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Graphs

DEPARTMENT OF ECONOMICS

RESEARCH MEMORANDUM

DISTANCE REGULARITY AND THE SPECTRUM OF GRAPHS

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FEW 582

Communicated by Prof.dr. M.H.C. Paardekooper

Distance-Regularity and the Spectrum of Graphs

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Abstract

We deal with the question: Can one see from the spectrum of a graph Γ whether it is distance-regular or not? Up till now the answer was not known when Γ has precisely 4 distinct eigenvalues (the diameter 3 case). We show that in this case the answer is negative. We also give positive answers in some special situations. For instance, if Γ has the spectrum of a distance-regular graph with diameter 3 and $\mu = 1$, then Γ is distance-regular. Our main tools are eigenvalue techniques for partitioned matrices.

1 Introduction

Many properties of graphs can be recognized from the spectrum of its adjacency matrix. Such as bipartiteness, regularity and strong regularity. Here we deal with the question: Is a graph with the spectrum of a distance-regular graph distance-regular? In case the distance-regular graph has diameter 1 (complete graphs) or 2 (strongly regular graphs) the answer is affirmative. Hoffman [11] constructed a graph cospectral to (which means: with the same spectrum as) the Hamming 4-cube $H(4,2)$, but not distance-regular, showing that the answer is negative if the diameter is at least 4. We shall show (in Section 3) that for several distance-regular graphs with diameter 3, including the tetrahedral graphs $J(n,3)$ the answer is negative too. This solves a problem of Brouwer, Cohen and Neumaier [2] (p. 263) and disproves an old conjecture mentioned by Bose and Laskar [1], see also Cvetković, Doob and Sachs [5] (p. 183).
In Section 5 we give some positive answers to the above question for diameter bigger than 2, provided some additional requirement is fulfilled. For instance a graph with the spectrum of a distance-regular graph with diameter 3 and with the correct number of points at distance 2 for each vertex is distance-regular. This gives a common generalization of results of Bose and Laskar [1], Cvetković [4] and Laskar [12]. To prove these results we develop (in Section 4) a tool for proving regularity of a vertex partition of a graph based on its spectrum. But first we need some preliminary results on matrix partitions.

2 Matrix partitions

Throughout the paper A will be a symmetric real matrix whose rows and columns are indexed by $X = \{0, \ldots, n\}$. Let $\{X_0, \ldots, X_d\}$ be a partition of X. The *characteristic* *matrix* S is the $(n+1)\times(d+1)$ matrix whose j^{th} column is the characteristic vector of X_j $(j=0,\dots,d)$. Define $k_i=|X_i|$ and $K=\operatorname{diag}(k_0,\dots,k_d)$. Let A be partitioned according to $\{X_0,\dots,X_d\}$, that is

$$A=\begin{bmatrix} A_{0,0} & \dots & A_{0,d} \\ \vdots & & \vdots \\ A_{d,0} & \dots & A_{d,d} \end{bmatrix},$$

wherein $A_{i,j}$ denotes the submatrix (block) of A formed by rows in X_i and the columns in X_j. Let $b_{i,j}$ denote the average row sum of $A_{i,j}$. Then the matrix $B=(b_{i,j})$ is called the ***quotient matrix***. We easily have

$$KB=S^{\top}AS,\ S^{\top}S=K.$$

If the row sum of each block $A_{i,j}$ is constant then the partition is called *regular* and we have $A_{i,j}\underline{1}=b_{i,j}\underline{1}$ for $i,j=0,\dots,d$ ($\underline{1}$ denotes the all-one vector), so

$$AS=SB.$$

The following result is well-known and often applied, see [5],[10].

Lemma 2.1 *If, for a regular partition, v is an eigenvector of B for an eigenvalue λ, then Sv is an eigenvector of A for the same eigenvalue λ.*

Proof. $Bv=\lambda v$ implies $ASv=SBv=\lambda Sv$. □

Suppose A is the adjacency matrix of a connected graph Γ. Let γ be a vertex of Γ with local diameter d and let X_i denote the number of points at distance i from γ $(i=0,\dots,d)$. Then $\{X_0,\dots,X_d\}$ is called the *distance partition* of Γ around γ. Note that in this case we can compute K from B, since $k_0=1$, $k_ib_{i,i+1}=k_{i+1}b_{i+1,i}$ and $b_{i+1,i}\neq 0$ for $i=0,\dots,d-1$. If the distance partition is regular, Γ is called ***distance-regular around*** γ and the quotient matrix B is a tridiagonal matrix, called the ***intersection matrix*** of Γ with respect to γ. If Γ is distance-regular around each vertex with the same intersection matrix, then Γ is (by definition) a distance-regular graph with intersection array

$$\{b_{0,1},\dots,b_{d-1,d};b_{1,0},\dots,b_{d,d-1}\}.$$

Clearly the intersection array determines the intersection matrix, because B has constant row sum $k(=k_1=b_{0,1})$. Lemma 2.1 gives that for a distance-regular graph Γ, the eigenvalues of its intersection matrix B are also eigenvalues of its adjacency matrix A. In fact, the distinct eigenvalues of Γ are precisely the eigenvalues of B. Also the multiplicities (and hence the whole spectrum of Γ) can be expressed in terms of the intersection array. For these, and all other results on distance-regular graphs used in this paper, we refer to Brouwer, Cohen and Neumaier [2].

3 Switch partitions

In this section we describe a method to change adjacency in a given graph in order to obtain another graph with the same spectrum. Let A be the adjacency matrix of a graph. A *switch partition* $\{X_0, \ldots, X_d\}$ of A is a regular partition split into two parts $\{X_0, \ldots, X_{h-1}\}$, $\{X_h, \ldots, X_d\}$ such that $b_{i,j} \in \{0, k_j, \frac{1}{2}k_j\}$ whenever i and j are separated (that is, X_i and X_j lie in different parts). A separated pair $\{i, j\}$ is called a *switch pair* if $b_{i,j} = \frac{1}{2}k_j$.

Theorem 3.1 *Let Γ be a graph with a switch partition $\{X_0, \ldots, X_d\}$. Let Γ' be the graph obtained from Γ by switching, for each switch pair $\{i, j\}$, the adjacency relation between X_i and X_j to its complement (that is, edges become non-edges and non-edges become edges). Then Γ' has the same spectrum as Γ.*

Proof. Let A and A' be the adjacency matrices of Γ and Γ' respectively. With the given partition define

$$E = \begin{bmatrix} E_{0,0} & \ldots & E_{0,d} \\ \vdots & & \vdots \\ E_{d,0} & \ldots & E_{d,d} \end{bmatrix}, \text{ where } E_{i,j} = \begin{cases} J \text{ if } \{i,j\} \text{ is a switch pair,} \\ 2J \text{ if } b_{i,j} = k_j \text{ and } \{i,j\} \text{ is separated,} \\ O \text{ otherwise.} \end{cases}$$

(As usual, O is the zero matrix and J the all-one matrix.) Put $D = \text{diag}(D_0, \ldots, D_d)$, where $D_i = I$ if $i < h$ and $D_i = -I$ if $i \geq h$. Then we easily have that $A' = D(A - E)D$, that $\{X_0, \ldots, X_d\}$ is also regular for A', and that A and A' have the same quotient matrix. Therefore, by Lemma 2.1, the eigenvalues of A and A' with eigenvectors in the range of the characteristic matrix S coincide. Let v be an eigenvector of A with eigenvalue λ, perpendicular to the columns of S. Then

$$A'Dv = D(A - E)DDv = D(A - E)v = DAv - \underline{0} = \lambda Dv.$$

So also the remaining eigenvalues of A and A' coincide. $\square$

In some cases Γ' is isomorphic to Γ, but in many cases it isn't. The switching concept of Theorem 3.1 turned out to be not new. It was already known to Godsil and McKay [7], who used it to contruct lots of cospectral graphs. In case all separated pairs are switch pairs it is the same as Seidel switching, see [5] or [13].

Example 1. Consider the tetrahedral graph $J(n, 3)$ (the vertices are the unordered triples of an n-set Ω; triples are adjacent if they meet in two points). Let Q be a 4-subset of Ω and take $n \geq 6$. For $i = 0, \ldots, 3$ let X_i be the set of triples meeting Q in i points. This clearly defines a regular partition of $J(n, 3)$, moreover it is a switch partition for $h = 3$ ($\{2, 3\}$ is the only switch pair). The switching, explained above, produces a graph cospectral but not isomorphic to $J(n, 3)$. The new graph is not even distance-regular. Indeed, consider a vertex x in X_1 and a vertex y in X_3. Then after switching x and y have distance 2 with 2 common neighbours if the corresponding triples meet and 6 common neigbours otherwise.

Many other distance-regular graphs admit switch partitions producing different graphs. We give two more examples.

Example 2. The Gosset graph is the unique distance-regular graph on 56 vertices with intersection array $\{27, 10, 1; 1, 10, 27\}$. It can be constructed as follows. Take for the vertices twice the set of edges of the complete graph K_8. Vertices within a set are adjacent if the corresponding edges are disjoint and vertices from different sets are adjacent whenever the corresponding edges intersect in one point. The edges of K_8 can be partitioned into 7 classes of 4 non-intersecting edges. This gives a partition of the vertices of the Gosset graph into 14 classes of size 4 and it is easily checked that the two sets of vertices make it a switch partition. It is also easy to verify that, after switching, for each vertex there is no vertex at distance 3 any more. So we have obtained a graph with diameter 2 cospectral to a distance-regular graph with diameter 3.

Remark. The Gosset graph is an instance of a Taylor graph. This is a distance-regular graph with intersection array $\{rs, (r+1)(s-1)/2, 1; 1, (r+1)(s-1)/2, rs\}$. It is the same as a regular two-graph represented as a double cover of K_{rs+1}. If a Taylor graph admits a clique of size $s+1$, then any vertex not in the clique is adjacent to none or half of the vertices of this clique. This gives rise to a switch partition, and the local diameter of a vertex of the clique becomes 2 after switching (the case $n = 6$ of Example 1 is of this type). If the graph admits a partition into $(s+1)$-cliques, the global diameter becomes 2. Taylor graphs with this property have been constructed by Taylor [16] for $r = s^2$ whenever s is an odd prime power. For $s = 3$ we have Example 2.

Example 3. (By A.E. Brouwer, personal communication.) For a distance-regular graph with $k = 2\mu$ ($k = k_1 = b_{0,1}$ and $\mu = b_{2,1}$) the distance partition (with respect to any vertex) is a switch partition. This applies for instance to distance-regular graphs with intersection array $\{2\mu, 2\mu-1, \mu, 1; 1, \mu, 2\mu-1, 2\mu\}$, the so called Hadamard graphs. For $\mu = 2$ we get the array of the Hamming 4-cube, and switching leads to the mentioned example of Hoffman.

In [9] Haemers and Spence determined all graphs cospectral to distance-regular graphs up to 27 vertices. Many, but not all, can be obtained by switching. Amoung these graphs there is one cospectral but not isomorphic to the cubic lattice graph $H(3, 3)$.

4 Regularity and eigenvalues

In this section we give some eigenvalue tools for proving regularity of partitions. The first result is proved in Haemers [8], Section 1.2 (see also [2], Section 3.3).

Theorem 4.1 *Given a symmetric partitioned matrix A. Let S and B denote the corresponding characteristic and quotient matrix, respectively. Let $\lambda_0 \geq \ldots \geq \lambda_n$ be the eigenvalues of A. Then B has real eigenvalues $\mu_0 \geq \ldots \geq \mu_d$ (say). Denote the respective eigenvectors by $v_0, \ldots, v_d$. Then the following holds:*

(i) $\lambda_i \geq \mu_i \geq \lambda_{n-d+i}$ $(0 \leq i \leq d)$.

(ii) If for some integer k $(0 \leq k \leq d)$, we have $\lambda_i = \mu_i$ for $i = 0, \ldots, k$ (or $\mu_i = \lambda_{n-d+i}$ for $i = k, \ldots, d$) then Sv_i is an eigenvector of A with eigenvalue μ_i for $i = 0, \ldots, k$ (respectively for $i = k, \ldots, d$).

(iii) If, for some integer k $(0 \leq k \leq d+1)$, we have $\lambda_i = \mu_i$ for $i = 0, \ldots, k-1$ and $\mu_i = \lambda_{n-d+i}$ for $i = k, \ldots, d$ then the partition is regular.

Thus we have a tool for proving regularity of a partition using eigenvalues. If we want to prove distance-regularity of a graph Γ, we wish to apply (iii) to its distance partitions. This, however, will hardly ever work if the diameter is bigger than 2, since if Γ is connected, the quotient matrix B has $d+1$ distinct eigenvalues (see Theorem 4.3), whilst all but the largest eigenvalue of the adjacency matrix A have in general a multiplicity bigger than 1, in which case equality in (i) can only hold for μ_0, μ_1 and μ_d. So we need a result like (iii) in terms of these three eigenvalues only.

Lemma 4.2 *With the hypotheses of Theorem 4.1, let A be a block tridiagonal matrix (i.e. $A_{i,j} = O$ if $|i-j| > 1$) and let $v_i = [v_{i,0}, \ldots, v_{i,d}]^\top$ denote an eigenvector of μ_i $(0 \leq i \leq d)$. If $\mu_0 = \lambda_0$, $\mu_1 = \lambda_1$ and $\mu_d = \lambda_n$ and if any three consecutive rows of $[v_0\ v_1\ v_d]$ are independent, then the partition is regular.*

Proof. By (ii) of Theorem 4.1 $ASv_i = \mu_i S v_i$ for $i = 0, 1, d$. By considering the ℓ^{th} block row of A we get

$$v_{i,\ell-1} A_{\ell,\ell-1} \underline{1} + v_{i,\ell} A_{\ell,\ell} \underline{1} + v_{i,\ell+1} A_{\ell,\ell+1} \underline{1} = \mu_i v_{i,\ell} \underline{1} \quad \text{for } i = 0, 1, d,$$

(wherein the undefined terms have to be taken equal to zero). Since, for $i = 0, 1, d$ and $j = \ell-1, \ell, \ell+1$, the matrix $(v_{i,j})$ is non-singular, we find $A_{\ell,j}\underline{1} \in \langle \underline{1} \rangle$ for $j = \ell-1, \ell, \ell+1$ (and hence for $j = 0, \ldots, d$). So the partition is regular. □

Theorem 4.3 *Let Γ be a connected graph with adjacency matrix A and eigenvalues $\lambda_0 \geq \ldots \geq \lambda_n$. Let $\{X_0, \ldots, X_m\}$ be a partition of the vertices of Γ, such that there are no edges between X_i and X_j if $|i-j| > 1$. Let B be the corresponding quotient matrix. Then B has $d+1$ distinct real eigenvalues $\mu_0 > \ldots > \mu_d$ (say) and the following holds.*

(i) $\lambda_0 \geq \mu_0$, $\lambda_1 \geq \mu_1$, $\lambda_n \leq \mu_d$.

(ii) If $\lambda_0 = \mu_0$, $\lambda_1 = \mu_1$ and $\lambda_n = \mu_d$, then the partition is regular.

Proof. Because Γ is connected, $b_{i,i+1} > 0$ for $i = 0, \ldots, d-1$. Hence, for any real number x, the upper right $d \times d$ submatrix of $B - xI$ is non-singular. Therefore no eigenvalue has multiplicity bigger than 1. Result (i) is part of Theorem 4.1. To prove (ii), we use Lemma 4.2 and show that every three consecutive rows of $[v_0\ v_1\ v_d]$ are independent. This will be a consequence of the following claims:

1. All entries of v_0 can be taken positive.
   Indeed, B is non-negative and, since Γ is connected, irreducible. Hence by the Perron-Frobenius theorem μ_0 has a positive eigenvector.

2. For $i = 0, \ldots, d$, the eigenvector v_i has exactly i sign changes.
   This follows from the theory of tridiagonal matrices (see for instance Stoer and Bulirsch [15], Section 6.6.1): Let $p_j(x)$ denote the leading principal $j \times j$ minor of $xI - B$ for $j = 1, \ldots, d$ and put $p_0(x) = 1$. Then we may take

$$v_{i,j} = \frac{p_j(\mu_i)}{b_{0,1} \cdots b_{j-1,j}} \quad \text{for } i, j = 0, \ldots, d.$$

   Moreover, the polynomials p_j form a Sturm sequence. This implies that $p_j(\mu_i)$ has exactly i sign changes when j runs from 0 to d, proving Claim 2.

3. The sequence $(\frac{v_{1,0}}{v_{0,0}}, \ldots, \frac{v_{1,d}}{v_{0,d}})$ is strictly monotonic.
   Write $\alpha_j = \frac{v_{1,j}}{v_{0,j}}$ for $j = 0, \ldots, d$. From $Bv_i = \mu_i v_i$ it follows

$$v_{i,j-1} b_{j,j-1} + v_{i,j} b_{j,j} + v_{i,j+1} b_{j,j+1} = \mu_i v_{i,j} \quad \text{for } i = 0, 1,\ j = 1, \ldots, d-1.$$

   This gives for $j = 1, \ldots, d-1$

$$(\alpha_j - \alpha_{j-1}) v_{0,j-1} b_{j,j-1} + (\alpha_j - \alpha_{j+1}) v_{0,j+1} b_{j,j+1} = (\mu_0 - \mu_1) v_{1,j},$$

   showing that $\alpha_j > \alpha_{j+1}$ if $\alpha_{j-1} > \alpha_j$ and $v_{1,j} \geq 0$ (using that $v_{0,j\pm 1}$ and $b_{j,j\pm 1}$ are positive). Similarly we get (in case $j = 0$)

$$(\alpha_0 - \alpha_1) v_{0,1} b_{0,1} = (\mu_0 - \mu_1) v_{1,0} > 0$$

   (using $v_{1,0} = 1$). Hence $\alpha_0 > \alpha_1$. Thus we have, by induction, that the sequence $\alpha_0, \alpha_1, \ldots$ is strictly decreasing until v_1 changes sign. Analoguesly it follows that the sequence $\alpha_d, \alpha_{d-1}, \ldots$ is strictly increasing until the first sign change of v_1. Since v_1 has just one sign change, the claim follows.

Now, after dividing the j^{th} row of $[v_0\ v_1\ v_d]$ by $v_{0,j}$ for $j = 0, \ldots, d$, v_0 becomes constant, v_1 becomes strictly monotonic and v_d remains alternating. This implies that dependence of three consecutive rows is impossible. □

Remark. Since Γ is connected, regularity of the partition means that X_0 (and also X_d) is a completely regular code.

5 Distance-regularity from the spectrum

Assume Γ' is a graph on $n+1$ vertices with spectrum $\Sigma = \{\mu_0^{f_0}, \ldots, \mu_d^{f_d}\}$ (the eigenvalues are in decreasing order; exponents denote multiplicities). Suppose there exitst a feasible intersection matrix B for a distance regular graph Γ giving the same spectrum Σ. (See [2], Section 4.1.D for a precise definition of "feasible". So we do not require that Γ actually exists. It will, however, be convenient to talk about properties of Γ, though they are in fact properties of B.) Since Γ is regular (of degree $k = \mu_0$) and connected, Σ satifies

$$f_0 = 1,\ \sum_{i=0}^{d} f_i = n+1,\ (n+1)\mu_0 = \sum_{i=0}^{d} f_i \mu_i^2\ . \tag{1}$$

This in turn implies that Γ' is regular of degree μ_0 and connected with diameter at most d (see for example [5]; proofs are, however, not difficult, for instance regularity follows from the third equation of (1) by applying Theorem 4.1(iii) to the trivial partition with only one class of the adjacency matrix of Γ'). For a given vertex γ of Γ', let B' denote the quotient matrix with respect to the distance partition around γ, let $k'_0, \ldots, k'_d$ be the sizes of the partition classes and let $\mu'_0 \geq \ldots \geq \mu'_d$ be the eigenvalues of B' (note that $\mu'_0 = \mu_0 = k = k_1 = k'_1$). We know that the intersection matrix B of Γ has eigenvalues $\mu_0, \ldots, \mu_d$. So, if we can prove $B' = B$, then by Theorem 4.3(ii) Γ' is distance-regular around γ (with the same intersection array as Γ). Some entries of B' and B coincide trivially: $b'_{0,0} = b_{0,0} = 0$, $b'_{1,0} = b_{1,0} = 1$ and $b'_{0,1} = b_{0,1} = \mu_0 (= k)$. The following lemma shows that we don't have to go all the way for proving $B' = B$.

Lemma 5.1 *If $k'_i = k_i$ for $i = 2, \ldots, d-1$ and $b'_{i,i} = b_{i,i}$ for $i = 1, \ldots, d-2$, then $B' = B$.*

Proof. Clearly $k'_i = k_i$ for $i = 0, \ldots, d$. Using $b'_{0,0} = 0$, $b'_{0,1} = k$, $b'_{i,i-1}k_i = b'_{i-1,i}k_{i-1}$, $b'_{i,i+1} = k - b'_{i,i} - b'_{i,i-1}$ and the same formulas without the prime we find that $b'_{i,j} = b_{i,j}$ if i or j is not equal to d or $d-1$. Define $x = b'_{d-1,d-1} - b_{d-1,d-1}$ and $E = [0, \ldots, 0, 1, -\frac{k_{d-1}}{k_d}]^\top [0, \ldots, 0, 1, -1]$, then

$$B' = B + xE. \tag{2}$$

Next we want to apply inequalities for eigenvalues. Therefore we prefer symmetric matrices and multiply the above equation by $K^{\frac{1}{2}}$ on the left by $K^{-\frac{1}{2}}$ on the right (where $K = diag(k_0, \ldots, k_d)$). Then (2) becomes $\widetilde{B}' = \widetilde{B} + x\widetilde{E}$. Clearly the matrices are symmetric now, the eigenvalues haven't changed and $\widetilde{E}$ is positive semi-definite. Denote the eigenvectors of B and $\widetilde{B}$ by v_i and $\tilde{v}_i$ $(= K^{\frac{1}{2}}v_i)$, respectively $(i = 0, \ldots, d)$. Then $\tilde{v}_0$ is also an eigenvector of $\widetilde{B}'$ for the eigenvalue $k (= \mu'_0 = \mu_0)$, since $v_0 (= \underline{1})$ is an eigenvector of B' for the eigenvalue k. If $x > 0$ we find (using $\tilde{v}_0 \perp \tilde{v}_1$)

$$\mu'_1 \geq \frac{\tilde{v}_1^\top \widetilde{B}' \tilde{v}_1}{\tilde{v}_1^\top \tilde{v}_1} = \mu_1 + x\frac{\tilde{v}_1^\top \widetilde{E} \tilde{v}_1}{\tilde{v}_1^\top \tilde{v}_1} \geq \mu_1.$$

Theorem 4.3(i) gives $\mu'_1 \leq \mu_1$, hence $\widetilde{E}\widetilde{v}_1 = 0$. Similarly, $x < 0$ implies $\widetilde{E}\widetilde{v}_d = 0$. If $\widetilde{E}\widetilde{v}_i = 0$ then $Ev_i = 0$, which yields $v_{i,d-1} = v_{i,d}$. But we saw in proving Theorem 4.3 that this is impossible if $i = 1$ or d. So $x = 0$ and $B = B'$. □

For a strongly regular Γ the lemma gives that always $B' = B$, showing that strong regularity can be recognized from the spectrum. An other direct consequence is the following result.

Theorem 5.2 *Suppose Γ' has the spectrum of a bipartite distance-regular graph Γ with diameter d, and suppose that for each vertex γ of Γ' the number k'_i of vertices at distance i from γ equals k_i (i.e. k'_i has the required value) for $4 \leq i \leq d$, then Γ' is distance-regular with the same intersection array as Γ.*

Proof. If Γ is biparte then so is Γ'. Therefore $b'_{i,i} = 0 = b_{i,i}$ for $i = 0, \ldots, d$ and $\sum_{i \text{ even}} k'_i = \sum_{i \text{ odd}} k'_i = \sum_{i \text{ even}} k_i = \sum_{i \text{ odd}} k_i = \frac{n+1}{2}$. Hence $k'_i = k_i$ for $i = 0, \ldots, d$ and Lemma 5.1 applies. □

In particular we find the known result that a graph cospectral to a bipartite distance-regular graph with diameter 3 is such a graph.
Since Γ' is regular of degree μ_0, its adjacency matrix A satisfies

$$(A - \mu_1 I) \cdots (A - \mu_d I) \in \langle J \rangle . \tag{3}$$

Together with the well-known fact that $(A^j)_{i,i}$ equals the number of closed walks of length j from i to i, this gives sometimes information on B'. Take $d = 3$, then (3) gives that A^3 has constant diagonal (because every lower power of A has). So the number of oriented triangles through any vertex equals

$$(A^3)_{0,0} = \frac{1}{n+1}\mathrm{tr}(A^3) = \frac{1}{n+1}\sum_{i=0}^{3} f_i \mu_i^3.$$

Hence $b'_{1,1} = \frac{1}{k(n+1)} \sum_{i=0}^{3} f_i \mu_i^3 = b_{1,1}$. Of course, as we saw in Section 3, we cannot determine all $b'_{i,j}$. But if we require that every vertex of Γ' has the right number of vertices at distance 2, we can.

Theorem 5.3 *Let Γ' be a graph with the spectrum of a distance-regular graph Γ with diameter 3 and k_2 vertices at distance 2 from a given vertex.*

(i) Each vertex of Γ' has at least k_2 vertices at distance 2.

(ii) If equality holds for some vertex γ, then Γ' is distance-regular around γ having the same intersection matrix as Γ.

(iii) If equality holds for all vertices then Γ' is distance-regular.

Proof. We shall prove (ii) with the weaker condition that Γ' has at most k_2 vertices at distance 2, then we get (i) for free. Let $\{X_0, X_1, X_2, X_3\}$ be the distance partition around γ. Extend X_2 with some vertices of X_3 until $|X_2| = k_2$. Then $|X_i| = k_i$ for $i = 0, \ldots, 3$ and the partition still satisfies the condition of Theorem 4.2. Now Lemma 5.1 gives $B' = B$, proving (ii), (i) and (iii). □

This generalizes theorems of Bose and Laskar [1] (who proved the result for tetrahedral graphs), Laskar [10] and Cvetković [4] (who proved it for the cubic lattice graph).

Remark. For $d = 3$, k_2 can be expressed in termes of Σ as follows:

$$k_2 = \frac{k(k-1-\theta_3)^2}{\theta_4 - \theta_3^2 - k}\ , \text{ where } \theta_j = \frac{1}{k(n+1)} \sum_{i=0}^{3} f_i \mu_i^j \text{ and } k = \mu_0.$$

So, in the above theorem, we can replace k_2 by this expression. If we do so, it is even conceivable that the result remains valid for an arbitrary connected regular graph with precisely 4 distinct eigenvalues.

Corollary 5.4 *If* Γ *has diameter* 3 *and* $\mu(= b_{2,1}) = 1$, *then* Γ' *is distance-regular.*

Proof. With respect to any vertex γ of Γ' we have $k_2' b_{2,1}' = k b_{1,2}' = k b_{1,2} = k_2 b_{2,1} = k_2$. Clearly $b_{1,2}' \geq 1$, hence $k_2' \leq k_2$ and Theorem 5.3 applies. □

A lot of feasible intersection arrays correspond to graphs satisfying the condition of Corollary 5.4. For example the point graph of a generalized hexagon has $d = 3$ and $\mu = 1$ and hence it can be recognized from the spectrum whether a graph is the point graph of a generalized hexagon. The following example shows the use of our result.

Example. The spectrum $\{(q^2-q)^1, q^{\frac{1}{2}q(q-1)(q^2-q+1)}, (-1)^{q^3}, (-q)^{\frac{1}{2}q(q-3)(q^2-q+1)}\}$ is for $q > 2$ the spectrum of a distance-regular graph with intersection array $\{q^2-q, q^2-q-2, q+1; 1, 1, q^2-2q\}$. Corollary 5.4 gives that a graph with that spectrum must be such a distance-regular graph. The adjacency matrix E of a projective plane of order q^2 with a polarity with q^3+1 absolute points has spectrum $\{(q^2+1)^1, q^{\frac{1}{2}q(q^3+2q-1)}, (-q)^{\frac{1}{2}q(q^3+1)}\}$. The submatrix A of E induced by the non-absolute points is symmetric with zero-diagonal and therefore the adjacency matrix of some graph Γ. An easy eigenvalue property (see [8], Theorem 1.3.3) shows that Γ has the above spectrum, hence Γ is distance-regular. This gives the unitary nonisotropics graphs (from the Hermitian polarity).

Other graphs for which Corollary 5.4 applies are distance-regular graphs with diameter 3 and girth 5, such as the Sylvester graph and the Perkel graph. We also find non-existence results. For example $\{5^1, (1+\sqrt{2})^{20}, (1-\sqrt{2})^{20}, -3^{15}\}$ is not the spectrum of a graph, since it belongs to an intersection array of a distance-regular graph with diameter 3 and girth 5 that does not exist (see Fon-der-Flaass [6]). These last examples are also special cases of the following result of Brouwer and Haemers [3].

Theorem 5.5 *If Γ' has the spectrum of a distance-regular graph with diameter d and girth $g \geq 2d-1$, then Γ' is such a distance-regular graph.*

Proof. The girth of a regular graph is determined by its spectrum (see [5], but again, proving it is an easy exercise). So Γ' has girth at least $2d-1$. Now we easily have $k'_i = k_i = k(k-1)^{i-1}$ for $i = 1, \ldots, d-1$. Moreover $b'_{i,i} = b_{i,i} = 0$ for $i = 1, \ldots, d-2$. Now Lemma 5.1 gives the result. □

As is mentioned in [3], the last result shows for example that the Coxeter graph is characterized by its spectrum.
We end with a remark about graphs for which distance-regularity is forced by its spectrum. If such a graph admits a switch partition, switching doesn't change the eigenvalues and we find another distance-regular graph. For strongly regular graphs a lot of examples are known, mostly from Seidel switching. There are also some examples for bipartite distance regular graphs with diameter 3. These are incidence graphs of symmetric block designs and there exist designs, for instance the recently discovered designs of Spence [14], with the required structure. Thus Spence finds many designs with the same parameters.

Acknowledgement. I thank A.E. Brouwer and E. Spence for many fruitful conversations about the subject of this paper and for their remarks on the manuscript.

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