Delta_a^k \leq 0 \\ &\qquad \lambda^k \geq 0 \end{aligned}$$

which is equivalent to (recall (4.5)):

$$\begin{aligned} &\text{Maximize } \sum_{k=1}^{r} \lambda^k \sum_{j=1}^{n} c_j \hat{x}_j^k \\ &\text{s.t.} \quad \sum_{k=1}^{r} \lambda^k = 1 \\ &\qquad \sum_{k=1}^{r} \lambda^k \sum_{j=1}^{n} A_j \hat{x}_j^k \leq a \\ &\qquad \lambda^k \geq 0 \end{aligned} \tag{6.1}$$

Now let $(\bar{\lambda}^1, \ldots, \bar{\lambda}^r)$ be an optimal solution to (6.1). Its solution value is equal to v_r, the optimal value of D_r. If we define

$$\bar{x}_j := \sum_{k=1}^{r} \bar{\lambda}^k \hat{x}_j^k , \quad j = 1, \ldots, n \tag{6.2}$$

then $(\bar{x}_1, \ldots, \bar{x}_n)$ is feasible to (2.1), due to the convexity properties, and it has solution value v_r.

Summarizing, we can derive a globally feasible solution by a convex combination of previously generated divisional solutions. The weighting factors are exactly the optimal dual variables associated to D_r, and the solution value is equal to the optimal value of D_r.

An alternative way to generate a globally feasible solution, with value $\geq v_r$ proceeds as follows. Compute $(j=1, \ldots, n)$

$$\bar{a}_j^r := \sum_{k=1}^{r} \bar{\lambda}^k A_j \hat{x}_j^k, \quad \bar{b}_j^r := \sum_{k=1}^{r} \bar{\lambda}^k y_j^k \tag{6.3}$$

where $(\bar{\lambda}^1, \ldots, \bar{\lambda}^r)$ is again an optimal dual solution to (D_r). Now solve for each $j = 1, \ldots, n$ the problem

$$\begin{aligned} &\text{Maximize } c_j x_j \\ &\text{s.t.} \quad A_j x_j \leq \bar{a}_j^r \\ &\qquad\ \ B_j x_j \leq \bar{b}_j^r \\ &\qquad\ \ x_j \in X_j \end{aligned} \tag{6.4}$$

Because $\bar{x}_j$ (as defined by formula (6.2)) is feasible to (6.4), the optimal value of (6.4) is at least $c_j \bar{x}_j$. Hence the sum of the optimal values of each of the problems (6.4) is at least $\sum_{j=1}^{n} c_j \bar{x}_j = v_r$.

7. Economic interpretation

As noted before, the original problem (2.1) can be viewed as a model for a divisionalized organization. The blocks $D_j x_j < d_j$, $x_j > 0$ ($j=1,\ldots,n$) are associated with divisions and the common constraints (*) and (**) reflect the interdependencies (e.g. allocation of common resources) between them. Furthermore, we presume that there is a central unit at the top level of the organization, that is aware of these interdependencies but does not have complete information on the divisional constraints. Identifying the objective function of (2.1) with the goal of the firm, to be strived after by the central unit, we observe that the central unit cannot immediately realize that goal due to the lack of knowledge on divisions. Instead, a planning procedure by which top-management gathers information, must be applied. Below we describe such a procedure. It is based on the decomposition algorithm as outlined in the previous sections.

The essential feature of the method is that

- the (*) interdependencies are coordinated by prices, while
- the (**) interdependencies are coordinated by direct allocations cq. budgets.

Hence, we provide for a mixed price-budget directive planning procedure.

For explanatory purposes, we will assume that the (*) and (**) interdependencies are due to the common use of certain resources by all divisions. The goal of the firm is assumed to be profit maximization.

The planning procedure is formed by a number of planning sessions. At the start of a new planning session, the central unit sends a price π^r for the (*) resources and allocations (or budgets) $y_1^r,\ldots,y_n^r$

for the (**) resources to the divisions. These are asked to compute their maximum profit $\Phi_j(y_j^r, \pi^r)$, the required amount $A_j \hat{x}_j^r$ of (*) resources under the price π^r and a marginal valuation $\hat{u}_j^r$ of the budget y_j^r for (**) resources ($j = 1,\ldots,n$).
Formally, each division solves its problem

$$\begin{aligned} &\text{Maximize } (c_j - \pi^r A_j)x_j \\ &\text{s.t.} \qquad B_j x_j \leq y_j^r \\ &\qquad\qquad x_j \in X_j \end{aligned} \tag{7.1}$$

and reports the maximum objective function value $\Phi_j(y_j^r, \pi^r)$, the optimal dual variable $\hat{u}_j^r$ with respect to the constraint $B_j\, x_j \leq y_j^r$ and the quantity $A_j\, \hat{x}_j^r$, where $\hat{x}_j^r$ is the optimal solution to (7.1).

Based on the divisional responses, and information gathered in previous sections, the central unit can derive a lower bound v_r and an upper bound w_r for the maximum attainable profit (i.e. the optimal value of the original problem (2.1)).
The maximum attainable profit lies between these bounds, so the central unit can terminate the procedure as soon as their difference has become small enough. The central unit finds a partitioning of the (*) and (**) resouces under which the divisions together will at least realize a profit of v_r, if the previous claims for (*) resources, i.e. $A_j\, \hat{x}_j^k$, $k = 1,\ldots, r$, and the previous budgets for (**) resources, i.e. y_j^k, $k = 1,\ldots, r$, are adequately weighted. Weighting factors with this property, which apply to both types of resources and are uniform over the divisions, can be obtained by a slight modification of the computational procedure for v_r. Formally, let $\bar{\lambda}^1,\ldots, \bar{\lambda}^r$ be these weighting factors. Now the central unit must compute ($j = 1,\ldots,n$)

$$\bar{a}_j^r := \sum_{k=1}^{r} \bar{\lambda}^k A_j \hat{x}_j^k, \quad \bar{b}_j^r := \sum_{k=1}^{r} \bar{\lambda}^k y_j^k .$$

If each division optimally uses these final amounts of common resources, it solves the problem

$$\begin{array}{lrl} \text{Maximize} & c_j x_j & \\ \text{s.t.} & A_j x_j & \leq \bar{a}_j^r \\ & B_j x_j & \leq \bar{b}_j^r \\ & x_j & \in X_j \end{array}$$

The sum of the divisional profits will be at least v_r.

On the other hand, if the central unit is not satisfied with the bounds v_r and w_r, it can update the prices and allocations leading to new bounds v_{r+1} and w_{r+1}. It will hold that

$$v_r \leq v_{r+1} \leq \text{OPT} \leq w_{r+1} \leq w_r$$

where OPT denotes the maximum attainable profit. So the more information the central unit gathers, the better it approximates the maximum attainable profit.

Now we will pay some attention to the way the central unit derives the prices, budgets and profit estimates. As a matter of fact, the prices π^r and the lower bounds v_r on the one hand, and the allocations $y_1^r, \ldots, y_n^r$ and the upper bounds w_r on the other hand follow from two separate computations. Firstly, we will consider the prices π^r together with the lower bounds v_r.

Suppose we are at the end of the r-th planning session. So the central unit has at hand the divisional responses of session 1,2,..., r. From the divisional responses of each session separately, the central unit can form a linear function which approximates the profit as function of the internal prices π. Hence, combining the divisional information as collected in all previous sessions up till now, the central unit derives a piecewise linear approximation of the profit function with respect to changes in the prices for (*) resources. The collected information is used in a "pessimistic" way, since the approximating profit function lies "below" the correct profit function. As a result, the estimated profit v^r is a lower bound for the actual maximum attainable profit. In the course of the procedure, the piecewise linear approximating function becomes better and better, hence giving rice to improved lower bounds v_{r+1}, v_{r+2} etc.

The determination of subsequent w_r and $y_1^r, \ldots, y_n^r$ proceeds in a similar way. The only difference is that here the central unit works with an improving piecewise linear approximation for the profit function with respect to changes in the allocations for (**) resources, which always lies above the correct profit function. Hence a decreasing sequence of upper bounds w^r is the result.

8. Conclusion

Our main purpose has been to derive a two-level decomposition method with prices for some resources and allocations for other resources occurring simultaneously. Furthermore, we were particularly interested in the computation of the prices and budgets. As a matter of fact, this computation provided for an upper and lower bound for the optimal value of the problem at hand. Moreover, the lower bound, which increases during the iteration sequence, is directly associated to a globally feasible solution. In other words, during the iterations globally feasible solutions can be obtained with increasing solution value.

The author is aware that, from a computational point of view, several improving modifications could be incorporated. However, the contribution of the present paper is that it provides for the correct mathematical foundation of mixed coordination by prices and budgets in a general two-level organization. Each of the subproblems, at the top level as well as at the divisional level, has a clear appealing economic interpretation in terms of a planning procedure. Secondly, the mixed use of prices and budgets is a most realistic option when comparing the present planning procedure with planning in real-world organizations (e.g. see Atkins (1973), Obel (1981)).

References

Atkins, D.R. (1973). Managerial decentralization and decomposition in mathematical programming. Operational Research Quarterly, 25, no. 4, pp. 615-624.

Atkins, D.R. (1979). Decentralizations by joint price and budget controls: the problem and an important special case. University of British Columbia. Working paper.

Dirickx, Y.M.I. and L.P. Jennergren (1979). Systems analysis by multilevel methods: with applications to economics and management. New York: Wiley.

Geoffrion, A.M. (1970). Primal resource-directive approaches for optimizing nonlinear decomposable systems. Operations Research. May-June 1970.

Lasdon, L.S. (1970). Optimization theory for large systems. London: Collier-MacMillan.

Obel, B. (1978). A note on mixed procedures for decomposing linear programming problems. Math. Operationsforsch. Statist., Ser. Optimization, 9, no. 4, pp. 537-544.

Zangwill, W.I. (1969). Nonlinear programming: a unified approach. Englewood Cliffs N.Y.: Prentice Hall.

Appendix A. Proof theorem 1

Consider the problem

$$
\begin{aligned}
&\text{Maximize } px \\
&\text{s.t.} \quad Ax \leq a \\
&\qquad\quad Bx \leq b \\
&\qquad\qquad x \geq 0
\end{aligned}
\tag{A1}
$$

with optimal solution $\hat{x}$. Due to duality theory for LP, there exist $\hat{u}$, $\hat{v}$ that solve the dual problem i.e.

$$
\begin{aligned}
&\text{Minimize } ua + vb \\
&\text{s.t.} \quad uA + vB \geq p \\
&\qquad\qquad u,v \geq 0
\end{aligned}
\tag{A2}
$$

Furthermore, it holds that $\hat{u}(A\hat{x}-a) = 0$ and $\hat{x}$ is an optimal solution to

$$
\begin{aligned}
&\text{Maximize } (p-\hat{u}A)x \\
&\text{s.t.} \qquad B\,x \leq b \\
&\qquad\qquad\quad x \geq 0
\end{aligned}
\tag{A3}
$$

Now define:

$$L(x,u) := px - u(Ax-a) \ , \ Bx \leq b,\ x \geq 0,\ u \geq 0.$$

Then: $(\hat{x},\hat{u})$ is a saddle point of $L(x,u)$.

Hence, we must prove that

$$L(x,\hat{u}) < L(\hat{x},\hat{u}) < L(\hat{x},u)$$

for all x with $Bx < b$, $x > 0$, all $u > 0$.
As $\hat{x}$ solves (A3), we have $(p-\hat{u}A)x < (p-\hat{u}A)\hat{x}$, so $L(x,\hat{u}) < L(\hat{x},\hat{u})$.
Secondly, $\hat{u}(A\hat{x}-a) = 0$ whereas $-u(A\hat{x}-a) > 0$, so $L(\hat{x},\hat{u}) < L(\hat{x},u)$.
Summarizing, we have shown that if (A1) has an optimal solution $\hat{x}$, then there exist an $\hat{u} > 0$ such that $(\hat{x},\hat{u})$ is a saddlepoint of $L(x,u)$.

Reversely, if $L(x,u)$ has a saddlepoint $(\hat{x},\hat{u})$ then $\hat{x}$ solves the original problem (A1). This statement is even true for convex programming problems, see Lasdon (1970, p. 85).

Appendix B. Further relaxation of problem (4.2)

We will show that for each $k = 1,\ldots, r$ the following inequality holds:

$$\Phi_j(y_j^k,\pi) > \Phi_j(y_j^k,\pi^k) + (\pi^k-\pi)\ A_j\hat{x}_j^k \qquad \text{(B1)}$$

with (obviously) equality for $\pi = \pi^k$.
Recall the definition of $\Phi_j(y_j,\pi)$ (section 4).
For all x_j satisfying $B_jx_j < y_j^k$, $x_j \in X_j$, it holds that

$$\Phi_j(y_j^k,\pi) > (c_j-\pi^kA_j)x_j + (\pi^k-\pi)A_jx_j$$

In particular

$$\Phi_j(y_j^k,\pi) \geq (c_j-\pi^k A_j)\hat{x}_j^k + (\pi^k-\pi)A_j\hat{x}_j^k$$

which is equivalent to

$$\Phi_j(y_j^k,\pi) \geq \Phi_j(y_j^k,\pi^k) + (\pi^k-\pi)A_j\hat{x}_j^k \text{ .}$$

Now that we have proved (B1), it easily follows that

$$\sum_{j=1}^{n} \Phi_j(y_j^k,\pi) + \pi a \geq \sum_{j=1}^{n} (\Phi_j(y_j^k,\pi^k) + (\pi^k-\pi)A_j\hat{x}_j^k)$$

so

$$\sum_{j=1}^{n} \Phi_j(y_j^k,\pi) + \pi a \geq f^k + (\pi-\pi^k)\Delta_a^k \qquad \text{(B2)}$$

where $\quad f^k := \sum_{j=1}^{n} \Phi_j(y_j^k,\pi^k) + \pi^k a,\ \Delta_a^k := a - \sum_{j=1}^{n} A_j\hat{x}_j^k$,

(B2) is exactly inequality (4.4).

Appendix C. A convergence proof for the algorithm as presented in section 5

In this appendix we will prove convergence of the algorithm as presented in section 5. Two more assumptions are required. After the proof, a sufficient condition for one of them is given.

Theorem:

If we assume that

1. the sequence $(\pi^r)_1^\infty$, is bounded,

2. the $\hat{u}_j^r$ are uniformly bounded (i.e. $0 < \hat{u}_j^r < M$),

then the algorithm of section 5 converges in the sense that

$w_r - v_r \downarrow 0, \ r \to \infty.$

Proof:

We already know that $v_r < v_{r+1} < v_D = w_P < w_{r+1} < w_r$. In the sequel we will prove that $w_r - v_r \downarrow 0$ on a subset of indices. Of course, this implies that

$$\lim_{r\to\infty} v_r = v_D = w_P = \lim_{r\to\infty} w_r .$$

The reasoning proceeds as follows.

The sequence $(\pi^r, y_1^r, \ldots, y_n^r)_1^\infty$ converges on a subset of indices, as the π^r and all y_j^r come from bounded sets. (The boundedness of Y follows from the compactness of each X_j and the global restriction $\sum_{j=1}^n y_j < b$.) The convergent sub-sequences are denoted by

$(\pi^{r_s})_{s=1}^\infty$ and $(y_j^{r_s})_{s=1}^\infty$, $j = 1, \ldots, n$.

$$\text{As } v_{r_{s+1}} > f^{r_s} + (\pi^{r_{s+1}} - \pi^{r_s})\Delta_a^{r_s}$$

$$\text{and } w_{r_{s+1}} < f^{r_s} + \sum_{j=1}^n \hat{u}_j^{r_s}(y_j^{r_{s+1}} - y_j^{r_s}),$$

we have

$$0 < w_{r_{s+1}} - v_{r_{s+1}} < (\pi^{r_{s+1}} - \pi^{r_s})\Delta_a^{r_s} + \sum_{j=1}^{n} \hat{u}_j^{r_s}(y_j^{r_{s+1}} - y_j^{r_s})$$

For $s \to \infty$, the right-hand side of this expression converges to 0 as $\Delta_a^{r_s}$ and $\hat{u}_j^{r_s}$ are bounded. (The boundedness of $\Delta_a^{r_s}$ is simply due to the boundedness of the sets X_j, $j = 1,\ldots, n$.) Hence

$$v_{r_s} - w_{r_s} \downarrow 0 \ , \quad s \to \infty$$

and the proof is completed.

Now we present a sufficient condition for assumption 1 in the theorem.

Suppose that there exists a known, feasible solution $\tilde{x}_1,\ldots, \tilde{x}_n$ such that $\sum_{j=1}^{n} A_j \tilde{x}_j < a$. The knowledge of this "interior point" can be used as follows.

From (3.1) it is clear that

$$v_D \geq \min_{\pi \geq 0} L(x,\pi) \text{ for every } \underline{\text{fixed}} \text{ feasible } x$$

As a consequence, we are allowed to add the constraint $v \geq L(\tilde{x},\pi)$ to D_r, $r = 1,2\ldots$ Each D_r remains to be a relaxed dual problem.

With respect to the algorithm, we suppose that, upon initialitation, D_r with $r = 0$ has only one constraint, viz.

$$v \geq \sum_{j=1}^{n} c_j \tilde{x}_j + \pi(a - \sum_{j=1}^{n} A_j \tilde{x}_j)$$

which will be maintained throughout the subsequent iterations. An immediate consequence of this modification is that the sequence $(\pi^r)_1^\infty$

will be bounded. To prove this, define: $\tilde{P} := \sum_{j=1}^{n} c_j \tilde{x}_j$; $\pi_i^r :=$ the i-th component of π^r; $\tilde{\Delta}_i$:= the i-th component of $a - \sum_{j=1}^{n} A_j \tilde{x}_j$.
Now we have:

$$0 \leq \pi^{r+1} \tilde{\Delta}_i \leq \sum_i \pi_i^{r+1} \tilde{\Delta}_i \leq v_r - \tilde{P} \leq v_D - \tilde{P}$$

so $$0 \leq \pi_i^{r+1} \leq (v_D - \tilde{P})/(\tilde{\Delta}_i), \text{ as } \tilde{\Delta}_i > 0$$

In other words, all future π_i^r are bounded.

IN 1984 REEDS VERSCHENEN

138 G.J. Cuypers, J.P.C. Kleijnen en J.W.M. van Rooyen
Testing the Mean of an Asymetric Population:
Four Procedures Evaluated

139 T. Wansbeek en A. Kapteyn
Estimation in a linear model with serially correlated errors when observations are missing

140 A. Kapteyn, S. van de Geer, H. van de Stadt, T. Wansbeek
Interdependent preferences: an econometric analysis

141 W.J.H. van Groenendaal
Discrete and continuous univariate modelling

142 J.P.C. Kleijnen, P. Cremers, F. van Belle
The power of weighted and ordinary least squares with estimated unequal variances in experimental design

143 J.P.C. Kleijnen
Superefficient estimation of power functions in simulation experiments

144 P.A. Bekker, D.S.G. Pollock
Identification of linear stochastic models with covariance restrictions.

145 Max D. Merbis, Aart J. de Zeeuw
From structural form to state-space form

146 T.M. Doup and A.J.J. Talman
A new variable dimension simplicial algorithm to find equilibria on the product space of unit simplices.

147 G. van der Laan, A.J.J. Talman and L. Van der Heyden
Variable dimension algorithms for unproper labellings.

148 G.J.C.Th. van Schijndel
Dynamic firm behaviour and financial leverage clienteles

149 M. Plattel, J. Peil
The ethico-political and theoretical reconstruction of contemporary economic doctrines

150 F.J.A.M. Hoes, C.W. Vroom
Japanese Business Policy: The Cash Flow Triangle
an exercise in sociological demystification

151 T.M. Doup, G. van der Laan and A.J.J. Talman
The $(2^{n+1}-2)$-ray algorithm: a new simplicial algorithm to compute economic equilibria

IN 1984 REEDS VERSCHENEN (vervolg)

152 A.L. Hempenius, P.G.H. Mulder
Total Mortality Analysis of the Rotterdam Sample of the Kaunas-Rotterdam Intervention Study (KRIS)

153 A. Kapteyn, P. Kooreman
A disaggregated analysis of the allocation of time within the household.

154 T. Wansbeek, A. Kapteyn
Statistically and Computationally Efficient Estimation of the Gravity Model.

155 P.F.P.M. Nederstigt
Over de kosten per ziekenhuisopname en levensduurmodellen

156 B.R. Meijboom
An input-output like corporate model including multiple technologies and make-or-buy decisions

157 P. Kooreman, A. Kapteyn
Estimation of Rationed and Unrationed Household Labor Supply Functions Using Flexible Functional Forms

158 R. Heuts, J. van Lieshout
An implementation of an inventory model with stochastic lead time

159 P.A. Bekker
Comment on: Identification in the Linear Errors in Variables Model

160 P. Meys
Functies en vormen van de burgerlijke staat
Over parlementarisme, corporatisme en autoritair etatisme

161 J.P.C. Kleijnen, H.M.M.T. Denis, R.M.G. Kerckhoffs
Efficient estimation of power functions

162 H.L. Theuns
The emergence of research on third world tourism: 1945 to 1970;
An introductory essay cum bibliography

163 F. Boekema, L. Verhoef
De "Grijze" sector zwart op wit
Werklozenprojecten en ondersteunende instanties in Nederland in kaart gebracht

164 G. van der Laan, A.J.J. Talman, L. Van der Heyden
Shortest paths for simplicial algorithms

165 J.H.F. Schilderinck
Interregional structure of the European Community
Part II: Interregional input-output tables of the European Community 1959, 1965, 1970 and 1975.