Congestion models and weighted Bayesian potential games

Giovanni Facchini[1], Freek van Megen, Peter Borm and Stef Tijs[2]

April 24, 1995

Abstract

Games associated to congestion situations à la Rosenthal (1973) have pure Nash equilibria. This result implicitly relies on the existence of a potential function. In this paper we will provide a characterization of potential games in terms of coordination games and dummy games. Secondly, we extend Rosenthal's congestion model to an incomplete information setting, and show that the related Bayesian games are potential games and therefore have pure Bayesian equilibria.

[1]Department of Econometrics, Tilburg University. This author gratefully acknowledges financial support provided by the University of Trento, Italy.

[2]Department of Econometrics and CentER, Tilburg University, P.O.Box 90153, 5000 LE Tilburg, the Netherlands.

1 Introduction

The situation in which different agents make use of the same set of facilities and where the using costs are expressed in terms of a function depending on the number of users has been described by Rosenthal (1973). He also showed that the associated strategic game has a pure strategy Nash equilibrium. This result is implicitly due to the existence of a potential function for this class of games, as has been shown by Monderer and Shapley (1993).

In this paper we first derive a characterization of (weighted) potential games in terms of coordination and dummy games, which enables us to compute the dimension of the linear space of weighted potential games. In the second part we propose a generalization of Rosenthal's model, which gives the possibility to model broader classes of economic and real life situations. In fact we consider situations with incomplete information, in which an agent can be of several types and has, according to each type, a specific goal. On the other hand we will allow the different individuals to have different cost functions, introducing a vector of weights. A weighted congestion model has been proposed also by Milchtaich (1994) but, as will be shown later, the role of the weight vector in our model is quite different.

It turns out that the congestion games associated to weighted Bayesian congestion situations are Bayesian potential games and, under the common prior assumption, this implies the existence of a pure Bayesian equilibrium (van Heumen,Peleg, Tijs and Borm 1994). These results are illustrated by a booking game. The paper concludes with an example which shows that Bayesian potential games need not have a pure Bayesian equilibrium when the common prior assumption (Harsanyi (1967-68)) is violated. This was posed as an open question by van Heumen et al. (1994).

2 Potential Games

In this section we will provide a new characterization of weighted potential games, which have been introduced by Monderer and Shapley (1993). As a result of this characterization by means of coordination and dummy games, the dimension of the class of potential games is easily calculated.

2.1 A characterization of weighted potential games

Let $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ be a game in strategic form, where N is the finite set of players, A_i is the finite set of actions available to player i and $u_i : \prod_{i\in N} A_i \to \mathbb{R}$ is some von Neumann- Morgenstern utility function for player i. The game G is called a *weighted potential game* if there exists a function $P : \prod_{i\in N} A_i \to \mathbb{R}$ and a vector $w \in \mathbb{R}^N_{++}$ such that

$$u_i(a_i, a_{-i}) - u_i(a'_i, a_{-i}) = (P(a_i, a_{-i}) - P(a'_i, a_{-i}))w_i$$

for all $i \in N, a_i \in A_i, a'_i \in A_i$ and $a_{-i} \in A_{-i} := \prod_{j\in N\setminus\{i\}} A_j$.

We now consider the following two families of games: Γ_{WC} and Γ_D.
Let Γ_{WC} be the class of strategic form games $G = \langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle$ for which the utility function of player i is such that there exists a vector $w \in \mathbb{R}^N_{++}$ and a function $P : \prod_{i\in N} A_i \to \mathbb{R}$ with for each $i \in N$: $c_i = w_i P$. Such games are called the *weighted coordination games.*
Let Γ_D be the class of strategic form games $G = \langle N, \{A_i\}_{i\in N}, \{d_i\}_{i\in N}\rangle$ in which the utility function of a player does not depend on his own actions. So, for each $a_{-i} \in A_{-i}$, there exists a $k \in \mathbb{R}$ such that $d_i(a_i, a_{-i}) = k$ for each $a_i \in A_i$. These games are called *dummy games.*

In the following theorem we will use the above notions to characterize the class of weighted potential games.

Theorem 2.1 $G = \langle N, \{A_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ is a weighted potential game if and only if

$$u_i = c_i + d_i$$

for all $i \in N$ where c_i and d_i, are such that $\langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle \in \Gamma_{WC}$ and $\langle N, \{A_i\}_{i\in N}, \{d_i\}_{i\in N}\rangle \in \Gamma_D$. [1, 2]

Proof. We will just prove the only if part. Let $G = \langle N, \{A_i\}_{i\in N}, \{c_i\}_{i\in N}\rangle$ be a weighted potential game, then there exist $w \in \mathbb{R}^N_{++}$ and $P : \prod_{i\in N} A_i \to \mathbb{R}$ with :

$$u_i(a_i, a_{-i}) = w_i P(a_i, a_{-i}) + u_i(a'_i, a_{-i}) - w_i P(a'_i, a_{-i})$$

for all $i \in N, a_i \in A_i, a'_{-i} \in A_i$ and $a_{-i} \in A_{-i}$.

[1]Using the sum characterization it is easy to get a new axiomatic characterization of the potential maximizer, following the line originally proposed by Peleg, Potters and Tijs (1994).

[2]An analogous result has been recently obtained by Slade (1994) in a contest of oligopolistic competition , where the potential function is called "fictitious objective function".

Taking $c_i(a_i, a_{-i}) = w_i P(a_i, a_{-i})$ and $d_i(a_i, a_{-i}) = u_i(a_i, a_{-i}) - w_i P(a_i, a_{-i})$, it follows that $\langle N, \{A_i\}_{i \in N}, \{c_i\}_{i \in N} \rangle$ is a coordination game and $\langle N, \{A_i\}_{i \in N}, \{d_i\}_{i \in N} \rangle$ is a dummy game since $u_i(a_i, a_{-i}) - w_i P(a_i, a_{-i}) = u_i(a_i', a_{-i}) - w_i P(a_i', a_{-i})$ for all $i \in N$, $a_i \in A_i$, $a_i' \in A_i$ and $a_{-i} \in A_{-i}$. □

The notion of weighted potential game can be illustrated by means of the following

Example 2.2 In the following 2×2 game, which is a simplified version of Rousseau's stag-hunt game (1971), a player has to decide whether to cooperate to hunt a stag (action S) or to go off on his own and hunt rabbits (action R).

$$\begin{array}{c} \\ S \\ R \end{array} \begin{array}{c} \begin{array}{cc} S & R \end{array} \\ \left[\begin{array}{cc} 10, 20 & 0, 6 \\ 3, 0 & 3, 6 \end{array} \right] \end{array}$$

If the weight vector is $w = (1, 2)$, then a weighted potential exists and is given by

$$P = \left[\begin{array}{cc} 10 & 3 \\ 3 & 6 \end{array} \right]$$

For player 1 the payoff matrix is

$$\underbrace{\left[\begin{array}{cc} 10 & 0 \\ 3 & 3 \end{array} \right]}_{\text{w-pot. game}} = 1 \underbrace{\left[\begin{array}{cc} 10 & 3 \\ 3 & 6 \end{array} \right]}_{\text{w-coord. game}} + \underbrace{\left[\begin{array}{cc} 0 & -3 \\ 0 & -3 \end{array} \right]}_{\text{dummy game}}$$

and likewise for player 2

$$\underbrace{\left[\begin{array}{cc} 20 & 6 \\ 0 & 6 \end{array} \right]}_{\text{w-pot. game}} = 2 \underbrace{\left[\begin{array}{cc} 10 & 3 \\ 3 & 6 \end{array} \right]}_{\text{w-coord. game}} + \underbrace{\left[\begin{array}{cc} 0 & 0 \\ -6 & -6 \end{array} \right]}_{\text{dummy game}}$$

2.2 On the dimension of the linear space of potential games

Consider the family $\Gamma^{N,m}$ of strategic form games with fixed player set $N = \{1, ..., n\}$ and fixed action space $A = \prod_{i \in N} A_i$ with $m_i = |A_i|$ and $m = (m_1, \ldots, m_n)$. Clearly the family $\Gamma^{N,m}$ can be identified with the function space $(\mathbb{R}^N)^{\Pi_{i \in N} A_i}$ of maps from $\prod_{i \in N} A_i$ into $\mathbb{R}^N$ in a natural sense, according to the fact that the game is "known" if for every action profile $a \in \prod_{i \in N} A_i$ the utility vector $(u_1(a), u_2(a), ..., u_n(a))$ is given. Therefore we have that for the family $\Gamma^{N,m}$

$$\dim(\Gamma^{N,m}) = \dim(\mathbb{R}^N)^{\Pi_{i \in N} A_i} = n \prod_{i \in N} m_i.$$

In theorem 2.1 we have characterized (weighted) potential games as the sum of coordination games and dummy games. Using that result, we will derive the dimension of the linear space of potential games.

Let $P\Gamma^{N,m} \subset \Gamma^{N,m}$ denote the subclass of potential games with N players and $m = (m_1, ..., m_n)$, where $m_i = |A_i|$. As a corollary of theorem 2.1 we have that

$$P\Gamma^{N,m} = \Gamma_C^{N,m} + \Gamma_D^{N,m} \qquad (*)$$

where $\Gamma_C^{N,m}$ is the class of *coordination games* and $\Gamma_D^{N,m}$ is the class of *dummy games*. We can now prove the following

Theorem 2.3 For the linear space of potential games$P\Gamma^{N,m}$:

$$\dim\ P\Gamma^{N,m} = \prod_{i=1}^{n} m_i + \sum_{i=1}^{n} (\prod_{j \neq i} m_j) \quad - 1$$

Proof. Because of $(*)$ we have that

$\dim(P\Gamma^{N,m}) = \dim(\Gamma_C^{N,m}) + \dim(\Gamma_D^{N,m}) - \dim(\Gamma_C^{N,m} \cap \Gamma_D^{N,m})$. The dimensions of the right hand side of the equation can be easily computed, identifying $\Gamma_C^{N,m}$ with the function space $(\mathbb{R})^{\Pi_{i \in N} A_i}$, and $\Gamma_D^{N,m}$ with the function space $(\mathbb{R})^{\Pi_{i \neq 1} A_i} \times \ldots \times (\mathbb{R})^{\Pi_{i \neq n} A_i}$.
Then $\dim((\mathbb{R})^{\Pi_{i \in N} A_i}) = \Pi_{i \in N} m_i$ and $\dim((\mathbb{R})^{\Pi_{i \neq 1} A_i} \times \ldots \times (\mathbb{R})^{\Pi_{i \neq n} A_i}) = \sum_{i \in N} \prod_{j \neq i} m_j$.
Now it suffices to show that $\dim(\Gamma_D^{N,m} \cap \Gamma_C^{N,m}) = 1$. Using the definition of coordination and dummy games a game $\langle N, \{A_i\}_{i \in N}, \{u_i\}_{i \in N} \rangle$ in $(\Gamma_D^{N,m} \cap \Gamma_C^{N,m})$ has the property that there exist $u : \prod_{i \in N} A_i \to \mathbb{R}$ such that $u_i(a) = u(a)$ for all $i \in N, a \in \prod_{i \in N} A_i$ because it is a coordination game and $u(a) = u(b)$ for all $a, b \in \prod_{i \in N} A_i$ since it also is a dummy game. It means that $(\Gamma_D^{N,m} \cap \Gamma_C^{N,m})$ can be identified with $\mathbb{R}$. □

Remark 2.4 It is straightforward to show that the same result holds even in the computation of the dimension of the linear space of weighted potential games with fixed weight vector.

3 Congestion situations and Bayesian potential games

Rosenthal (1973) considers congestion situations where each agent wants to achieve an individual objective by choosing a suitable subset of a set M of common facilities. The using cost of each separate facility depends on the number of users.

Congestion situations give rise to potential games and, conversely, each finite potential game can be derived from a congestion situation (Monderer and Shapley, 1993). An important property of potential games is the existence of a pure Nash equilibrium. In this section we look at a general type of congestion situation which gives rise to Bayesian potential games with pure Bayesian equilibria. Our congestion model constitutes a generalization of Rosenthal's one.

Example 3.1 (The highway game). Consider the network of roads depicted in Fig. 1. Agent 1 is in city A and has to go to city B. Agent 2 is instead in city C and, depending on his type, has to drive either to A or to B. The cost each player has to afford to get to his target depends on the time spent on the road, which is obviously an increasing function of the number of users of the same street [3]. For example, as it is depicted in Figure 1, using the facility AC alone requires 3 units of time, while if the users are two, the time required increases up to 8 units etc. Let agent 1 be a businessman and agent 2 be retired, then the disutility of wasting time for agent 1 is bigger than that for agent 2 and this can be modelled using a vector of weights, like for example $w = (2, 1)$. Furthermore we assume that when each agent reaches his own target he gets a reward which depends on his type and on the the roads he has used. To give an intuition to this second point, one might imagine that choosing a certain road gives the driver the opportunity to enjoy a very nice landscape, something not possible otherwise. In our example we will assume in particular that the rewards are given by

$$r_1(C, AB) = 5 \quad r_2(A, CA) = 5 \quad r_2(B, CB) = 2$$
$$r_1(C, ABC) = 4 \quad r_2(A, CBA) = 4 \quad r_2(B, CAB) = 3$$

[3] As we will see, this assumption is not required to prove our results.

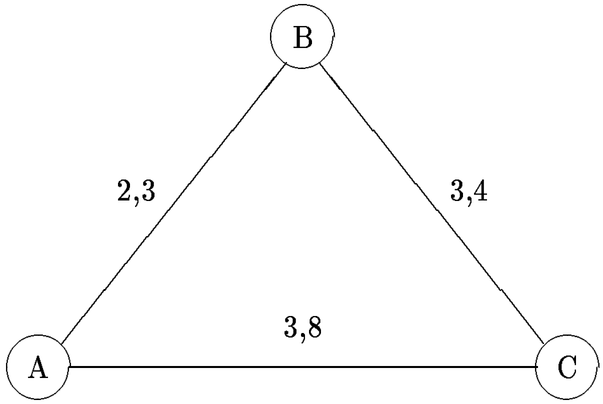

Figure 1: The highway game

and zero otherwise, where $r_1(C, AB) = 5$ means that the utility of player 1, who has to reach city C will increase by 5 units if he uses facility AB. The question to be answered is: what is a self enforcing way for each player to reach his own target?

The general model underlying this kind of situation is called *weighted Bayesian congestion situation* and can be described as follows:

$$[N, M, \{T_i\}_{i \in N}, p, \{r_i\}_{i \in N}, \{c_k\}_{k \in M}, w]$$

where

- $N = \{1, 2, \ldots, n\}$ is the finite set of *players*.
- $M = \{1, 2, \ldots, m\}$ is the finite set of *facilities*.
- T_i is the finite set of *types* of users $i \in N$, which specify the goal of each player.
- $p \in \Delta(T)$ is a *probability measure* on $T := \prod_{i \in N} T_i$.
- $r_i : 2^M \times T_i \to \mathbb{R}$; $\quad r_i(a_i, t_i)$ is the *reward* of player i for using the facilities in $a_i \in 2^M$ if his type is t_i.
- $c_k : \{0, 1, \ldots, |N|\} \to \mathbb{R}_+$ is the *cost function* depending on the number of users of facility k.
- $w \in \mathbb{R}^N_{++}$ is a interpreted as follows: player i has costs $w_i c_k(\ell), \ell \in \{0, 1, \ldots, |N|\}$ for factor k if there are ℓ users.

We are now going to define a Bayesian game (with common prior) corresponding to the weighted congestion situation described above. The general form of a Bayesian game G is given by

$$G = \langle N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, p, \{u_i\}_{i\in N}\rangle$$

where $N, \{T_i\}_{i\in N}$ and p play the obvious roles and the set of actions is defined by $A_i := 2^M$ for all players $i \in N$ and the utility function $u_i : (2^M)^N \times T \to \mathbb{R}$ for all $i \in N$ by

$$u_i(a,t) = r_i(a_i,t_i) - w_i \sum_{k\in a_i} c_k(n_k(a_1,\dots,a_n)) \qquad (**)$$

for all $a \in (2^M)^N$ and $t \in T$, where $n_k(a_1,\dots,a_n)$ is the number of users of facility k according to the chosen facility sets. It means that in our model the role of the weights is to extend Rosenthal's framework allowing different cost functions for each player. The problem of how to model a player specific contribution to the congestion has been considered also by Milchtaich in a recent paper (1994). In his framework however,where the weights are used to model the fact that a car and a heavy truck play different roles in inducing a congestion, it is impossible to guarantee the existence of a pure strategy Nash equilibrium if not all weights are equal.

Formally, given a Bayesian game $G = \langle N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, p, \{u_i\}_{i\in N}\rangle$ a strategy of player i is a map $x_i : T_i \to A_i$. A strategy profile $x \in X := \prod_{i\in N} X_i$ is called a (pure) *Bayesian equilibrium* of the game G if for for all $i \in N, t_i \in T_i$ and $a_i \in A_i$:

$$\sum_{t_{-i}\in T_{-i}} p(t_{-i}|t_i)u_i(\{x_j(t_j)\}_{j\in N},t) \geq \sum_{t_{-i}\in T_{-i}} p(t_{-i}|t_i)u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}},a_i),t)$$

where $p(t_{-i}|t_i)$ is the conditional probability[4] player i puts on t_{-i}, assuming that his own type is t_i.

For a Bayesian game $G = \langle N, \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, p, \{u_i\}_{i\in N},\rangle$ the corresponding ex ante game $\widehat{G}$ is defined by

$$\widehat{G} = \langle X_1,\dots,X_n,\ \widehat{u}_1,\dots,\widehat{u}_n\rangle$$

[4]This conditional probability can be defined if the assumption is made that every player puts positive probability on each of his types. We restrict ourselves to games for which this is the case.

where for all $i \in N, X_i = (A_i)^{T_i}$ is the strategy set for player i and $\hat{u}_i(x) = \sum_{t\in T} p(t)\ u_i((x_j(t_j))_{j\in N}, t)$ is the payoff function.

Harsanyi (1968, II, p. 321) proved the following theorem:

Theorem 3.2 For any Bayesian game G with common prior x is a Bayesian equilibrium of G if and only if x is a Nash equilibrium of the ex ante game $\hat{G}$.

In theorem 3.4 it will be shown that the game associated to a weighted Bayesian congestion situation is a weighted Bayesian potential game in the sense of the following

Definition 3.3 Let G be a Bayesian game. G is called a weighted Bayesian potential game if there exist a function $q : A \times T \to \mathbb{R}$ and a vector $w \in \mathbb{R}^N_{++}$ such that, for every $i \in N, a \in A, b_i \in A_i$, and $t \in T$

$$u_i(a,t) - u_i((a_{-i}, b_i), t) = w_i(q(a,t) - q((a_{-i}, b_i), t))$$

The function q is called a weighted potential for G.

Theorem 3.4 Let $G = \langle N, \{A_i\}_{i;nN}, \{T_i\}_{i\in N}, p, \{u_i\}_{i\in N}\rangle$ be a weighted Bayesian game arising from a weighted Bayesian congestion situation $[N, M, \{T_i\}_{i\in N}, p, \{r_i\}_{i\in N}, \{c_k\}_{k\in M}, w]$. Then G is a weighted Bayesian potential game.

Proof. Define

$$q(a,t) = \sum_{i\in N} \frac{r_i(a_i, t_i)}{w_i} - \sum_{k\in M} \sum_{\ell=0}^{n_k(a)} c_k(\ell)$$

Then, using $(**)$

$$w_i q(a,t) - u_i(a,t) = \sum_{j\neq i} \frac{r_j(a_j, t_j) w_i}{w_j} - w_i \sum_{k\in M} \sum_{\ell=0}^{n_k(a_{-i})} c_k(\ell)$$

This means that $w_i q(a,t) - u_i(a,t)$ does not depend on the action a_i. Therefore q is a weighted potential for G. □

Remark 3.5 As we have said, Monderer and Shapley (1993) have proved that each finite potential game can be derived from a congestion situation in the complete information case. We do not know yet if this is true also in presence of incomplete information.

Remark 3.6 Our result can be easily extended to consider type dependent weights, but this could give rise to interpretation problems (see the highway game for an example). Therefore we prefer to confine all the private information present in our model to the type dependent targets.

We now apply the previous results to the highway model and obtain the associated *highway game*, which is described as follows:

$$\langle\{1,2\},\{AC,CA,CB,ABC,CBA,CAB\},\{\{C\}\},\{A,B\}\},p,\{u_1,u_2\}\rangle$$

where the common prior p will be specified later and u_1, u_2 are the utility functions of the players. The cost matrices are, depending on the type of each player

$$\begin{array}{cc} & \begin{array}{cc} A & B \end{array} \\ C \begin{array}{c} AC \\ ABC \end{array} & \begin{array}{cc} \begin{array}{cc} CA & CBA \end{array} & \begin{array}{cc} CB & CAB \end{array} \\ \left[\begin{array}{cc} 11,3 & 1,1 \\ 6,-2 & 10,3 \end{array}\right] & \left[\begin{array}{cc} 1,1 & 11,7 \\ 8,2 & 8,3 \end{array}\right] \end{array} \end{array}$$

In other words, if player 2 likes to go to city A (he is of type A) using road CA and player 1 uses the same facility to go from A to C, then player 1 supports a disutility of 11=16-5 and player 2 of 3=8-5.

Consider now the following (common) prior

$$p = \begin{array}{c} \\ C \end{array} \begin{array}{c} \begin{array}{cc} A & B \end{array} \\ \left[\begin{array}{cc} \frac{1}{2} & \frac{1}{2} \end{array}\right] \end{array}$$

The associated ex ante game is therefore:

$$\begin{array}{c} \\ AC \\ ABC \end{array} \begin{array}{c} \begin{array}{cccc} CA,CB & CA,CAB & CBA,CB & CBA,CAB \end{array} \\ \left[\begin{array}{cccc} 6,2 & 11,5 & 1,1^* & 6,4 \\ 7,0 & 7,0.5 & 9,2.5 & 9,3 \end{array}\right] \end{array}$$

There is only one pure strategy Nash equilibrium, which suggest to player 1 always to use road AC while player 2 has to use road CBA, if his type is "A", and road CB if his type is B. Considering the associated ex ante potential, this strategy profile turns out to be also the potential minimizer.

$$\begin{array}{c} \\ AC \\ ABC \end{array} \begin{array}{c} \begin{array}{cccc} CA,CB & CA,CAB & CBA,CB & CBA,CAB \end{array} \\ \left[\begin{array}{cccc} 0 & 3 & -1^* & 2 \\ 0.5 & 1 & 3 & 3.5 \end{array}\right] \end{array}$$

4 Inconsistent priors

In this paper we have considered a weighted congestion model, which has been associated to a weighted Bayesian potential game. It is well known (see van Heumen, Peleg, Tijs and Borm, (1994)) that every Bayesian potential game with common prior has a pure strategy equilibrium. In the same paper the problem whether each Bayesian potential game has a pure equilibrium is posed as an open question. It turns out that this need not be the case as can be seen in the next example. To show our result we first state the following

Lemma 4.1 Let $G = \langle \{A_i\}_{i\in N}, \{T_i\}_{i\in N}, \{p_i\}_{i\in N}, \{u_i\}_{i\in N}\rangle$ be a general Bayesian game where p_i is the probability measure of player I over $T := \prod_{i\in N} T_i$.
Let $\hat{G}$ be the game associated to G where, for every $i \in N, X_i$ is the set of pure strategies of player i and for every $x \in X, i \in N$,

$$\hat{u}_i(x) := \sum_{t\in T} p_i(t)u_i(\{x_j(t_j)\}_{j\in N}, t)$$

then if $\{x_i\}_{i\in N}$ is a Bayesian equilibrium of G, $\{x_i\}_{i\in N}$ is a Nash equilibrium of the ex-ante game $\widehat{G}$ associated to G.

Proof. By definition of a Bayesian equilibrium, we have that for all $t_i \in T_i$, $a_i \in A_i$,

$$\sum_{t_{-i}} p_i(t_{-i}|t_i)u_i(\{x_j(t_j)\}_{j\in N}, t) \geq \sum_{t_{-i}} p_i(t_{-i}|t_i)u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}}, a_i), t)$$

Then

$$\sum_{t_{-i}}(\sum_{s_{-i}} p_i(s_{-i}, t_i))p_i(t_{-i}|t_i)u_i(\{x_j(t_j)\}_{j\in N}, t) \geq$$
$$\sum_{t_{-i}}(\sum_{s_{-i}} p_i(s_{-i}, t_i))p_i(t_{-i}|t_i)u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}}, a_i), t)$$

so for each $t_i \in T_i$, $a_i \in A_i$

$$\sum_{t_{-i}} p_i(t)u_i(\{x_j\}_{j\in N}, t) \geq \sum_{t_{-i}} p_i(t)u_i((\{x_j\}_{j\in N\setminus\{i\}}, a_i), t)$$

It means that

$$\sum_t p_i(t) u_i(\{x_j(t_j)\}_{j\in N}, t)) \geq \sum_t p_i(t) u_i((\{x_j(t_j)\}_{j\in N\setminus\{i\}}, a_i), t)$$

and thus for all $y_i \in X_i$

$$\widehat{u}_i(x) \geq \widehat{u}_i(y_i, x_{-i})$$

□

Now we look at a specific Bayesian potential game with inconsistent priors. There are 2 players, 1 and 2. Each player has 2 different types $T_1 = \{\alpha, \beta\}, T_2 = \{\gamma, \delta\}$. The priors p_1, p_2 are given by

$$p_1 = \begin{array}{c} \\ \alpha \\ \beta \end{array} \begin{array}{c} \begin{array}{cc} \gamma & \delta \end{array} \\ \begin{bmatrix} 0 & \frac{3}{4} \\ \frac{1}{4} & 0 \end{bmatrix} \end{array} \qquad p_2 = \begin{array}{c} \\ \alpha \\ \beta \end{array} \begin{array}{c} \begin{array}{cc} \gamma & \delta \end{array} \\ \begin{bmatrix} \frac{1}{3} & 0 \\ 0 & \frac{2}{3} \end{bmatrix} \end{array}$$

and the payoff matrices are given typewise:

$$\begin{array}{ccc} & \gamma & \delta \\ & \begin{array}{cc} L & R \end{array} & \begin{array}{cc} L & R \end{array} \\ \alpha \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 1,1 & 0,0 \\ 0,0 & 1,1 \end{bmatrix} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} \\ \beta \begin{array}{c} T \\ B \end{array} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} & \begin{bmatrix} 0,0 & 1,1 \\ 1,1 & 0,0 \end{bmatrix} \end{array}$$

The corresponding $\hat{G}$ game is given by

$$\begin{array}{c} \\ TT \\ TB \\ BT \\ BB \end{array} \begin{array}{c} \begin{array}{cccc} LL & LR & RL & RR \end{array} \\ \begin{bmatrix} 0, \frac{1}{3} & \frac{3}{4}, 1 & \frac{1}{4}, 0 & 1, \frac{2}{3} \\ \frac{1}{4}, 1 & 1, \frac{1}{3} & 0, \frac{2}{3} & \frac{3}{4}, 0 \\ \frac{3}{4}, 0 & 0, \frac{2}{3} & 1, \frac{1}{3} & \frac{1}{4}, 1 \\ 1, \frac{2}{3} & \frac{1}{4}, 0 & \frac{3}{4}, 1 & 0, \frac{1}{3} \end{bmatrix} \end{array}$$

It is easy to show that there are no pure Nash equilibria in this game. Then using Lemma 4.1 the Bayesian game does not have a pure Bayesian equilibrium.

Acknowledgements. We gratefully thank Eric van Damme and Vincent Feltkamp for many helpful comments.

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